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ON COMBINED FLEXURE AND TORSION, AND THE FLEXURAL BUCKLING OF A TWISTED BAR*

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1. Introduction. When a straight uniform slender bar is twisted, the straight form becomes unstable at a certain value of the twisting couple, and the center line of the bar becomes a space curve. Elements of the bar are bent about both principal axes of section, and the buckled form thus possesses strain energy of flexure as well as of torsion. If the bar is twisted to the critical configuration, and its end sections then held against further rotation, the jump to the buckled form means the appearance of flexural energy at the expense of the torsional energy. The occurrence of the *flexure* must therefore produce some relief of the *torsion*, that is, it must modify the amount of twist.

It proves to be impossible to account for the transference of strain energy from that of torsion to that of flexure if the strain energy is represented in the accepted form of the theory of small bending and torsion of thin bars—

$$\frac{1}{2}\int_0^l (EI_1u''^2 + EI_2v''^2 + GC\beta'^2)dz,$$

where EI_1, EI_2, GC are the flexural and torsional rigidities, u, v the components of deflection parallel to the principal axes of the section, and β the torsional rotation, as functions of the axial co-ordinate z. Coincidence of shear center and centroid is assumed, and secondary effects of non-uniform torsion[1] are disregarded, for simplicity. If for instance this form is used in the potential energy, and the differential equations of the bar buckled from a state of simple torsion by couples M_3 are found by means of the theorem of stationary potential energy, the correct equations[2]

$$EI_1u'' + M_3v' = 0, \qquad EI_2v'' - M_3u' = 0, \qquad M_3 = GC\beta'$$

are not obtained. The terms M_3v', M_3u' in the first two fail to appear. These equations are nevertheless easily derived directly as conditions of equilibrium.

The comparison with the corresponding problem of the bar under thrust is useful. The bar is compressed to the critical state, and the ends held against further approach. The bar jumps over to the bent form, and energy of bending appears. But

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[1] J. N. Goodier, (i) *The buckling of compressed bars by torsion and flexure*, Cornell University Engineering Experiment Station, Bulletin 27, 1942; (ii) *Flexural torsional buckling of bars of open section*, Bulletin 28, 1942.

[2] S. Timoskenko, *Theory of elastic stability*, McGraw-Hill, 1936, p. 168, or (1) (ii) equations 2, 3, 7.

the transition to the bent form involves a lengthening of the bar, and some of the compressional strain energy is thus released to supply the energy of flexure. The Euler problem has been analysed from this point of view by R. V. Southwell.[3]

This lengthening of the bar is of the second order in the derivative of the bending displacement with respect to the axial coordinate. It can be disregarded in writing down the differential equation of equilibrium, but not in energy methods. It is natural to look for something analogous in the torsional problem by investigating the nature of combined torsion and flexure to a higher order of small quantities than formerly. This is done in what follows and the required new terms in the strain energy are found. At the same time the nature of combined torsion and flexure is clarified, and the energy method is made available for more difficult problems of buckling from a twisted state such as those of non-uniform bars.

2. Finite bending and torsion of a thin bar. Let the axis (of centroids) of the undeformed straight bar lie along the z-axis of fixed cartesian axes u, v, z. The bar is now subjected to small bending and twisting. Its axis becomes a space curve, consisting of points of co-ordinates u, v, z. Even if the deflection (u, v) is small, the geometrical torsion of this curve is not small. The bending may be in one plane (the osculating plane) at one point, and in a perpendicular plane at another.

The geometrical torsion τ_c of the curve is distinct from the torsion τ of the bar. When the deflection (u, v) is prescribed the space curve of centroids is definite, with definite curvature and torsion. The cross sections of the bar must be in the normal planes of this curve, but the torsion of the bar remains indefinite until the orientations of the principal axes in these planes are specified.

