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# On common fixed point theorems

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The aim of this paper is to extend a recent result of S.A. Husain and V.M. Sehgal [*Bull. Austral. Math. Soc.* 13 (1975), 261-267]. The condition that the function should be continuous in Husain and Sehgal, *op. cit.*, is replaced by a semicontinuity condition. Moreover, a different proof is given, the last part of which requires no continuity or semicontinuity condition whatsoever.

In a recent paper Husain and Sehgal [*1*] proved some results on fixed and common fixed points under the condition of continuity of the function. In this paper the theorem has been proved under considerably weaker conditions.

Let $R^+$ denote the set of nonnegative reals. Let $\psi$ denote a family of mappings such that each $\phi \in \psi$, $\phi : (R^+)^5 \to R^+$, and $\phi$ is upper semicontinuous and nondecreasing in each coordinate variable. Also let $\gamma(t) = \phi(t, t, a_1t, a_2t, t)$ where $\gamma$ is a function $\gamma : R^+ \to R^+$ where $a_1 + a_2 = 3$.

LEMMA 1. *For every* $t > 0$, $\gamma(t) < t$ *if and only if* $\lim_{n\to\infty} \gamma^n(t) = 0$.

Proof (Necessity). Since $\phi$ is upper semicontinuous, then $\gamma$ is upper semicontinuous. Assume $\lim_{n\to\infty} \gamma^n(t) = A$ where $A \neq 0$. Then

$$A = \lim_{n\to\infty} \gamma^{n+1}(t) \le \gamma \lim_{n\to\infty} \gamma^n(t) = \gamma(A) < A ;$$

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that is, $A < A$ , a contradiction. Therefore $A = 0$ .

(Sufficiency). Since $\phi$ is nondecreasing, then $\gamma$ is nondecreasing. Given $\lim_{n\to\infty} \gamma^n(t) = 0$ , assume $\gamma(t) > t$ for some $t > 0$ . Then $\gamma^n(t) > t$ for some $t > 0$ for $n = 1, 2, 3, \ldots$ . Thus $\lim_{n\to\infty} \gamma^n(t) \not\to 0$ , a contradiction. Also if $\gamma(t) = t$ for some $t > 0$ , then $\lim_{n\to\infty} \gamma^n(t) \not\to 0$ . Hence, for all $t > 0$ , $\gamma(t) < t$ .

THEOREM 1. *Let* $(X, d)$ *be a complete metric space and let* $S$ *and* $T$ *be selfmappings of* $X$ . *Suppose there exists a* $\phi \in \psi$ *such that for all* $x, y \in X$ ,

(1) $$d\big(S(x), T(y)\big) \le \phi\big(d(x, y), d\big(x, S(x)\big), d\big(x, T(y)\big), d\big(y, S(x)\big), d\big(y, T(y)\big)\big)$$

*where* $\phi$ *satisfies the condition: for any* $t > 0$ ,

(2) $$\phi\big(t, t, a_1 t, a_2 t, t\big) < t \text{ where } a_1 + a_2 = 3 .$$

*Then there exists a* $z \in X$ *such that* $z$ *is a unique common fixed point of* $S$ *and* $T$ .

Proof. Since $\gamma(t) = \phi(t, t, t, 2t, t) < t$ , then, by Lemma 1, $\lim_{n\to\infty} \gamma^n(t) = 0$ . Now let $x_0 \in X$ be any point. Then define a sequence of iterates $\{x_n\}$ in the following way:

$$x_1 = S(x_0),\ x_2 = T(x_1),\ x_3 = S(x_2) \ldots x_{2n} = T(x_{2n-1}),\ x_{2n+1} = S(x_{2n}),\ \ldots .$$

Claim. $d(x_1, x_2) \le d(x_0, x_1)$ .

Assume $d(x_0, x_1) < d(x_1, x_2)$ . Then using the triangular inequality, $d(x_0, x_2) < 2d(x_1, x_2)$ . Let $r = d(x_1, x_2)$ . Then

$$r \le \phi\big(d(x_0, x_1), d(x_1, x_2), d(x_1, x_1), d(x_0, x_2), d(x_0, x_1)\big) < \phi(r, r, r, 2r, r) .$$

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By (2), $\phi(r, r, r, 2r, r) < r$ and thus $r < r$, a contradiction: therefore, $d(x_1, x_2) \le d(x_0, x_1)$ and

$$d(x_1, x_2) \le \phi\left(d(x_0, x_1), d(x_0, x_1), 0, 2d(x_0, x_1), d(x_0, x_1)\right) \le \gamma d(x_0, x_1) .$$

Similarly, $d(x_2, x_3) \le \gamma\left(d(x_1, x_2)\right) \le \gamma^2\left(d(x_0, x_1)\right)$ and in general, $d(x_n, x_{n+1}) \le \gamma^n\left(d(x_0, x_1)\right)$. Since $\lim_{n\to\infty} \gamma^n(t) = 0$ for $t > 0$; therefore,

(3) $$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 .$$

In order to show $\{x_n\}$ is Cauchy, it is sufficient to show that $\{x_{2n}\}$ is Cauchy. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ such that for each integer $2k$, $k \in I^+$, there exist integers $2n(k)$ and $2m(k)$ with $2k \le 2n(k) < 2m(k)$ such that

