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On common fixed points for $\alpha - F$-contractions and applications

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Abstract

In this paper, we introduce the concept of modified F-contractions via α-admissible pair of mappings. We also provide several common fixed point results in the setting of metric spaces. Moreover, we present some illustrated examples and we give two applications on a dynamic programming and an integral equation.

Keywords: Metric space, α-admissible mappings, F-contraction, common fixed point.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

In 2012, Wasrdowski [27] defined a new class of contractions named as F-contractions. First, let $\mathfrak{F}$ be the set of functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is increasing, i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

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(F_2) For any sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers, $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) There exists $k \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

As examples of $F \in \mathfrak{F}$, we may cite $F(y) = \ln(y)$, $F(y) = \ln(y) + y$ and $F(y) = -\frac{1}{\sqrt{y}}$ for $y > 0$. For more details, refer to [27]. The concept of an F-contraction is defined as follows:

Definition 1.1 ([27])**.** Let (X, d) be a metric space. A self-mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{1.1}$$

for all $x, y \in X$ and for some $F \in \mathcal{F}$.

The main result of Wardowski [27], which is a generalization of the Banach contraction principle [7], is stated as follows.

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then, T has a unique fixed point.*

Recently, the concept of an F-contraction and Theorem 1.2 have been generalized in many directions, for more details see [1, 11, 16, 18, 19, 22].

On the other hand, Samet et al. [21] introduced the class of α-admissible mappings.

Definition 1.3 ([21])**.** For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be given mappings. We say that T is $\alpha-$admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{1.2}$$

This concept has been considered in many papers, see [4, 5, 12–15]. Very recently, Aydi [3] generalized Definition 1.3 and introduced the following.

Definition 1.4. For a nonempty set X, let $A, B : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that (A, B) is a generalized α-admissible pair if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Ax, By) \geq 1 \quad \text{and } \alpha(By, Ax) \geq 1. \tag{1.3}$$

Starting from the work of Wardowski [27], the goal of this paper is to modify, extend and improve the notion of F-contraction via α-admissible pair of mappings and to prove some common fixed point results for this type of contractions. We will support the obtained theorems by some concrete examples. Two illustrated applications on a dynamic programming and an integral equation are also provided.

2. Main results

We introduce the concept of an $\alpha - F$-contraction as follows:

Definition 2.1. Let (X, d) be a metric space and $A, B : X \to X$ be self mappings. The pair (A, B) is $\alpha - F$-contractive if there exists $\tau > 0$ such that for all $x, y \in X$ with $\alpha(x, y) \geq 1$

$$d(Ax, By) > 0 \Rightarrow \tau + F(d(Ax, By)) \leq F(M(x, y)), \tag{2.1}$$

where $F \in \mathfrak{F}$ and

$$M(x, y) = \max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{2}\}. \tag{2.2}$$

In the case where $F(t) = \ln(t)$ for $t > 0$, equation (2.1) becomes

$$d(Ax, By) \leq e^{-\tau} M(x, y) = kM(x, y) \tag{2.3}$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$, $Ax \neq By$ and $k = e^{-\tau} < 1$. Note that (2.3) is also satisfied for all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $Ax = By$.

Now, let us prove the following main theorem.

Theorem 2.2. *Let (X, d) be a complete metric space and $A, B : X \to X$ be such that (A, B) is $\alpha - F$-contractive. Suppose that*

(i) (A, B) is a generalized α-admissible pair;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$ and $\alpha(Ax_0, x_0) \geq 1$;
(iii) A and B are continuous.

Then, A and B have a common fixed point.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$ and $\alpha(Ax_0, x_0) \geq 1$. Take $x_1 = Ax_0$ and $x_2 = Bx_1$. By induction, we construct a sequence $\{x_n\}$ such that

$$x_{2n} = Bx_{2n-1} \quad \text{and} \quad x_{2n+1} = Ax_{2n} \quad \forall\ n = 1, 2, \dots\ . \tag{2.4}$$

Let $a_n = d(x_n, x_{n+1})$ for $n \geq 0$.

We split the proof of our result into several steps:

Step 1: $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \geq 0$.

