JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 128, No. 1, pp. 203–221, January 2006 (© 2006)
DOI: 10.1007/s10957-005-7565-y

On Coincidence of Feedback Nash Equilibria and Stackelberg Equilibria in Economic Applications of Differential Games[1]

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Communicated by G. Feichtinger

Abstract. The scope of the applicability of the feedback Stackelberg equilibrium concept in differential games is investigated. First, conditions for obtaining the coincidence between the stationary feedback Nash equilibrium and the stationary feedback Stackelberg equilibrium are given in terms of the instantaneous payoff functions of the players and the state equations of the game. Second, a class of differential games representing the underlying structure of a good number of economic applications of differential games is defined; for this class of differential games, it is shown that the stationary feedback Stackelberg equilibrium coincides with the stationary feedback Nash equilibrium. The conclusion is that the feedback Stackelberg solution is generally not useful to investigate leadership in the framework of a differential game, at least for a good number of economic applications.

Key Words. Differential games, stationary feedback Nash equilibrium, stationary feedback Stackelberg equilibrium.

[1]This paper was presented at the 8th Viennese Workshop on Optimal Control, Dynamic Games, and Nonlinear Dynamics: Theory and Applications in Economics and OR/MS, Vienna, Austria, May 14–16, 2003, at the Seminar of the Instituto Complutense de Analisis Economico, Madrid, Spain, June 20, 2003, and at the Sevilla Workshop on Dynamic Economics and the Environment, Sevilla, Spain, July 2–3, 2003. The author is grateful to the participants in these sessions, in particular F.J. Andre and J. Ruiz, for their comments. Five referees provided particularly helpful suggestions. Financial support from the Ministerio de Ciencia y Tecnologia under Grant BEC2000-1432 is gratefully acknowledged.
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1. Introduction

Differential games have been used in economics to study the strategic interdependence among agents in a dynamic framework; see Ref. 1 for an excellent survey of differential games applications in economics and management science. Of the numerous economic applications of differential games, we mention here the ones developed in Refs. 2–5; in these papers, it is recognized explicitly that the stationary feedback Nash equilibrium (SFNE) coincides with the stationary feedback Stackelberg equilibrium (SFSE).

The aim of this paper is to investigate how general this coincidence is, or in other words to evaluate the scope of the applicability of the feedback Stackelberg equilibrium concept in differential games. This issue has been addressed already by Başar and Olsder in Ref. 6, pp. 415–417, although these authors focus on linear-quadratic games. Here, we extend their analysis looking for general conditions that yield the coincidence between the two equilibria. In our analysis, these conditions are defined in terms of the primitive functions of the game, i.e., the instantaneous payoff functions and the state equations. Our results show that, when these conditions are satisfied, the instantaneous reaction functions of the players are orthogonal with respect to the control variables so that the players optimal policy depends only on the state variable; the first movement advantage disappears yielding the coincidence between the two equilibria. This result is a consequence of the continuous-time framework in which differential games are defined. For these games, the variation of the state variable is given by a differential equation; i.e., the variation is instantaneous so that the interdependence that appears in the game through this variation cannot be captured in the first-order conditions (f.o.c.) of the maximization of the right-hand side Hamilton–Jacobi–Bellman equations. The result is that, if there does not exist another kind of interdependence in the game, the reaction functions are orthogonal with respect to the control variables.

In order to evaluate how general this coincidence is in economic applications of differential games, we define a class of differential games that represents the underlying structure of a good number of economic applications of differential games; as a corollary of our previous result, we show that, for this class of differential games, both equilibria are identical independently of which player is the leader of the game. Thus, our conclusion is that the feedback Stackelberg solution cannot generally be used to investigate leadership in the framework of a continuous-time differential game. However, this does not mean that leadership cannot be investigated in a dynamic framework, since the coincidence does not occur in the global Stackelberg solution or in discrete-time difference games.

The paper is organized as follows. In Section 2, the SFNE and SFSE definitions are presented and the conditions for the coincidence of both equilibria are defined. The stationary global Stackelberg equilibrium concept is also discussed. In Section 3, the scope of the applicability of the SFSE in economic applications of differential games is addressed. Section 4 concludes the paper.

2. Coincidence of the Feedback Nash and Stackelberg Equilibria

Let us consider a differential game with two players, $h=1,2$; $u^h(t)$ is the vector of control variables of player h, $u^h(t)\in U^h$, with $U^h\subseteq R^{m^h}$; $x(t)$ is the vector of state variables, $x(t)\in L$, with $\mathrm{L}\subseteq R^n$. Both players live infinitely and have strictly concave, twice differentiable utility functions that depend on the state and control variables, $v^h(x(t),u^1(t),u^2(t))$, $h=1,2$. The dynamics of the state variables is defined by the functions $f_i(x(t),u^1(t),u^2(t))$, $i=1,\dots,n$, which are twice differentiable. Then, the differential game between the two players can be represented by the following pair of interdependent optimal control problems:

$$\max_{u^h(t)\in U^h} J^h=\int_0^\infty e^{-rt}v^h(x(t),u^1(t),u^2(t))dt, \tag{1}$$

$$\text{s.t.}\quad \dot{x}_i(t)=f_i(x(t),u^1(t),u^2(t)),\quad x_i(0)=x_{i0}, \tag{2}$$

where r is the players discount rate, $h=1,2$, and $i=1,\dots,n$.