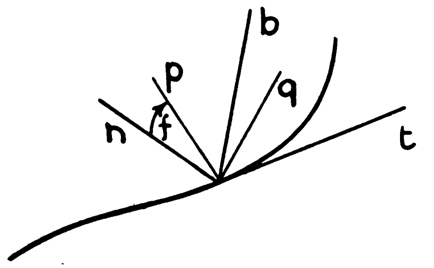

Fig. 1.

In Fig. 1 the tangent, normal and binormal at P are indicated by t, n, b.[4] As the origin of the triad moves along the curve with unit speed, it has a component τ_c of angular velocity about t, and a component κ (the curvature) about b, right handed rotations looking along the positive axes being reckoned positive. Define an angle γ such that $\tau_c = d\gamma/ds$ (s being arc length increasing in the sense of t) and $\gamma = 0$ at some chosen reference section $s = s_0$, as for instance one end of the bar.

Let f be the angle which one principal axis p (Fig. 1) of the cross section at P makes with the principal normal n, positive when this axis is obtained from n by positive rotation about t. Let f_0 be its value at $s = s_0$. Then the rate of rotation of the tpq-triad about t is given by $\tau_c + df/ds$ or $(d\gamma/ds) + (df/ds)$ and this is by definition the torsion of the bar.[5]

Accordingly if the bar is bent but not twisted, $\gamma + f$ is a constant along the bar and in fact $\gamma + f = f_0$, or $f = f_0 - \gamma$. From this state we may derive a twisted form of the

[3] *Introduction to the theory of elasticity*, 2nd ed., Oxford University Press, 1941, p. 443.

[4] The notation and conventions are those of C. E. Weatherburn, *Differential geometry*, vol. 1, Cambridge University Press, Cambridge 1939, p. 15.

[5] The discussion thus far, except for the introduction of the angle γ, corresponds with that of A. E. H. Love, *Mathematical theory of elasticity*, 4th ed., Cambridge University Press, 1934, Ch. XVIII. The further development is different.

bent bar by introducing an angle of twist ϕ with $\phi=0$ at $s=s_0$, so that $f=f_0-\gamma+\phi$. The torsion of the bar is now $d\phi/ds$. The bent and twisted form of the bar is completely specified by the curve of centroids, which defines γ, and the angle $f_0+\phi$ which can be assigned independently.

In the elementary theory of bending, the curvature is related to the bending moments by means of components along the principal axes of cross sections. If κ_1, κ_2 denote these components along p and q (Fig. 1), we have (κ can be regarded as an angular velocity about b)

$$\kappa_1 = \kappa \sin f, \qquad \kappa_2 = \kappa \cos f \tag{1}$$

or

$$\kappa_1 = \kappa \sin (f_0 - \gamma + \phi), \qquad \kappa_2 = \kappa \cos (f_0 - \gamma + \phi). \tag{2}$$

We have also

$$\tau = d\phi/ds. \tag{3}$$

But

$$\kappa = (u''^2 + v''^2 + z''^2)^{1/2}, \tag{4}$$

primes denoting differentiation with respect to s. Also γ is defined through $d\gamma/ds=\tau_c$ and we have

$$\tau_c = \kappa^{-2} \begin{vmatrix} u' & v' & z' \\ u'' & v'' & z'' \\ u''' & v''' & z''' \end{vmatrix}. \tag{5}$$

With these formulas the bending and torsion of the bar are completely specified by the deflection (u, v as given functions of z) and the angles f_0 and ϕ. The orientations of the principal normal and binormal are defined by the deflection curve, and the orientations of the principal axes relative to these are defined by f_0 and ϕ. The formulas (2) and (3) may be used to specify not only the deformed state of the bar, but also an initial "bent and twisted" but unstressed state. The differences between the values of κ_1, κ_2 and τ then represent the changes of curvature and torsion to which the components of bending moment, and the twisting moment, will be respectively proportional.