We have $\alpha(x_0, x_1) \geq 1$ and $\alpha(x_1, x_0) \geq 1$. (A, B) is a generalized α-admissible pair of mappings, so

$$\alpha(x_1, x_2) = \alpha(Ax_0, Bx_1) \geq 1 \quad \text{and} \quad \alpha(x_2, x_1) = \alpha(Bx_1, Ax_0) \geq 1.$$

We also have

$$\alpha(x_3, x_2) = \alpha(Ax_2, Bx_1) \geq 1 \quad \text{and} \quad \alpha(x_2, x_3) = \alpha(Bx_1, Ax_2) \geq 1.$$

Similar to above, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_{n+1}, x_n) \geq 1 \quad \text{for all } n = 0, 1, \dots\ . \tag{2.5}$$

Step 2: We shall prove

$$\lim_{n\to\infty} a_n = 0. \tag{2.6}$$

If $d(x_{2n}, x_{2n+1}) = 0$ for some n, then we prove that $d(x_{2n+1}, x_{2n+2}) = 0$. We argue by contradiction that, $d(x_{2n+1}, x_{2n+2}) = d(Ax_{2n}, Bx_{2n+1}) > 0$.

From (2.1) and (2.5) (that is, $\alpha(x_{2n}, x_{2n+1}) \geq 1$), by the triangular inequality, we have

$$\tau + F(d(x_{2n+1}, x_{2n+2})) = \tau + F(d(Ax_{2n}, Bx_{2n+1})) \leq F(M(x_{2n}, x_{2n+1})),$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Ax_{2n}), d(x_{2n+1}, Bx_{2n+1}), \\ &\qquad \frac{d(x_{2n}, Bx_{2n+1}) + d(x_{2n+1}, Ax_{2n})}{2}\} \\ &= \max\{0, 0, d(x_{2n+1}, x_{2n+2}), \frac{1}{2} d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Then,

$$\tau + F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n+1}, x_{2n+2})).$$

This implies that

$$F(d(x_{2n+1}, x_{2n+2})) < F(d(x_{2n+1}, x_{2n+2})).$$

From (F_1),

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction.

Finally, we have $x_{2n} = x_{2n+1} = x_{2n+2}$. Then $x_{2n} = x_{2n+m}$, for all $m = 0, 1, \ldots$.

We have $x_{2n} = x_{2n+1} = Ax_{2n}$ and $x_{2n} = x_{2n+2} = Bx_{2n+1} = Bx_{2n}$. Hence x_{2n} is a common fixed point of A and B.

Similarly, if $d(x_{2n+1}, x_{2n+2}) = 0$ for some n, we find that x_{2n+1} is a common fixed point of A and B and this completes the proof.

Suppose now that $d(x_n, x_{n+1}) > 0$ for all $n \geq 0$.

Since $d(x_{2n}, x_{2n+1}) = d(Ax_{2n}, Bx_{2n-1}) > 0$, by (2.1) and (2.5) (that is, $\alpha(x_{2n}, x_{2n-1}) \geq 1$), we have

$$\tau + F(d(x_{2n+1}, x_{2n})) = \tau + F(d(Ax_{2n}, Bx_{2n-1})) \leq F(M(x_{2n}, x_{2n-1})),$$

where

$$\begin{aligned} M(x_{2n}, x_{2n-1}) &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \\ &\quad \frac{d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \frac{1}{2} d(x_{2n-1}, x_{2n+1})\} \\ &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\}. \end{aligned}$$

Therefore, by (F_1)

$$d(x_{2n+1}, x_{2n}) < \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\}.$$

If $\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+1})$, then

$$0 < d(x_{2n+1}, x_{2n}) < d(x_{2n}, x_{2n+1}),$$

which is a contradiction. Thus, for all $n \geq 0$,

$$F(d(x_{2n+1}, x_{2n})) \leq F(d(x_{2n}, x_{2n-1})) - \tau. \tag{2.7}$$

Again, we have $d(x_{2n+1}, x_{2n+2}) = d(Ax_{2n}, Bx_{2n+1}) > 0$. Then, by (2.1) and (2.5) (that is, $\alpha(x_{2n}, x_{2n+1}) \geq 1$), we get

$$\tau + F(d(x_{2n+1}, x_{2n+2})) = \tau + F(d(Ax_{2n}, Bx_{2n+1})) \leq F(M(x_{2n}, x_{2n+1})),$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2} d(x_{2n}, x_{2n+2})\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

Then, by (F_1)

$$d(x_{2n+1}, x_{2n+2}) < \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.$$

If $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, then

$$0 < d(x_{2n+1}, x_{2n+2}) < \psi(d(x_{2n+1}, x_{2n+2})\}) \leq d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction. Thus

$$F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})) - \tau \quad \text{for all } n \geq 0. \tag{2.8}$$

Combining (2.7) and (2.8), we get

$$F(a_n) \leq F(a_{n-1}) - \tau \quad \text{for all } n \geq 1. \tag{2.9}$$

We have

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \ldots \leq F(a_0) - n\tau \quad \text{for all } n \geq 1. \tag{2.10}$$

From (2.10), we obtain $\lim_{n\to\infty} F(a_n) = -\infty$. Applying (F_2), we get

$$\lim_{n\to\infty} a_n = 0. \tag{2.11}$$

Step 3: We shall prove that $\{x_n\}$ is a Cauchy sequence.