For the information structure, we consider a feedback or closed-loop information structure.[3] For this kind of information structure, the control of player – at each point in time is a function of both the time and the state of the system $x(t)$ at that time. Thus, we can write the feedback strategies as $u^h=u^h(t,x(t))$, which are defined on $[0,\infty)\times L$. However, given that, for infinite-horizon autonomous differential games, the equilibrium strategies are stationary, we work with stationary feedback strategies $u^h=u^h(x)$, defined only on L. Formally, we have the following definition.

Definition 2.1. The stationary feedback strategy space for player h is the set

$$S^h=\{u^h(x(t)):u^h(x(t)) \text{ is continuous in } x(t) \text{ and } u^h(x(t))\in U^h\}.$$

[3]In the traditional differential games literature (Ref. 6), feedback strategies are known as well as Markov strategies. For expositional simplicity, we consider infinite-horizon autonomous problems, although our result is general and works for any kind of differential game nonautonomous or with a finite-time horizon that satisfies the conditions of Definition 2.4.

The feedback strategies describe decision rules that prescribe a value for the control variable as a function of the observed value of the state variables.

2.1. Stationary Feedback Nash Equilibrium. It is now straightforward to present the definition of stationary feedback Nash equilibrium.

Definition 2.2. A stationary feedback Nash equilibrium (SFNE) is a pair of stationary feedback strategies $(u^{1^*}, u^{2^*}) \in S^1 \times S^2$ such that, for every possible initial condition (t_0, x_0),

$$J^h(u^{h^*}, u^{j^*}) \geq J^h(u^h, u^{j^*}),$$

for every $u^h \in S^h$, with $h, j = 1, 2, h \neq j$.

Feedback strategies provide a strongly time consistent (i.e., subgame perfect) equilibrium.[4] Subgame perfectness requires that, for every subgame, the restriction of a strategy pair (u^{1^*}, u^{2^*}) to the subgame remains an equilibrium in that subgame.

Next, we explain how the dynamic programming approach can be used to calculate this equilibrium. For the SFNE, the equilibrium strategies (u^{1^*}, u^{2^*}) must satisfy the following Hamilton–Jacobi–Bellman (HJB) equations[5]

$$rV^h(x) = \max_{u^h \in U^h} \left\{ v^h(x, u^1, u^2) + \sum_{i=1}^{n} \frac{\partial V^h(x)}{\partial x_i} f_i(x, u^1, u^2) \right\}, \quad h = 1, 2, \tag{3}$$

where $V^h(x)$ is the value for player h of the game that starts at x. Note that, although in the general case V^h is also a function of t, it can be shown that, for infinite-horizon autonomous problems, the value functions do not depend on t; see Ref. 8, pp. 262–263.

In order to make operative (3), let us introduce the vector of the players instantaneous reaction functions,

$$T^h(x; u^j, \nabla V^h(x)) = \arg\max_{u^h \in U^h} \left\{ v^h(x, u^1, u^2) + \sum_{i=1}^{n} \frac{\partial V^h(x)}{\partial x_i} f_i(x, u^1, u^2) \right\}, \tag{4}$$

[4] Usually, open-loop strategies are weakly time consistent in the case of simultaneous play (Nash) or time inconsistent in the case of sequential play (Stackelberg).
[5] In order to simplify the notation, we eliminate t as argument of the state and control variables.

where $h, j = 1, 2, h \neq j$ and $\nabla V^h(x) = (\partial V^h(x)/\partial x_1, \dots, \partial V^h(x)/\partial x_n)$.

The maximization on the right-hand side of the above equation yields

$$\frac{\partial v^h(x, u^1, u^2)}{\partial u^{hk}} + \sum_{i=1}^{n} \frac{\partial V^h(x)}{\partial x_i} \frac{\partial f_i(x, u^1, u^2)}{\partial u^{hk}} = 0, \tag{5}$$

where $h = 1, 2$ and $k = 1, \dots, m^h$.

This system of $m^1 + m^2$ equations defines implicitly the reaction functions (4) of the two players. Let us assume that functions v^h and f_i have sufficient properties to find a unique solution to the equation system defined by (5); then, we can write the stationary feedback strategies as

$$u^{1^*} = \phi^1(x; \nabla V^1(x), \nabla V^2(x)), \quad u^{2^*} = \phi^2(x; \nabla V^1(x), \nabla V^2(x)),$$

or finally as functions whose arguments are the state variables,[6]

$$u^{1^*} = \chi^1(x) \quad \text{and} \quad u^{2^*} = \chi^2(x).$$

Substituting them into (3), we obtain

$$rV^h(x) = v^h(x, \chi^1(x), \chi^2(x)) + \sum_{i=1}^{n} \frac{\partial V^h(x)}{\partial x_i} f_i(x, \chi^1(x), \chi^2(x)), \quad h = 1, 2. \tag{6}$$

The expression (6) defines a system of two partial differential equations. By solving this system and finding the value functions $(V^1(x), V^2(x))$, we can find also the equilibrium strategies.[7] Summarizing, we obtain the following proposition.