To illustrate this, and also the significance of f_0, let the bar be circular and in a horizontal plane, with the principal axis p of all cross sections also in the horizontal plane. Then we may take for the initial state $\gamma=f_0=\phi=\tau=\kappa_1=0$, $\kappa_2=\kappa=1/r$ where r is the radius of the circle. Let each cross section now be rotated by the same angle α about t. For the deformed state $f_0=\alpha$ and

$$\kappa_1 = r^{-1} \sin \alpha, \qquad \kappa_2 = r^{-1} \cos \alpha, \qquad \tau = 0.$$

The changes of the components of curvature are

$$r^{-1} \sin \alpha, \qquad r^{-1}(\cos \alpha - 1).$$

When α is small, the second, the change in κ_2, is negligible. The bending moment induced is proportional to $r^{-1}\alpha$, and corresponds to κ_1, that is, its axis is n, in the plane of the ring.[6]

[6] This problem is analysed from first principles in Timoshenko, *Strength of materials*, Part II, 2nd ed., Van Nostrand, 1941, p. 177.

3. Small bending and torsion of a straight bar. The formulas (2) and (3) must yield such expressions as d^2u/dz^2, d^2v/dz^2, $d\phi/dz$ as their principal parts for small deformation. The object of the present investigation is to obtain terms of higher order as well.

Let u', v', ϕ be small compared with 1, and let l be a suitable length such as the length of the bar, or the wavelength of a periodic deflection. The formulas (4) and (5) involve u'', v'', u''', v'''. If the greatest absolute value of u''' and v''' is denoted by η/l^2, u'' and v'' do not exceed η/l and u', v' do not exceed η, which is small. Let ϵ denote the largest absolute value of η and ϕ. Quantities not exceeding ϵ, ϵ/l, etc., or quantities differing from them only by terms involving higher powers of ϵ, will be denoted by $O(\epsilon)$, $O(\epsilon/l)$, etc.

The relation $u'^2+v'^2+z'^2=1$ yields $z'^2=1-O(\epsilon^2)$ and so $z'=1-O(\epsilon^2)$. It yields also

$$z'' = -(u'u''+v'v'')(1-u'^2-v'^2)^{-1/2} = O(\epsilon^2/l) \tag{6}$$

and

$$z''' = O(\epsilon^2/l^2). \tag{7}$$

Then (4) yields $\kappa=[u''^2+v''^2+O(\epsilon^4/l^2)]^{1/2}$. Since u''^2, v''^2 are $O(\epsilon^2/l^2)$ we have as an approximation

$$\kappa = (u''^2+v''^2)^{1/2} \tag{8}$$

in which the error is of order ϵ^2, relative to the part retained.

The determinant of (5) yields $u''v'''-u'''v''$ with an error of order ϵ^2. Then (5) becomes

$$\tau_c = (u''v'''-u'''v'')(u''^2+v''^2)^{-1} \tag{9}$$

with an error or order ϵ^2.

Now the right of (9) may be identified as $(d/ds)\tan^{-1}(v''/u'')$ and, in view of the equations defining γ ($d\gamma/ds=\tau_c$, $\gamma=0$ when $s=s_0$) we have

$$\gamma = \tan^{-1}\frac{v''}{u''} - \tan^{-1}\frac{v_0''}{u_0''}, \tag{10}$$

where u_0'', v_0'' are the values of u'', v'' at $s=s_0$. The inverse tangents are principal values. The values of $\sin\gamma$ and $\cos\gamma$ are required. From (10)

$$\tan\gamma = (u_0''v''-v_0''u'')(u_0''u''+v_0''v'')^{-1}$$

and therefore

$$\left.\begin{aligned}\sin\gamma &= \left(\frac{v''}{u''}-\frac{v_0''}{u_0''}\right)\\ \cos\gamma &= \left(1+\frac{v_0''v''}{u_0''u''}\right)\end{aligned}\right\}\times\left(1+\frac{v_0''^2}{u_0''^2}\right)^{-1/2}\left(1+\frac{v''^2}{u''^2}\right)^{-1/2} \tag{11}$$

The ambiguity of sign involved in obtaining the sine and cosine from the tangent is disposed of by the consideration that if v''/u'' slightly exceeds v_0''/u_0'', both being positive, γ must be a small positive angle.

