ON CLOSED SPHERICAL MOTIONS*

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I. Introduction. The study of one-parameter closed motions became an interesting subject in kinematics after the work of Jacob Steiner [4, p. 113–115] and A. Holditch [4]. During the second half of the nineteenth century there appeared many publications about Steiner's and Holditch's theorems, for example: C. Leudesdorf [6] and [7], A. B. Kempe [8], E. B. Elliott [5], [9] and [10]. Elliott wrote about spherical motions, whereas the others wrote only on planar motions.

In 1948, W. Blaschke defined the Steiner point and the Steiner vector for one-parameter closed spherical motions and gave the area formula equivalent to that of Steiner for the case of a sphere [3]. In order to obtain this, Blaschke integrated the geodesic curvature of a spherical closed curve in the formula for the Gauss–Bonnet theorem [2, p. 237]. A recent study in this field has been given in H. R. Müller's *Sphärische Kinematik* [1, p. 50–51].

In Sec. IV, with Theorem 3, we give a necessary and sufficient condition for the points of a moving sphere which pass around equal areas or draw the same curve on the fixed sphere K'. Hence, we show the invariants of one-parameter closed spherical motion using only the Steiner vector. In this way Holditch's and Steiner's well-known theorems can be proved more elegantly.

In Sec. III, Corollaries I, II and, in Sec. IV, their extensions to the spherical case are original.

II. Basic concepts. A. *Spherical motions.* A motion of a rigid body about a fixed point 0 uniquely defines a motion K/K' of the moving unit sphere K with the fixed center 0 over the fixed unit sphere K' of the same center.

Let $\{0; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{0; \vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ be two right-handed sets of orthogonal unit vectors that are rigidly linked to the spheres K and K' respectively, and denote by E, E' the matrices

$$E = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}, \qquad E' = \begin{bmatrix} \vec{e}_1' \\ \vec{e}_2' \\ \vec{e}_3' \end{bmatrix}. \tag{1}$$

We may then write

$$E = AE' \quad \text{or} \quad E' = A^{-1}E = A^TE \tag{2}$$

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where

$$A = [a_{ij}] \tag{3}$$

is a positive orthogonal 3×3 matrix, the exponent -1 indicates the inverse, and the superscript T the transpose. The elements a_{ij} of the matrix A will be regarded as functions of a real single parameter t, and we will write $A = A(t)$ to indicate that we restrict the discussion to one-parameter motions..

Definition. If the matrix function $A(t)$ is periodic, say

$$A(t + 2\pi) = A(t), \qquad \forall\ t \tag{4}$$

the motion K/K' is closed; otherwise it is open.

During the closed motion K/K', the orbits of the points of K and K' are closed curves; also the moving and fixed centrodes (polodes) are closed curves.

B. *The Pfaffian vector.* Since the matrix A is a positive orthogonal matrix we may write

$$AA^T = I \tag{5}$$

where I is the unit matrix. This equation, by differentiation with respect to t, yields

$$dA \cdot A^T + A \cdot dA^T = 0. \tag{6}$$

This relation shows that the matrix

$$\Omega = dA \cdot A^T \tag{7}$$

is antisymmetric. We may write

$$\Omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}. \tag{8}$$

Differentiation of the first of Eqs. (2) with respect to t yields

$$dE = \Omega E. \tag{9}$$

The position vector, with respect to K, of a point X of K can be expressed as

$$\vec{X} = X^T E \tag{10}$$

and its differential velocity, with respect to K', by

$$d\vec{X} = (dX + \Omega^T X)^T E. \tag{11}$$

If the point $\vec{X}$ is a fixed point on K then its differential velocity, with respect to K, is $dX^T \cdot E = 0$ and (11) reduces to

$$d\vec{X} = X^T \Omega E. \tag{12}$$

Now in order to rewrite (12) more meaningfully, we define a new vector $\vec{\omega}$ such that its components ω_1, ω_2, ω_3 are the nonzero elements of the matrix Ω; Eq. (12) then becomes

$$d\vec{X} = \vec{\omega} \times \vec{X} \tag{13}$$

where the cross denotes the vector product and $\vec{\omega}$ is called the *instantaneous Pfaffian* *(differential) vector* of the motion K/K' [1]. The Pfaffian vector $\vec{\omega}$ at a given instant t of a one-parameter motion on a sphere is an analogue to the Darboux vector in the differential geometry of space curves [2].

The direction of the vector $\vec{\omega}$ passes through the poles (the instantaneous centers of rotation) P and P' on the spheres K and K'. Denoting the position vectors of P and P' by $\vec{p}$ and $\vec{p}'$, we have

$$\vec{\omega} = \omega\vec{p} \quad \text{or} \quad \vec{\omega} = \omega\vec{p}' \tag{14}$$

where $\omega = |\vec{\omega}|$ is the instantaneous angular velocity of the motion K/K'.