[6] We ignore at this stage the constraint $u^h \in U^h$. A priori, it is difficult to establish the sufficient conditions that guarantee the existence and uniqueness of the maximization of the right-hand side of the HJB equation. This difficulty is explained by the presence in (5) of the partial derivatives $\partial V^h/\partial x_i$, whose sign is unknown. Nevertheless, the strict concavity of the player h utility function with respect to the vector u^h and the linearity of the transition equations seems to be sufficient conditions for getting the uniqueness of the solution.

[7] Notice that, as long as the value function of an infinite-horizon autonomous problem does not depend on the time, the system of equations (5), which defines implicitly the reaction functions, does not depend on the time either; consequently, the feedback strategies given by the solution to the partial differential equations (6) are going to be stationary, i.e., autonomous with respect to the time. Obviously, this in not necessarily true for nonautonomous problems.

Proposition 2.1. A stationary feedback Nash equilibrium (SFNE) is given by a solution of the first-order partial differential equation system (6) provided that $\chi^h(x) \in U^h$ for all $x \in L$ and $h = 1, 2$.

If $\chi^h(x) \notin U^h$, for all $x \in L$, the constraints are binding and the maximization problem presents corner solutions. In that case, we have to operate as follows. First, we have to divide the set L in different subsets, some of them closed and others open, using the conditions $\chi^h(x) \in \bar{U}^h, h = 1, 2$, which represent the binding constraints on the control variables. Second, for each of these subsets, a pair of partial differential equations must be built taking into account the binding constraints. Then, a SFNE is given by a solution to the set of pairs of partial differential equations originating by the partition of the set L, which probably would include the pair of partial differential equations (6).

2.2. Stationary Feedback Stackelberg Equilibrium. Let player 1 be the leader. Then, the definition of a stationary feedback Stackelberg equilibrium is the following.[8]

Definition 2.3. A stationary feedback Stackelberg equilibrium (SFSE), with player 1 as the leader, is a pair of stationary feedback strategies $(u^{1^*}, u^{2^*}) \in S^1 \times S^2$ such that, for every possible initial condition (t_0, x_0),

$$u^{1^*} = \arg \max_{u^1 \in U^1} J^1(u^1, T^2(x; u^1, \nabla V^2(x))),$$
$$u^{2^*} = T^2(x; u^{1^*}, \nabla V^2(x)).$$

Again, this equilibrium is subgame perfect and also dynamically consistent. Then, the equilibrium strategy of the leader must satisfy the following HJB equation:

$$rV^1(x) = \max_{u^1 \in U^1} \left\{ v^1(x, u^1, T^2(x; u^1, \nabla V^2(x))) + \sum_{i=1}^{n} \frac{\partial V^1(x)}{\partial x_i} f_i(x, u^1, T^2(x; u^1, \nabla V^2(x))) \right\}. \tag{7}$$

Thus, the instantaneous Stackelberg solution for the leader is

[8] We base our definition on the feedback Stackelberg solution proposed by Başar and Olsder; see Ref. 6, pp. 413–415. We discuss later the global Stackelberg solution mentioned also in Ref. 6, pp. 412–413, and more extensively in Ref. 1, pp.134–141.

$$u^{1^*} = \arg\max_{u^1 \in U^1} \left\{ v^1(x, u^1, T^2(x; u^1, \nabla V^2(x))) + \sum_{i=1}^{n} \frac{\partial V^1(x)}{\partial x_i} f_i(x, u^1, T^2(x; u^1, \nabla V^2(x))) \right\}. \tag{8}$$

The maximization of the right-hand side of the above equation yields

$$\frac{\partial v^1}{\partial u^{1k}} + \sum_{l=1}^{m^2} \frac{\partial v^1}{\partial u^{2l}} \frac{\partial T^{2l}}{\partial u^{1k}} + \sum_{i=1}^{n} \frac{\partial V^1}{\partial x_i} \times \left(\frac{\partial f_i}{\partial u^{1k}} + \sum_{l=1}^{m^2} \frac{\partial f_i}{\partial u^{2l}} \frac{\partial T^{2l}}{\partial u^{1k}} \right) = 0, \quad k = 1, \ldots, m^1. \tag{9}$$