4. Expressions for small curvature and torsion. Expanding the first of (2) in the form

$$\kappa_1 = \kappa\left\{(\sin f_0 \cos\gamma - \cos f_0 \sin\gamma)\left(1 - \frac{\phi^2}{2}\cdots\right) + (\cos f_0 \cos\gamma + \sin f_0 \sin\gamma)\left(\phi - \frac{\phi^3}{6}\cdots\right)\right\}$$

and substituting for κ, $\cos\gamma$, $\sin\gamma$ from (8), (11) we find

$$\kappa_1 = u''\sin(f_0+\delta) - v''\cos(f_0+\delta) + \phi[u''\cos(f_0+\delta) + v''\sin(f_0+\delta)] + \cdots \quad (12)$$

and similarly

$$\kappa_2 = u''\cos(f_0+\delta) + v''\sin(f_0+\delta) - \phi[u''\sin(f_0+\delta) - v''\cos(f_0+\delta)] + \cdots \quad (13)$$

where $\cos\delta = u_0''(u_0''^2 + v_0''^2)^{-1/2}$, $\sin\delta = v_0''(u_0''^2 + v_0''^2)^{-1/2}$. In these developments the errors are of order ϵ^2 relative to the leading terms. They are therefore accurate as far as explicitly carried.

Since $dz/ds = 1 - O(\epsilon^2)$, replacement of differentiation with respect to s by differentiation with respect to z, to any order, will involve errors of order ϵ^2. Thus the primes in the terms set out in (12) and (13) may be taken to indicate differentiation with respect to z, and the developments remain correct to this order. In the same way the torsion $d\phi/ds$ may be replaced by $d\phi/dz$ with an error of order ϵ^2.

The angle f_0, while significant of course when the axis of the bar is appreciably deflected, tends to become merely a rigid body rotation when the bar is nearly straight. In order to eliminate such a rigid-body rotation, we observe that there is as yet no connection between the u-axis and the principal axis p. If these axes coincide when the bar is undeformed, small torsion and bending, free of large rigid body rotations, will restrict the angle between them to be of the same order as ϕ. Then the direction cosines of p, relative to the u, v, z axes must be $1-O(\epsilon^2)$, $O(\epsilon)$, $O(\epsilon)$ at most.

The direction cosines of n, the principal normal, are u''/κ, v''/κ, z''/κ so that, if $\boldsymbol{n}$ is the unit vector along n, $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ unit vectors along the axes of u, v and z,

$$\boldsymbol{n} = \kappa^{-1}(u''\boldsymbol{i} + v''\boldsymbol{j} + z''\boldsymbol{k}).$$

The direction cosines of b, the binormal, are used as the coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ in

$$\boldsymbol{b} = \kappa^{-1}[(v'z'' - z'v'')\boldsymbol{i} + (z'u'' - u'z'')\boldsymbol{j} + (u'v'' - v'u'')\boldsymbol{k}]$$

where $\boldsymbol{b}$ is the unit vector along the binormal.

Since the principal axis p (Fig. 1) is in the plane of b and n, and is derived from n by a rotation f towards b, the unit vector along it is given by $\boldsymbol{n}\cos f + \boldsymbol{b}\sin f$ or

$$\begin{aligned}\kappa^{-1}[u''\cos f + (v'z'' - z'v'')\sin f]\boldsymbol{i}& \\ + \kappa^{-1}[v''\cos f + (z'u'' - u'z'')\sin f]\boldsymbol{j}& \\ + \kappa^{-1}[z''\cos f + (u'v'' - v'u'')\sin f]\boldsymbol{k}&\end{aligned} \quad (14)$$

and the coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ give the direction cosines of p.