From (2.11) and (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0. \tag{2.12}$$

By (2.10), we have for all $n = 1, 2, \ldots$

$$a_n^k F(a_n) - a_n^k F(a_0) \leq a_n^k (F(a_0) - n\tau) - a_n^k F(a_0) = -n\tau a_n^k \leq 0. \tag{2.13}$$

Letting $n \to \infty$ in (2.13), by (2.11) and (2.12), we obtain

$$\lim_{n\to\infty} n a_n^k = 0. \tag{2.14}$$

This implies that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$a_n \leq \frac{1}{n^{\frac{1}{k}}}. \tag{2.15}$$

Then for all $n \geq n_0$ and $p \in \mathbb{N}$

$$d(x_n, x_{n+p}) \leq \sum_{i=n}^{n+p-1} d(x_i, x_{i+1}) = \sum_{i=n}^{n+p-1} a_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Since $\sum_{n\geq 1} \frac{1}{n^{\frac{1}{k}}} < \infty$, $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$. Thus $\{x_n\}$ is a Cauchy sequence.

As (X, d) is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} d(x_n, u) = 0$.

Step 4: We shall prove that u is a common fixed point of A and B.

Having $\lim_{n\to\infty} d(x_n, u) = 0$, then $\lim_{n\to\infty} d(x_{2n}, u) = \lim_{n\to\infty} d(x_{2n+1}, u) = 0$. By continuity of A and B, we obtain that $\lim_{n\to\infty} d(x_{2n+1}, Au) = \lim_{n\to\infty} d(Ax_{2n}, Au) = 0$ and $\lim_{n\to\infty} d(x_{2n+2}, Bu) = \lim_{n\to\infty} d(Bx_{2n+1}, Bu) = 0$. Hence $Au = u = Bu$, that is, u is a common fixed point of A and B. The proof is completed.

□

Now, let Ψ be the family of continuous functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following condition: $\psi(t) < t$ for all $t > 0$. As in Definition 2.1, we introduce the concept of an $\alpha-\psi-F$-contraction as follows:

Definition 2.3. Let (X, d) be a metric space and $A, B : X \to X$ be self mappings. The pair (A, B) is $\alpha - \psi - F$-contractive if there exists $\tau > 0$ such that for all $x, y \in X$ with $\alpha(x, y) \geq 1$

$$d(Ax, By) > 0 \Rightarrow \tau + F(d(Ax, By)) \leq F(\psi(M(x, y))), \tag{2.16}$$

where $F \in \mathfrak{F}$, $\psi \in \Psi$ and $M(x, y)$ is defined by (2.2).

In the next result, the continuity hypothesis is replaced by the following property:

(H) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ and $\alpha(x, x_{n(k)}) \geq 1$ for all k.

So we have,

Theorem 2.4. *Let (X, d) be a complete metric space and $A, B : X \to X$ be given mappings such that (A, B) is $\alpha - \psi - F$-contractive. Suppose that*

(i) (A, B) is a generalized α-admissible pair;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$ and $\alpha(Ax_0, x_0) \geq 1$;
(iii) (H) holds.

Then A and B have a common fixed point.

Proof. Following the proof of Theorem 2.2, it is obvious that the sequence $\{x_n\}$ is Cauchy in (X, d) and converges to some $u \in X$. We shall show that $Au = u = Bu$. Suppose, on the contrary, $Au \neq u$ or $Bu \neq u$. If $x_n = Au$ and $x_n = Bu$ for arbitrary large n, so necessarily $Au = u = Bu$. So, we assume that $x_n \neq Au$ or $x_n \neq Bu$ for infinitely many n. Let us suppose that $x_n \neq Au$ for all $n \in \mathbb{N}$.
Since for all $k \in \mathbb{N}$, we have $d(Au, x_{2n(k)}) = d(Au, Bx_{2n(k)-1}) > 0$. Then, by assumption (iii) (that is, $\alpha(u, x_{2n(k)-1}) \geq 1$) and (2.16), we have the following

$$\tau + F(d(Au, Bx_{2n(k)-1})) = \tau + F(d(Au, Bx_{2n(k)-1})) \leq F(\psi(M(u, x_{2n(k)-1}))), \tag{2.17}$$

where

$$\begin{aligned} M(u, x_{2n(k)-1})) &= \max\{d(u, x_{2n(k)-1}), d(u, Au), d(x_{2n(k)-1}, Bx_{2n(k)-1}), \\ &\quad \frac{d(u, Bx_{2n(k)-1}) + d(x_{2n(k)-1}, Au)}{2}\} \\ &= \max\{d(u, x_{2n(k)-1}), d(u, Au), d(x_{2n(k)-1}, x_{2n(k)}), \\ &\quad \frac{d(u, x_{2n(k)}) + d(x_{2n(k)-1}, Au)}{2}\}. \end{aligned}$$