This equation defines implicitly the strategy of the leader. If we assume that this system of equations has a unique solution, the strategy of the leader can be written as a function of the state variables, $u^{1^*} = \psi(x)$. Using this strategy and the follower reaction functions, the strategy of the follower can be written as

$$u^{2^*} = T^2(x; u^{1^*}, \nabla V^2(x)) = T^2(x; \psi(x), \nabla V^2(x)) = \omega(x).$$

Substituting then into the HJB equations, we get

$$rV^h(x) = v^h(x, \psi(x), \omega(x)) + \sum_{i=1}^{n} \frac{\partial V^h(x)}{\partial x_i} f_i(x, \psi(x), \omega(x)), \quad h = 1, 2. \tag{10}$$

The expression (10) defines a system of two partial differential equations different from the one defined by (6). By solving it, the value functions $(V^1(x), V^2(x))$ are found, which allow us to calculate the equilibrium strategies.[9] Summarizing, we present the following result.

Proposition 2.2. A stationary feedback Stackelberg strategy (SFSE) is given by a solution of the first-order partial differential equation system (10) provided that $\psi(x) \in U^1$ and $\omega(x) \in U^2$ for all $x \in L$.

[9] As happens for the SFNE, the reaction functions of the follower do not depend explicitly on time; then, as equation (9), which defines implicitly the strategy of the leader u^1, does not depend on the time, the feedback strategies given by the solution to the partial differential equations (10) are going to be stationary as occurs for the SFNE.

At this point, we deal with corner solutions as we did for the SFNE.

Next, we investigate under what conditions the two equilibria coincide. In order to progress in this investigation, we give the following definition.

Definition 2.4. A differential game with orthogonal (instantaneous) reaction functions is such that the first derivatives $\partial v^h/\partial u^{hk}$ and $\partial f_i/\partial u^{hk}$ are independent of u^j for $h, j = 1, 2, h \neq j, k = 1, \dots, m^h$, and $i = 1, \dots, n$.[10]

Then, it is straightforward to show that the following proposition holds.

Proposition 2.3. For the class of differential games with orthogonal reaction functions, the leader first movement advantage disappears. Then, if the SFNE exists, there is no difference between that equilibrium and the SFSE independently of which player acts as the leader.

Proof. As we have seen, the stationary feedback Nash strategies must satisfy the conditions (5) which we can be rewritten as

$$\frac{\partial v^1(x, u^1, u^2)}{\partial u^{1k}} + \sum_{i=1}^{n} \frac{\partial V^1(x)}{\partial x_i} \frac{\partial f_i(x, u^1, u^2)}{\partial u^{1k}} = 0, \tag{11}$$

$$\frac{\partial v^2(x, u^1, u^2)}{\partial u^{2l}} + \sum_{i=1}^{n} \frac{\partial V^2(x)}{\partial x_i} \frac{\partial f_i(x, u^1, u^2)}{\partial u^{2l}} = 0, \tag{12}$$

where $k = 1, \dots, m^1$ and $l = 1, \dots, m^2$. But, if the conditions of Definition 2.4 are fulfilled, the subsystem (11) is independent of the vector u^2 of the control variables of player 2 and the subsystem (12) is independent of the vector u^1 of the control variables of player 1. This means that the reaction functions defined by each subsystem of equations depends only on the state variables; in other words, the reaction functions coincide with the stationary feedback Nash strategies. Thus, from (11), we obtain

$$u^{1^*} = T^1(x; \nabla V^1(x)) = \phi^1(x; \nabla V^1(x)) = \chi^1(x),$$

and from (12),

[10] The term orthogonal is used in this definition in the sense that the intersection point of the reaction functions in a graph when $U^h = R_+$ defines a right angle as if two orthogonal vectors were being represented.

$$u^{2^*} = T^2(x; \nabla V^2(x)) = \phi^2(x; \nabla V^2(x)) = \chi^2(x).$$

Now, let player 1 be the leader again. Using $u^{2^*} = T^2(x; \nabla V^2(x))$, we obtain that the $m^1 \times m^2$ matrix $D_{u^1} T^2(x; \nabla V^2(x))$ is the zero matrix; i.e., all the terms $\partial T^{2l}/\partial u^{1k}$ in the system of equations (9) are zero. Then, (9) is identical to (11), which implies that $\psi(x) = \chi^1(x)$; as the reaction functions of the follower do not depend on the control variables of the leader, we establish that $\omega(x) = \chi^2(x)$ from (12). The result is that there is no difference between the systems of partial differential equations (6) and (10); consequently, the SFSE coincides with the SFNE.