The first of these is of order 1 without restriction on f. The second may be represented as

$$O(l/\epsilon)\left[O(\epsilon/l)\cos f + O(1)O(\epsilon/l)\sin f - O(\epsilon)O(\epsilon^2/l)\sin f\right],$$

from which it is apparent that these direction cosines will not be small of order ϵ unless

$$\kappa'(v'' \cos f + u'' \sin f)$$

is small of this order. This expression may be developed, by the processes which led to (12) and (13) as

$$\kappa^{-2}v''\{u'' \cos (f_0 + \delta) + v'' \sin (f_0 + \delta) - \phi[u'' \sin (f_0 + \delta) - v'' \cos (f_0 + \delta)] + \cdots\}$$
$$+ \kappa^{-2}u''\{u'' \sin(f_0 + \delta) - v'' \cos (f_0 + \delta) + \phi[u'' \cos (f_0 + \delta) + v'' \sin (f_0 + \delta)] + \cdots\}$$

and will be small of order ϵ only if $f_0 + \delta$ is small of this order.

This result simplifies (12) and (13) to

$$\kappa_1 = - v'' + u''(\phi + f_0 + \delta), \qquad \kappa_2 = u'' + v''(\phi + f_0 + \delta), \tag{15}$$

and with $\tau = \phi'$ these constitute approximations to κ_1, κ_2, and τ with errors of order ϵ^2 relative to the leading terms. It is now implied of course that one principal axis (p) coincides with the u-axis in the undeformed state, and that in the deformation it rotates from it by an angle of the same order as u', v' and ϕ. This is the case if one section of the bar is fixed against rotation, or against rotation of the type ϕ only.

The third direction cosine in (14) is of order ϵ without further conditions.

5. An alternative torsional co-ordinate. The angle ϕ represents a rotation of the cross section about t, from the torsionless configuration associated with the deflection u, v. This torsionless state is far from being geometrically obvious, and the terminal values of ϕ and f corresponding to various types of simple end constraints are not immediately obtainable.

A representation of the torsion and flexure to the second order which does not suffer from these disadvantages is desirable. A straight bar (initially along the z-axis, Fig. 2) may be imagined brought to a bent and twisted state by supposing it cut into thin discs. Let a typical disc be translated without rotation so that its centroid is brought to its final position P on the deflected curve and the principal axes are brought to x_1, y_1 parallel to x, y. It must now be rotated so that the tangent at P to the deflection curve is normal to it, in accordance with the theory of flexure of thin bars. Let this rotation consist of a rotation about y_1 bringing x_1 to x_2 in the normal plane at P, followed by a rotation about x_2 bringing y_1 to y_2 in the normal plane. The configuration so produced is evidently a possible state of bending and torsion. The principal axis x_2 is still parallel to the plane xz. This configuration is to be used as a reference from which to measure the torsional rotation of cross sections. To the first order the torsion is zero, but to the second order it is not.

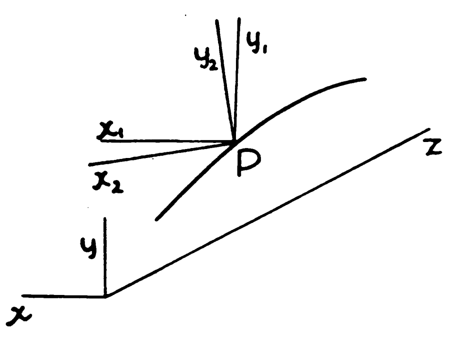

FIG. 2.

To determine its value, let the x, y axes in Fig. 2 correspond with the u, v axes, and let x_2 be the principal axis p. Then, in the proposed configuration, p is everywhere normal to the y, or v, axis. Thus the coefficient of j in (14), which represents the direction cosine of p with the v axis, must vanish, so that the value of f is determined by the equation

$$f_1 = \tan^{-1} \frac{v''}{u'z'' - z'u''} \cdot \tag{16}$$

The torsion of the bar is $\tau_c + df_1/ds$, τ_c being given by (5), and is thus expressed in terms of the derivatives of u and v. When expanded in powers of these derivatives its leading term is $u''v'$. This is an approximation to the torsion with error of order ϵ. Thus if ϕ_1 is the value of ϕ corresponding to this configuration $\phi_1' = u''v'[1+O(\epsilon)]$. Also, f_0 is obtained from (16) by putting u_0, v_0 for u, v and it is easily found that $\tan f_0 = -(v_0''/u_0'') + O(\epsilon^2)$. Since $\tan \delta = v_0''/u_0''$ it follows that $f_0 + \delta = O(\epsilon^2)$. This being so $f_0 + \delta$ in (15) ceases, for this particular configuration, to be significant, since its products with u'', v'' are of the order of the terms neglected.