(4) $$d(x_{2n(k)}, x_{2m(k)}) < \varepsilon .$$

Let, for each integer $2k$, $k \in I^+$, $2m(k)$ be the least integer exceeding $2n(k)$ satisfying (4); that is, $d(x_{2n(k)}, x_{2m(k)-2}) \le \varepsilon$ and

(5) $$d(x_{2n(k)}, x_{2m(k)}) > \varepsilon .$$

Let $d_n = d(x_n, x_{n+1})$. Then for each integer $2k$, $k \in I^+$,

$$\varepsilon < d(x_{2n(k)}, x_{2m(k)}) \le d(x_{2n(k)}, x_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1} .$$

By (3) and (5),

(6) $$d(x_{2n(k)}, x_{2m(k)}) \to \varepsilon \text{ as } k \to \infty .$$

It follows from the triangular inequality that

$$\left|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})\right| \le d_{2m(k)-1}$$

and

$$\left|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})\right| \le d_{2m(k)-1} + d_{2n(k)} .$$

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By (6), as $k \to \infty$,

$$d\left(x_{2n(k)}, x_{2m(k)-1}\right) \to \varepsilon$$

and

$$d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) \to \varepsilon .$$

Now let

$$p(2k) = d\left(x_{2n(k)}, x_{2m(k)}\right) ,$$

$$q(2k) = d\left(x_{2n(k)}, x_{2m(k)-1}\right) ,$$

and

$$r(2k) = d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) .$$

Then

$$p(2k) \le d_{2n(k)} + d\left(S\left(x_{2n(k)}\right), T\left(x_{2m(k)-1}\right)\right) .$$

By (1),

$$p(2k) < d_{2n(k)} + \phi\left(q(2k), d_{2n(k)}, p(2k), r(2k), d_{2m(k)-1}\right) .$$

Since $\phi$ is upper semicontinuous, as $n \to \infty$, it follows that $\varepsilon \le \phi(\varepsilon, 0, \varepsilon, \varepsilon, 0) \le \phi(\varepsilon, \varepsilon, \varepsilon, 2\varepsilon, \varepsilon) < \varepsilon$, a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence and hence by completeness, there is a $z \in X$ such that $x_n \to z$. We show that $z$ is a common fixed point of $S$ and $T$.

Since $\{x_n\}$ converges to $z$, therefore $\{x_{2n}\}$ and $\{x_{2n+1}\}$ both converge to $z$. Let $d(S(z), z) = \varepsilon > 0$. Thus we have, for $n \ge N$, $d\left(x_{2n}, z\right) < \frac{\varepsilon-\gamma(\varepsilon)}{4}$ and $d\left(x_{2n}, x_{2n+1}\right) \le \frac{\varepsilon-\gamma(\varepsilon)}{4}$. Therefore using the triangular inequality,

$$d\left(z, x_{2n-1}\right) \le d\left(z, x_{2n}\right) + d\left(x_{2n}, x_{2n-1}\right) < \frac{\varepsilon-\gamma(\varepsilon)}{4} + \frac{\varepsilon-\gamma(\varepsilon)}{4} = \frac{\varepsilon-\gamma(\varepsilon)}{2} ;$$

that is,

(7) $$d\left(z, x_{2n-1}\right) < \varepsilon .$$

Using the triangular inequality and (7) we have

(8) $$d(x_{2n-1}, S(z)) \le d(x_{2n-1}, z) + d(z, S(z)) < \varepsilon + \varepsilon ;$$

that is, $d(x_{2n-1}, S(z)) < 2\varepsilon$ .

Now

$$\varepsilon = d(S(z), z) \le d(S(z), T(x_{2n-1})) + d_{2n} + d(x_{2n+1}, z) .$$

By (1),

$$\varepsilon \le \phi(d(z, x_{2n-1}), d(z, S(z)), d(z, x_{2n}), d(x_{2n-1}, S(z)), d_{2n-1}) + d_{2n} + d(x_{2n+1}, z) .$$

By (7) and (8), $\varepsilon \le \phi(\varepsilon, \varepsilon, \varepsilon, 2\varepsilon, \varepsilon) + \frac{\varepsilon-\gamma(\varepsilon)}{4} + \frac{\varepsilon-\gamma(\varepsilon)}{4}$ ; that is, $\varepsilon \le \gamma(\varepsilon) + \frac{\varepsilon-\gamma(\varepsilon)}{2} = \frac{\gamma(\varepsilon)+\varepsilon}{2}$ . Since $\gamma(\varepsilon) < \varepsilon$ , then $\frac{\gamma(\varepsilon)+\varepsilon}{2} < \varepsilon$ ; that is, $\varepsilon < \varepsilon$ , a contradiction. Therefore $z = S(z)$ . Similarly $z = T(z)$ .

It remains to show that $z$ is a unique common fixed point. Let $z \ne y$ be two common fixed points of $S$ and $T$ . Then

$$d(z, y) = d(S(z), T(y)) \le \phi(d(z, y), d(z, S(z)), d(z, T(y)), d(y, S(z)), d(y, T(y)))$$
$$= \phi(d(z, y), 0, d(z, y), d(x, y), 0) < \gamma(d(z, y)) < d(z, y) .$$

Therefore $z = y$ and $S$ and $T$ have a unique common fixed point.

Taking $\phi$ to be continuous we get a slightly revised version of the result of Husain and Sehgal [1] as a corollary.

## References

[1] S.A. Husain and V.M. Sehgal, "On common fixed points for a family of mappings", *Bull. Austral. Math. Soc.* 13 (1975), 261-267.

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