The same kind of argument applies when player 2 is the leader. Using now $u^{1^*} = T^1(x; \nabla V^1(x))$, we obtain that the $m^2 \times m^1$ matrix $D_{u^2} T^1(x; \nabla V^1(x))$ is the zero matrix. The same argument applies also for nonautonomous problems. The only difference is that, in that case, the feedback strategies are nonstationary. Finally, notice that this argument works also for corner solutions as long as the pair of partial differential equations (6) is identical to the pair of partial differential equations (10). We recall that, for dealing with corner solutions for the SFNE, we use the conditions $\chi^h(x) \in \bar{U}^h$, $h = 1, 2$. Then, if $\psi(x)$ is equal to $\chi^1(x)$ and if $\omega(x)$ is equal to $\chi^2(x)$, for an interior solution, the set of pairs of partial differential equations originating by the partition of the set L defined by the binding constraints is going to be the same for both equilibria. □

Notice that, if the partial derivatives $\partial f_i/\partial u^{hk}$ are independent of u^j and there does not exist any strategic interdependence through the payoff functions ($\partial v^h/\partial u^{hk}$ is independent of u^j), then as the variation of the state variable is instantaneous, the reaction functions are orthogonal with respect to the control variables; the first movement advantage disappears, so that the two equilibria coincide. In Definition 2.4, we have established the conditions which yield this coincidence over the primitive functions of the game so that they can be checked directly before solving the system of partial differential equations (6). This means that the issue of the existence of the equilibria can be separated from the issue of their coincidence and they can be studied independently. Obviously, for the class of differential games with orthogonal reaction functions, if the SFNE does not exist, neither does the SFSE; then, the issue of the coincidence becomes irrelevant. Moreover, they are necessary and sufficient conditions.[11]

[11] Notice that the additive separability between u^1 and u^2 in v^h and f_i is not sufficient to yield orthogonal reaction functions. If v^h and f^i are additive separable, then we can write them as $v^h = F[v^{h1}(u^1; x) + v^{h2}(u^2; x)]$ and $f_i = G[f_{i1}(u^1; x) + f_{i2}(u^2; x)]$ so that only when F' and G' are constant will the conditions in Definition 2.4 be satisfied.

A straightforward consequence of the previous result is the following corollary.

Corollary 2.1. Let us assume that $\partial v^h/\partial u^{hk}$ and $\partial f_i/\partial u^{hk}$ are independent of u^j, but that $\partial v^j/\partial u^{jl}$ and $\partial f_i/\partial u^{jl}$ depend on u^h for $h, j = 1, 2$, $h \neq j$, $k = 1, \dots, m^h, l = 1, \dots, m^j$, and $i = 1, \dots, n$. Then, if the SFSE where player h is the leader and player j is the follower and the SFNE exist, those two equilibria are different. However, when the leader is player j, the SFSE coincides with the SFNE.

According to our results, the set of differential games can be divided in three classes: the class of differential games with orthogonal reaction functions; the class of differential games for which there is only one SFSE different from the SFNE; and the class of differential games that have two different SFSEs that are also different from the SFNE.[12]

Example 2.1. Next, we present an example with orthogonal reaction functions, Example 5.7 in Ref. 1. This example describes the joint exploitation of a pesticide. A pesticide possesses the property that its effectiveness declines with the accumulated number of doses. Let $u^h(t) \geq 0$ denote the firm h rate of application of the pesticide, $h = 1, 2$. Let $x(t) \geq 0$ stand for the effectiveness of the pesticide. Assume that the doses can be produced costlessly and that the profit rate of the firm h is

$$v^h(x(t), u^h(t)) = [u^h(t)x(t)]^\alpha,$$

where $0 < \alpha < 1/2$. We assume also that the decline in effectiveness is equal to the total application by both firms. Therefore, we have

$$\dot{x}(t) = f(x(t), u^1(t); u^2(t)) = \begin{cases} -u^1(t) - u^2(t), & \text{if } x(t) > 0, \\ 0, & \text{if } x(t) = 0. \end{cases} \tag{13}$$

Each firm seeks to maximize the discounted profit integral,

$$\max_{u^h \geq 0} \int_0^\infty e^{-rt}[u^h(t)x(t)]^\alpha dt,$$

[12] Some economic applications of differential games for which there exists only one feedback Stackelberg equilibrium different from the feedback Nash equilibrium can be found in the following references: Ref. 2 (capitalistic accumulation), Ref. 4 (optimal dynamic profit taxation) and Refs. 5, 9 and 10, (strategic Pigovian taxation with stock externalities).

subject to (13) and an initial condition $x(0)=x_0>0$, where r is the firm rate of discount and $h=1,2$.

It is straightforward that this differential game satisfies the conditions of Definition 2.4. Notice that neither $\partial v^h/\partial u^h$ nor $\partial f/\partial u^h$ depend on the control variable of the other firm. For the SFNE, the equilibrium strategies must satisfy the following HJB equations:

$$rV^h(x)=\max_{u^h\geq 0}\{(u^h x)^\alpha - V_x^h(x)(u^1+u^2)\}, \quad h=1,2. \tag{14}$$

The maximization on the right-hand side ignores the constraint on the control variable and yields directly the following stationary feedback strategies:

$$u^{h^*}=[\alpha x^\alpha / V_x^h(x)]^{1/(1-\alpha)}, \quad h=1,2. \tag{15}$$

Then, by substitution into the HJB equations, the following differential equation is obtained under the symmetry assumption and after some manipulations:

$$rV^h(x)[V_x^h(x)]^{\alpha/(1-\alpha)}=(1-2\alpha)[\alpha x]^{\alpha/(1-\alpha)}.$$