C. *Steiner vector and Steiner area formula on unit sphere.* Let (X) be the trajectory of an arbitrary fixed point X of K, and imagine this trajectory to be covered by a distribution of mass with density $\omega = |\vec{\omega}|$. The centroid of this mass distribution is defined as the Steiner point; it has the position vector

$$\vec{S}_x = \oint \omega\vec{X} \Big/ \oint \omega \tag{15}$$

where the integrations are taken along the closed curve (X) on K'. Similarly, the Steiner point of the centrode (P) on K is

$$\vec{S}_p = \oint \omega\cdot\vec{p} \Big/ \oint \omega \tag{16}$$

where the integrations are taken along the closed curve (P). The vector

$$\vec{V} = \oint \omega\cdot\vec{p} \tag{17}$$

in (16) will be called the Steiner vector of the orbit (X). Since (17) can be written as

$$\vec{V} = \oint \sum_{i=1}^{3} \omega\cdot p_i\cdot\vec{e}_i = \sum_{i=1}^{3} \left(\oint \omega\cdot p_i\right)\cdot\vec{e}_i\ ,$$

the components of $\vec{V}$ are

$$V_i = \oint \omega\cdot p_i = \oint \omega_i \qquad (i = 1, 2, 3). \tag{18}$$

THEOREM 1 (W. BLASCHKE). *Let (X) be the orbit, on K', of an arbitrary fixed point X of K. The spherical area bounded by the closed curve (X) may be calculated from*

$$f_x = 2\pi - \vec{V}\cdot\vec{X} - 2\pi\cdot n. \tag{19}$$

This theorem has been derived by Blaschke [3] from the Gauss–Bonnet area formula. The formula (19) is the equivalent of the Steiner plane area formula [4] for the case of the sphere, where $\vec{X}$ denotes the position vector of the curve (X), the dot $(\cdot)$ indicates the inner product and n is the number of rotations of the centrode (P) at the point X.

III. Two corollaries of Holditch's theorem for one-parameter closed planar motions. For one-parameter closed planar motions, Holditch's theorem [4] is as follows:

THEOREM 2. *Consider a closed planar convex curve (L) and the closed curve (X) described by a fixed point X of a straight line segment $(MN)^-$ of constant length, the endpoints of which are moving along (L). The area f between the curves (L) and (X) is then expressed by the formula*

$$f = \pi \cdot (MX)^{-} \cdot (NX)^{-} \tag{20}$$

where $(MX)^{-}$ *and* $(NX)^{-}$ *are the line segments on the segment* $(MN)^{-}$.

It follows that the area f does not depend on the curve (L) but only on the length of $(MN)^{-}$ and the position of the point X of $(MN)^{-}$. Two further results follow from this theorem.

COROLLARY I. *Consider a one-parameter closed planar motion and a fixed straight line k on the moving plane. Choosing four arbitrary fixed points M, N, X, Y on the line k, let two of them move on the same curve (L), while the other two describe different curves (X) and (Y). If the ring area between (L) and (X) is f, and the area between (L) and (Y) is f′, then the ratio f/f′ depends only on the relative positions of these four points* (Fig. 1).

Proof. Consider another point Y on the segment $(MN)^{-}$ with the orbit (Y) on the same fixed plane. According to (20), the area f' between the curves (L) and (Y) is

$$f' = \pi \cdot (MY)^{-} \cdot (NY)^{-}. \tag{21}$$

Then, joining (20) and (21), one can obtain

$$\frac{f}{f'} = \left[\frac{(MX)^{-}}{(MY)^{-}}\right]^2 \cdot \frac{(MY)^{-}}{(MX)^{-}} \cdot \frac{(NX)^{-}}{(NY)^{-}}. \tag{22}$$

This invariant (22) does not depend on the curve (L) and the length of $(MN)^{-}$; it depends only on the choice of the points X and Y on $(MN)^{-}$. Since $X \neq Y$ it follows that $(MY^{-})/(MX^{-}) \neq 1$. Denote $\lambda = ((MY)^{-}/(MX)^{-}) \cdot ((NX)^{-}/(NY)^{-})$; λ is the cross ratio of the four points M, N, X, Y, i.e. $\lambda = (MNXY)$.

COROLLARY II. *Let M, N, A, B be four points of the moving plane which pass around equal areas, and let the segments* $(MN)^{-}$, $(AB)^{-}$ *meet in X. Then the necessary and sufficient condition for M, N, A, B to lie on the same circle of the moving plane is*

$$(MX)^{-} \cdot (NX)^{-} = (AX)^{-} \cdot (BX)^{-}. \tag{23}$$

The center of this circle is the Steiner point of the motion.