Now consider an arbitrary state of (small) flexure and torsion specified by u, v, ϕ. It may be derived from the reference state just defined simply by rotating cross-sections about t in order to convert ϕ_1 to ϕ. Let β be the amount of such rotation. Then $\phi - \phi_1 = \beta$, and $\tau = \phi' = \beta' + \phi_1'$, that is

$$\tau = \beta' + u''v' \tag{17}$$

with error of order ϵ^2.

Let s now be measured from one end of the bar so that $s_0 = 0$. Then ϕ_1 like ϕ is zero at $s = 0$ and $\phi_1 = \int_0^s u''v'ds$. Thus $\phi = \beta + \int_0^s u''v'ds$ and the integral is of order ϵ^2. Moreover $f_0 + \delta$ is not altered by the rotation β so that it is still of order ϵ^2. The first of (15) becomes in consequence

$$\kappa_1 = -v'' + u''\left(\beta + \int_0^s u''v'ds\right).$$

The first term is or order ϵ/l, $u''\beta$ is of order ϵ^2/l and $u''\int_0^s u''v'ds$ is of order ϵ^3/l. Therefore, with an error of order ϵ^2 the new formulas for the components of curvature are

$$\kappa_1 = -v'' + \beta u'', \qquad \kappa_2 = u'' + \beta v''. \tag{18}$$

These with (17) give an alternative representation of the torsion and flexure, convenient because f_0 and δ have been eliminated, and β is relatively easily envisaged-being the angle by which the cross section must be rotated, about the deflected, tangent, to bring p from the position parallel to the axial plane in which it originally lies, to its final position. At fixed ends β is clearly zero.

6. Energy considerations. The strain energy is given by

$$\frac{1}{2}\int_0^l (EI_1\kappa_2^2 + EI_2\kappa_1^2 + GC\tau^2)dz. \tag{19}$$

The integration with respect to z rather than s will involve an error of order ϵ^2.

Consider now the problem referred to in the introduction—the straight bar twisted until it buckles. Let the state just prior to buckling be

$$\beta = B, \qquad u = 0, \qquad v = 0,$$

and after buckling

$$\beta = B + \beta_1, \qquad u = u_1, \qquad v = v_1.$$

Then B is small in the sense of ϕ in the preceding analysis. But β_1, u_1, v_1 are to be true infinitesimals, since we seek a buckled form which comes to the straight form as a limit. Thus they are to approach zero after a fixed value has been assigned to B.

The expressions (17) and (18) are now used in (19), and terms to the second order in u_1, v_1, β_1 and their derivatives, without regard to B, are retained. The result is

$$\frac{1}{2}\int_0^l [EI_1(u_1''^2 + 2Bu_1''v_1'') + EI_2(v_1''^2 - 2Bu_1''v_1'') + GC(B'^2 + 2B'\beta_1' + \beta_1'^2 + 2Bu_1''v_1')]dz. \tag{20}$$

Let M_3 be the critical torsional couple GCB'. On buckling, some work is done by this couple, but exactly how much, in terms of β_1, u_1, v_1 depends on the end constraints of the bar.

If the ends are in bearings which constrain the axis of the bar to remain fixed in direction at the ends—i.e., the ends are "built-in" with respect to flexure—the rotation of one end may be set as zero, and that of the other about the axis is then the value of β_1 at that end. The potential energy of M_3 in the buckled form is $-M_3\int_0^l\beta_1' dz$ referred to the twisted but unbuckled form as zero. The total potential energy is thus this term together with (20), omitting $\frac{1}{2}\int_0^l GCB'^2 dz$ which is the energy of the unbuckled twisted form.