Guessing a value function $V^h(x)=ax^{2\alpha}$, the parameter a can be calculated yielding the following stationary feedback strategies:

$$u^{h^*}=rx/(2-4\alpha), \quad h=1,2. \tag{16}$$

Finally, it is easy to check that

$$u^{h^*}\in U^h, \quad \text{for all } x\in L.$$

Notice that, in this case, $L=U^h=R_+$ so that, according to (16),

$$u^{h^*}\geq 0, \quad \text{for all } x\geq 0;$$

we can conclude that the stationary feedback strategies (16) are the equilibrium strategies.

In order to compute the SFSE, we assume that the firm 1 acts as the leader. Then, taking into account the firm 2 reaction function (15), the HJB equation for the leader optimization problem is

$$rV^1(x)=\max_{u^1\geq 0}\left\{(u^1 x)^\alpha - V_x^1(x)\left(u^1+\left[\frac{\alpha x^\alpha}{V^2(x)}\right]^{1/(1-\alpha)}\right)\right\}.$$

The maximization on the right-hand side of this equation yields the same strategy as the maximization on the right-hand side of equation (14).

This coincidence occurs because the reaction functions (15) are independent of the control variable of the other player. In fact, in this class of differential games, the maximization on the right-hand side of the HJB equation yields directly the optimal policy of the player. Obviously, this result does not depend on the symmetry assumption.

2.3. Stationary Global Stackelberg Equilibrium. Other Stackelberg equilibrium concepts have been proposed in the literature of differential games: the open-loop Stackelberg equilibrium and the global Stackelberg equilibrium. In general, the first is not time consistent. Thus, it is not a plausible equilibrium concept in situations where economic agents cannot credibly commit to an entire control path.[13] On the other hand, the global Stackelberg equilibrium has been proposed to avoid this drawback. Let player 1 be the leader again. It announces to the follower, the strategy that it will use throughout the game. Let this strategy be denoted by $u^1 = \phi^1(x)$. The follower, taking this strategy as given, seeks to maximize his payoff. The strategy chosen by the follower must satisfy the HJB equation

$$rV^2(x) = \max_{u^2 \in U^2} \left\{ v^2(x, \phi^1(x), u^2) + \sum_{i=1}^{n} \frac{\partial V^2(x)}{\partial x_i} f_i(x, \phi^1(x), u^2) \right\}.$$

The maximization on the right-hand side yields

$$\frac{\partial v^2(x, \phi^1(x), u^2)}{\partial u^{2k}} + \sum_{i=1}^{n} \frac{\partial V^2(x)}{\partial x_i} \frac{\partial f_i(x, \phi^1(x), u^2)}{\partial u^{2k}} = 0,$$

where $k = 1, \ldots, m^2$. From this system, the stationary feedback strategies can be obtained, $u^{2^*} = \phi^2(x; \phi^1(x), \nabla V^2(x))$. Then, by substitution into the HJB equation, we get

$$\begin{aligned} rV^2(x) = {} & v^2(x, \phi^1(x), \phi^2(x; \phi^1(x), \nabla V^2(x))) \\ & + \sum_{i=1}^{n} \frac{\partial V^2(x)}{\partial x_i} f_i(x, \phi^1(x), \phi^2(x; \phi^1(x), \nabla V^2(x))). \end{aligned} \tag{17}$$

These strategies reduce to $u^2 = \phi^2(x; \phi^1(x))$ if a solution for this HJB equation can be found. Then, the leader, using this strategy, chooses among all possible strategies $\phi^1(x)$ the one that maximizes its objective

[13] We do not discuss this equilibrium concept in this paper; for a thorough discussion, see Section 5.2 in Ref. 1. We use also this reference for the presentation of the global Stackelberg equilibrium developed in this subsection.

function. Usually, the space of functions from which the leader can choose the strategy ϕ^1 is restricted in order to make this equilibrium concept operative.

It is easy to conclude that, for this kind of equilibrium, Proposition 2.3 no longer applies. If the conditions in Definition 2.4 hold, we get

$$u^{2^*} = \phi^2(x; \nabla V^2(x)),$$

so that Eq. (17) is written as

$$rV^2(x) = v^2(x, \phi^1(x), \phi^2(x; \nabla V^2(x))) + \sum_{i=1}^{n} \frac{\partial V^2(x)}{\partial x_i} f_i(x, \phi^1(x), \phi^2(x; \nabla V^2(x))).$$

Then, if a value function $V^2(x)$ that satisfies the HJB equation can be found, we would finally obtain that

$$u^{2^*} = \phi^2(x; \phi^1(x))$$

as long as the solution for the value function depends on $\phi^1(x)$. Thus, the follower equilibrium strategy would be seen to be affected by the decisions taken by the leader and the stationary global Stackelberg equilibrium would be different from the SFNE.