Proof. All the segments $(MN)^{-}$, $(AB)^{-}$, $(MA)^{-}$, $(NA)^{-}$, $(MB)^{-}$, $(NB)^{-}$ are constant. Using Eq. (20), one obtains the following value for f:

$$f = \pi \cdot (AX)^{-} \cdot (BX)^{-}. \tag{24}$$

Comparison of (24) and (20) yields Eq. (23).

This proposition is a particular case of the Steiner theorem [4, p. 115].

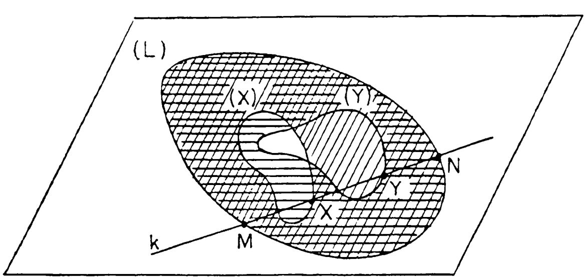

FIG. 1

IV. Generalizations to spherical motion. During the one-parameter closed motion K/K', two fixed points M, N of K generally draw two closed curves on the fixed sphere K'. Let these two curves encircle the spherical areas F_m and F_n respectively. Consider another fixed point X of K on the arc $\breve{(MN)}$ of a great circle on K of given length. During the same motion the point X also draws another closed curve (X) on the sphere K'. Denote by F_x the area surrounded by (X). In order to simplify the calculations one can suppose also that the total rotation numbers n of the moving centrode at the points M, N and X are the same. This can always be achieved by choosing the length of $\breve{(MN)}$ sufficiently small. Therefore according to (19)

$$\begin{aligned} F_m &= 2\pi(1 - n) - \vec{M}\cdot\vec{V}, \\ F_n &= 2\pi(1 - n) - \vec{N}\cdot\vec{V}, \\ F_x &= 2\pi(1 - n) - \vec{X}\cdot\vec{V}, \end{aligned} \tag{25}$$

where the vectors $\vec{M}, \vec{N}$ and $\vec{X}$ are the position vectors of the points M, N and X respectively. On the other hand, since

$$\vec{N} = \vec{M} + \overrightarrow{(MN)}$$

and

$$\vec{X} = \vec{M} + \overrightarrow{(MX)}$$

(25) becomes

$$\begin{aligned} F_n &= F_m + \vec{V}\cdot\overrightarrow{(NM)}, \\ F_x &= F_m + \vec{V}\cdot\overrightarrow{(XM)} = F_n + \vec{V}\cdot\overrightarrow{(XN)}. \end{aligned} \tag{26}$$

Thus one obtains

$$F_x = \tfrac{1}{2}\cdot\{F_m + F_n + (\overrightarrow{(XM)} + \overrightarrow{(XN)})\cdot\vec{V}\}. \tag{27}$$

This is equivalent to the Holditch formula [4]. Two similar formulas had been obtained by Elliott [5] and Müller [1, p. 50]. But (27) contains the Steiner vector of the motion and so it is more useful in obtaining some more extensions.

The most important special case of Holditch's theorem is the case of $F_m = F_n$. Now let us discuss the necessary and sufficient conditions for this case. In this special case, the ends M and N pass round equal areas or they draw the same curve (Γ) on K'. For this case, according to the first equality of (26), one can write

$$\vec{V}\cdot\overrightarrow{(MN)} = 0. \tag{28}$$

Hence there is the theorem below:

THEOREM 3. *During the one-parameter closed motion K/K', $\vec{V}\cdot\overrightarrow{(MN)} = 0$ is the necessary and sufficient condition that the two ends M, N of the moving arc go around either the same spherical curve or two curves of equal area.*

Now, more generally, one can ask for the locus of all the points of K which pass round either the same spherical curve or different spherical curves of equal area. According to Theorem 3, for each pair of points of this sort, the directions are orthogonal to the Steiner vector $\vec{V}$ of the motion K/K'. Therefore all of these points must lie on the same plane whose normal is $\vec{V}$ or, in other words, the locus required for any given value of area is a circle whose spherical center is located on the Steiner vector $\vec{V}$. Therefore there is the following theorem, equivalent to Steiner's theorem in the extended form in Müller's book [1, Theorem 21, p. 51].

THEOREM 4. *Consider the areas surrounded by different points of the moving sphere K that are not all on the same great circle; for the equality of these areas the necessary and sufficient condition is that these points must lie on the same plane whose normal is the Steiner vector* $\vec{V}$ *of the motion* K/K'.