If the potential energy is now varied by varying u_1 to $u_1+\epsilon_1\eta_1(z)$ the coefficient of ϵ_1 in the variation of the potential energy is

$$\int_0^l [-EI_2Bv_1'' + EI_1(u_1'' + Bv_1'') + GCB'v_1']\eta_1' dz$$

and this must vanish if the buckled state is a possible state of equilibrium. Since Bv_1'' is small compared with u_1'', on account of the smallness of B, the conclusion is that the equation

$$EI_1u_1'' + M_3v_1' = 0 \tag{21}$$

must be satisfied. Similarly variation of v_1 yields

$$EI_2v_1'' - M_3u_1' = 0. \tag{22}$$

Variation of β_1 yields $GCB'+GC\beta_1' - M_3=0$, that is $\beta_1' = 0$. Equations (21) and (22) are identical with the equations obtainable by direct equilibrium considerations. They are derived in this manner here in order to show that the terms M_3v_1', $-M_3u_1'$ arise from terms in the strain energy of torsion which are of higher order than the term $\frac{1}{2}\int_0^l GC\beta'^2dz$ hitherto accepted. It is to be expected therefore that in (17) and (18) the terms of the second order will be required in energy calculations in other problems where torsional loads cause, or contribute to, buckling.

When the equilibrium of the straight twisted form is neutral, the work done by M_3 during buckling is equal to the gain of strain energy. Then

$$M_3\int_0^l \beta_1' dz = \frac{1}{2}\int_0^l [EI_1(u_1''^2 + 2Bu_1''v_1'') + EI_2(v_1''^2 - 2Bu_1''v_1'') + GC(2B'\beta_1' + \beta_1'^2 + 2B'u_1''v_1')]dz. \tag{23}$$

The term $Bu_1''v_1''$ in the flexural terms is small compared with $u_1''^2$ or $v_1''^2$ and will be dropped. Introducing $M_3 = GCB'$ the resulting equation yields

$$M_3 = -\frac{1}{2}\frac{\int_0^l (EI_1u_1''^2 + EI_2v_1''^2 + GC\beta_1'^2)dz}{\int_0^l u_1''v_1' dz}. \tag{24}$$

Now equations (21), (22) (after one differentiation) together with $\beta_1' = 0$ are the Euler differential equations for the functions u_1, v_1, β_1 making the right of (24) a minimum. Since $\beta_1' = 0$ the term $GC\beta_1'^2$ in the numerator of (24) may be dropped. The critical M_3 is the least value of the right of (24) with or without this term. Without it the equation may be interpreted as showing that the energy of flexure which appears when buckling occurs is accounted for by a decrease of torsional energy of amount $M_3\int_0^l u_1''v_1' dz$.

The same equation is suitable for the approximate determination of the critical torque by the Rayleigh method—assuming simple plausible forms for u_1 and v_1 and adjusting the parameters of these forms to obtain a least value of M_3. This method is applicable to non-uniform bars.

Equation (23) would in general require modification if the ends are not "built-in," for instance if they are attached to Hooke's joints. For then the work of M_3 is not done merely on a rotation $\int_0^l \beta_1' dz$. Certain terms of higher order must be added to β_1', and these can be of the same order as $u_1''v_1'$. Such terms would be significant in (24). Nevertheless (24) is appropriate in the Rayleigh method whatever the end constraints, for its minimizing conditions are the differential equations of equilibrium which must be satisfied irrespective of end constraints.

There are expressions other than the right of (24) which yield the critical M_3 as a minimum value. If (21) and (22) are multiplied respectively by u_1'', v_1'', integrated along the bar, and added, the result yields another in the form

$$M_3 = \frac{\int_0^l (EI_1u_1''^2 + EI_2v_1''^2)dz}{\int_0^l (u_1'v_1'' - v_1'u_1'')dz}.$$