Example 2.2. To illustrate this conclusion, we go back to Example 2.1. Restricting the space of strategies to the linear ones, $u^1 = bx$ with $b > 0$, the follower seeks a solution to the HJB equation

$$rV^2(x) = \max_{u^2 \geq 0} \{(u^2 x)^\alpha - V_x^2(x)(bx + u^2)\}.$$

The maximization on the right-hand side yields the same stationary feedback strategy as the one obtained for the SFNE [see (15)]; but now, after substituting it into the HJB equation we find a different expression which depends on b,

$$rV^2(x)[V_x^2(x)]^{\alpha/(1-\alpha)} + b[V_x^2(x)]^{1/(1-\alpha)} x = (1-\alpha)[\alpha x]^{\alpha/(1-\alpha)}.$$

Applying this differential equation to the value function

$$V^2(x) = ax^{2\alpha},$$

the equilibrium stationary feedback strategies for the follower can be obtained,

$$u^{2^*} = (r + 2b\alpha)x/2(1-\alpha).$$

Notice that this strategy is different from the one obtained for the SFNE [see (16)].

In order to calculate b, the following problem must be solved

$$\max_{b\in R^{++}} \int_0^\infty e^{-rt}(bx^2)^\alpha dt,$$
$$\text{s.t.} \quad \dot{x}=-[b+(r+2b\alpha)/2(1-\alpha)]x, \quad x(0)=x_0>0.$$

It is easy to check (see Ref. 1, p. 173) that the SFSE for this game displays the property of time consistency. This is a consequence of the fact that the leader is restricted to play stationary strategies as long as it has to choose a constant control variable b. The main objection to this approach is well known. According to this approach, a game with feedback strategies for the follower and open-loop strategies for the leader is solved, but not a game in which both players have a feedback or closed-loop information structure.

3. Leadership in Economic Applications of Differential Games

In this section, we evaluate the scope of the applicability of the SFSE in differential games in order to have an idea of the scope of Proposition 2.3. In this paper, we focus on only economic applications of differential games. In order to address this issue, we are going to define a class of differential games representing the underlying structure of a good number of economic applications.

This class of differential games is characterized by state equations which depend linearly on the control variables,

$$f_i(x,u^1,u^2)=f_i(x)+\alpha^{i1}u^1+\alpha^{i2}u^2, \quad i=1,\dots,n, \tag{18}$$

where $\alpha^{i1}\in R^{m^1}$ and $\alpha^{i2}\in R^{m^2}$ are two vectors of constants. Moreover, the payoff functions of the players do not depend on the other player control variables,

$$v^h(x,u^1,u^2)=v^h(x,u^h), \quad h=1,2. \tag{19}$$

For this class of differential games, the strategic interdependence between the players appears only through the state variable, so that the payoffs of a player do not depend directly on the control variable of the other player.

Clearly, the class of differential games defined by (18) and (19) is a subset of the class of differential games with orthogonal reaction functions; for this reason, the conditions in Definition 2.4 are going to be satisfied. Then, from Proposition 2.3, we obtain the following corollary.

Corollary 3.1. For the class of differential games defined by (18) and (19), the leader first movement advantage disappears. Then, if the SFNE exists, there is no difference between that equilibrium and the SFSE independently of which player acts as the leader.[14]

Finally, we are going to present an economic applications of differential games to show how common the underlying structure presented in this section is in economics and also to show that it makes sense to model an economic problem as a differential game with orthogonal reaction functions.

Fershtman and Kamien (Ref. 7) extend the analysis of duopolistic competition to a dynamic setting in a natural way assuming that the price does not adjust instantaneously to the level given by the demand function for a given level of output. This adjustment takes time and the rate of change of price is a function of the gap between the current market price and the price indicated by the demand function for the currently produced quantities. Accordingly, the price is governed by the differential equation

$$\dot{p} = s(a - u^1 - u^2 - p), \tag{20}$$

where $a-(u^1+u^2)$ is the price on the demand function for the given level of output u^1+u^2 and s denotes the speed at which the price converges to its level on the demand function.

The authors assume a quadratic cost function,

$$C(u^h) = cu^h + (1/2)(u^h)^2,$$

where $u^h \geq 0$ is the firm h output rate and $c<a$, so that the firm h profit function is given by

$$\pi^h(p, u^h) = pu^h - cu^h - (1/2)(u^h)^2,$$

[14]A list of economic applications that belong to this class of differential games (25 references) is available from the author upon request. Many examples in this list are linear-quadratic differential games, although each of them has a different structure. However, our conclusion about the equivalence of both equilibria is more general and applies to any differential game that satisfies conditions in Definition 2.4.