This theorem contains also the generalization of the Corollary II in case of planar motion to the spherical case. In the case of $F_m = F_n$, let two ends M, N of a moving arc $(MN)^{\smile}$ go around the same spherical closed curve (Γ) of K'. Then the spherical ring area F between the closed curves (Γ) and (X) can be expressed as follows:

$$F = F_n - F_x \quad \text{or} \quad F = F_m - F_x$$

and, according to (26),

$$F = \vec{V}\cdot(NX)^{\to} \quad \text{or} \quad F = \vec{V}\cdot(MX)^{\to}. \tag{29}$$

This, the spherical analogue of (20), shows that the ring area on the sphere depends on the Steiner vector $\vec{V}$ or the closed curve (Γ) of K'. In spite of this, Corollary I can be mentioned for the spherical case. In order to show this invariant property of the motion K/K', let us rewrite (29) analytically. For this we can choose any special rectangular coordinate system in K because (29) is an inner product, and an inner product is independent of coordinate transformations. For example, let $(MN)^{\smile}$ be on the great circle of $(X_1 0 X_3)$ and $M = (1, 0, 0)$. Then the central angles of $(MX)^{\smile}$, $(XN)^{\smile}$ and $(MN)^{\smile}$ are ρ_1, ρ_2 and ρ_3 respectively. Thus $X = (\cos \rho_1, 0, \sin \rho_1)$ and $N = (\cos \rho, 0, \sin \rho)$. Hence (29) reduces to

$$F = 2\cdot\frac{\sin(\rho_1/2)\cdot\sin(\rho_2/2)}{\sin(\rho/2)}\cdot V_3 . \tag{30}$$

And, since $\sin(\rho_1/2) = (MX)^{-}/2$, $\sin(\rho_2/2) = (NX)^{-}/2$, and $\sin(\rho/2) = (MN)^{-}/2$, (30) becomes

$$F = \frac{(MX)^{-}\cdot(NX)^{-}}{(MN)^{-}}\cdot V_3 \tag{31}$$

where V_3 has been defined in (18).

Now, consider another point Y on the arc $(MN)^{\smile}$, such that while the point X draws its orbit (X), Y draws another orbit (Y) on the same sphere K'. The area F' between the curves (Γ) and (Y), according to (31), can be expressed as follows

$$F' = \frac{(MY)^{-}\cdot(NY)^{-}}{(MN)^{-}}\cdot V_3 . \tag{31'}$$

Then, (31) and (31') give

$$\frac{F}{F'} = \left[\frac{(MX)^{-}}{(MY)^{-}}\right]^2\cdot\frac{(MY)^{-}\cdot(NX)^{-}}{(MX)^{-}\cdot(NY)^{-}} \tag{32}$$

where $\mu = (MY)^{-}\cdot(NX)^{-}/(MX)^{-}\cdot(NY)^{-}$ can be expressed another way: let the bisectors of the angles $(XMY)^{\wedge}$, $(XNY)^{\wedge}$ met $(XY)^{-}$ at the points M', N' respectively. Since $(XMY)^{\wedge} = (XNY)^{\wedge}$ (Fig. 2) $(MM')^{-}$ and $(NN')^{-}$ meet on the great circle arc

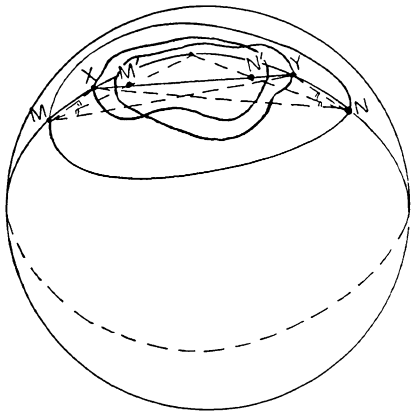

FIG. 2

$(MN)^{\smile}$. Hence $\mu = (M'N'XY)$, is the cross ratio of the collinear points M', N', X, Y. Therefore (32) becomes

$$\frac{F}{F'} = \left[\frac{(MX)^{-}}{MY}\right]^2 \cdot \mu.$$

Hence one can give the following theorems:

THEOREM 5. *Let the two ends M, N of a moving arc $(MN)^{\smile}$ with constant length go around the same convex simple curve (Γ) on K'. If one chooses a fixed point X on the arc $(MN)^{\smile}$, X describes a closed curve (X) on the same sphere, while M and N move on the curve (Γ). The spherical ring area F between the two closed curves (Γ) and (X) can be expressed by* (31).

THEOREM 6. *For all closed curves of K' which have the same component $V_3 = \oint \omega_3$, the ring area F only depends on the length of $(MN)^{-}$ and the position of the point X on $(MN)^{\smile}$.*

THEOREM 7. *Consider a one-parameter closed spherical motion K/K' and a fixed great circle on the moving sphere K. Choosing four arbitrary fixed points M, N, X, Y on this great circle, let two of them move on the same curve (Γ), while the other two describe different curves (X) and (Y). If the ring area between (Γ) and (X) is F, and the area between (Γ) and (Y) is F', then the ratio F/F' depends only on the relative positions of these four points M, N, X, Y.*

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