where $p \geq 0$. Then, the objective function of each firm is

$$\max_{u^h \geq 0} \int_0^\infty e^{-rt}[pu^h - cu^h - (1/2)(u^h)^2]dt,$$

subject to (20) and a given initial price. In this way, the duopolistic competition with sticky prices can be analyzed resorting to a differential game with orthogonal reaction functions.[15] For this problem, the associated HJB equations are given by

$$rV^h(p) = \max_{u^h \geq 0}\{pu^h - cu^h - (1/2)(u^h)^2 + sV_p^h(p)(a - u^h - u^j - p\}, \tag{21}$$

with $h, j = 1, 2, h \neq j$, and the f.o.c. for an interior solution yield

$$MC = c + u^{h^*} = p - sV_p^h(p) = MR, \quad h = 1, 2.$$

This condition is the well-known "marginal cost equals marginal revenue," but in this case the marginal revenue has two elements: the price that is the instantaneous marginal revenue and $-sV_p^h(p)$, the long run effect of an incremental change in the output rate. This condition can also be written as

$$u^{h^*} = p - c - sV_p^h(p). \tag{22}$$

Notice that the reaction function of the firms does not depend on the output rate of the other firm, so that (22) defines directly the optimal strategies of the firms.

By substitution (under the symmetry assumption) into the HJB equation, the following differential equation is obtained:[16]

$$(3/2)s^2V_p^h(p)^2 + s(a + 2c - 3p)V_p^h(p) - rV^h(p) = -(1/2)(p - c)^2. \tag{23}$$

This equation is satisfied by a quadratic value function,

$$V^h(p) = (1/2)Kp^2 - Ep + g,$$

which yields the following strategies from (22):

$$u^{h^*} = (1 - sK)p + (sE - c),$$

[15] It is easy to check that this game satisfies the conditions of Definition 2.4.

[16] In fact, the authors show that only a symmetric SFNE can be asymptotically stable. See Appendix 3 in Ref. 7.

where

$$K = \left[r + 6s - \sqrt{(r+6s)^2 - 12s^2} \right] / 6s^2,$$

$$E = (-asK + c - 2sKc)/(r - 3s^2K + 3s),$$

and

$$1 - sK > 0, \quad sE - c < 0.$$

In this case, it is clear that there exists a critical price

$$\hat{p} = (c - sE)/(1 - sK) > 0$$

such that, if $p < \hat{p}$, the nonnegativity constraint on the output rate is not satisfied. This leads to a partition of the set $L = R_+$ in an open subset defined by $p > \hat{p}$ and a closed subset defined by $0 \leq p \leq \hat{p}$, so that the SFNE must satisfy (23) for $p > \hat{p}$ and the HJB equation

$$rV^h(p) = sV_p^h(p)(a - p), \tag{24}$$

for $0 \leq p \leq \hat{p}$. Fershtman and Kamien propose a solution also for this equation, so that finally the equilibrium strategies of the SFNE are given by[17]

$$u^{h*} = \begin{cases} 0, & 0 \leq p \leq \hat{p}, \\ (1 - sK)p + (sE - c), & p > \hat{p}. \end{cases}$$

Notice that this strategy (the higher the price, the higher the output rate) makes sense from an economic perspective, since with sticky prices high production will pull down the market price slowly so that the firm can take advantage of the high prices that are going to prevail in the market for a while.

Now, we show that the SFSE coincides with the SFNE. In order to calculate the SFSE of the game, we assume that firm 1 acts as the leader, the HJB equation for the leader being

$$rV^1(p) = \max_{u^1 \geq 0} \{ pu^1 - cu^1 - (1/2)(u^1)^2 + sV_p^1(p) \\ \times (a - u^1 - (p - c - sV_p^2(p)) - p) \}.$$

[17] See details in Ref. 7.

The maximization of the right-hand side of the above equation yields the same optimal policy as the maximization of the right-hand side of equation (21), so that we get the same differential equation as the one obtained for the SFNE and, consequently, the same partition of the set L for the state variable.

4. Conclusions

In this paper, we have investigated the scope of the applicability of the feedback Stackelberg equilibrium concept in differential games. As a result of our investigation, we have found a set of necessary and sufficient conditions that yields the coincidence between the stationary feedback Stackelberg equilibrium (SFSE) and the stationary feedback Nash equilibrium (SFNE). When these conditions are present in a differential game, the instantaneous reaction functions are orthogonal and the first movement advantage disappears.

In order to avoid this limitation, the global Stackelberg equilibrium has been proposed. However, this procedure presents also some problems. Another alternative is to change the framework and use a discrete-time approach for the analysis of leadership in a dynamic setting. It is easy to check that the limitation of differential games to capture through the f.o.c. the interdependence among the players disappears in discrete-time difference games. For difference games, the optimal strategy of a player at time t depends on the state variable value at time $t+1$; then, as the state variable values at time $t+1$ depend on the control variable values of the other player at time t, we finally find that the optimal policy of a player at time t depends on the control variable values of the other player also at time t, even if the conditions in Definition 2.4 are satisfied by the utility functions and state equations. In other words, the set of conditions on the primitive functions of the game that yield orthogonal reaction functions in a differential game has not the same effect in a difference game.

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