We know that

$$\lim_{n\to\infty} d(u, x_{2n(k)-1}) = \lim_{n\to\infty} d(x_{2n(k)-1}, x_{2n(k)}) = \lim_{n\to\infty} d(u, x_{2n(k)}) = 0,$$

and

$$\lim_{n\to\infty} d(x_{2n(k)-1}, Au) = d(u, Au).$$

On the other hand, by (2.17) and (F_1), we have for all $k \in \mathbb{N}$

$$d(Au, x_{2n(k)}) = d(Au, Bx_{2n(k)-1}) < \psi(M(u, x_{2n(k)-1})), \quad \text{for all } k \in \mathbb{N}. \tag{2.18}$$

Referring to above limits and using the continuity of ψ in (2.18), as $k \to \infty$, we get

$$d(Au, u) \leq \psi(d(u, Au)).$$

Having in mind that $\psi(t) < t$ for all $t > 0$, so the above inequality becomes

$$0 < d(Au, u) \leq \psi(d(u, Au)) < d(Au, u),$$

which is a contradiction. Hence, we find that u is a fixed point of A. Similarly, we find that u is a fixed point of B. Thus, u is a common fixed point of A and B. □

We provide the following example.

Example 2.5. Take $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = |x - y|$. Consider the mappings $A, B : X \to X$ given by

$$Ax = \begin{cases} \frac{x}{3} & \text{if } x \in [0,1] \\ 2x - 2 & \text{if } x > 1 \end{cases} \qquad \text{and} \quad Bx = \begin{cases} 0 & \text{if } x \in [0,1] \\ x & \text{if } x > 1. \end{cases}$$

Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 + \cos(x^2 + y) & \text{if } x, y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{9}{10}t$, $F(t) = \ln(t^2 + t)$ for all $t > 0$ and $\tau = \ln \frac{3}{2}$. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By definition of α, this implies that $x, y \in [0, 1]$. Thus,

$$\alpha(Ax, By) = \alpha(\frac{x}{3}, 0) = 2 + \cos(\frac{x^2}{9}) \geq 1 \,\text{and}\, \alpha(By, Ax) = \alpha(0, \frac{x}{3}) = 2 + \cos(\frac{x}{3}) \geq 1.$$

Then, (A, B) is a generalized α-admissible pair.

Note that A and B are noncontinuous mappings. Now, we show that (H) is verified. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_{n+}) \geq 1$, for all n and $x_n \to u \in X$. Then, $\{x_n\} \subset [0, 1]$. Consequently, $u \in [0, 1]$. Thus, $\alpha(x_n, u) = 2 + \cos(x_n^2 + u) \geq 1$ and $\alpha(u, x_n) = 2 + \cos(u^2 + x_n) \geq 1$ for all n. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$ and $\alpha(Ax_0, x_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, A1) = \alpha(1, \frac{1}{3}) = 2 + \cos(\frac{4}{3}) \geq 1$ and $\alpha(A1, 1) = 2 + \cos(\frac{10}{9}) \geq 1$.

Now, we show that (A, B) is $\alpha - \psi - F-$contractive. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. So, $x, y \in [0, 1]$. In this case, we have $d(Ax, By) = |Ax - By| = \frac{1}{3}x$. We also have

$$\begin{aligned} M(x, y) &= \max\{|x - y|, \frac{2}{3}x, y, \frac{1}{2}(x + |\frac{1}{3}x - y|)\} \\ &= \begin{cases} x - y, & 0 \leq y \leq \frac{1}{3}x \\ \frac{2}{3}x, & \frac{1}{3}x < y \leq \frac{2}{3}x \\ y, & \frac{2}{3}x < y \leq 1. \end{cases} \end{aligned}$$

It is easy that

$$d(Ax, By) \leq \frac{2}{3}\psi(M(x, y)),$$

for all $x, y \in X$ such that $\alpha(x, y) \geq 1$. Therefore

$$\begin{aligned} d(Ax, By)(1 + d(Ax, By)) &\leq \frac{2}{3}\psi(M(x, y))(1 + \frac{2}{3}\psi(M(x, y))) \\ &\leq \frac{2}{3}\psi(M(x, y))(1 + \psi(M(x, y))). \end{aligned}$$

Thus, for $x, y \in X$ such that $\alpha(x, y) \geq 1$ and $d(Ax, By) > 0$, we have

$$\tau + F(d(Ax, By)) \leq F(\psi(M(x, y))).$$

Hence, all hypotheses of Theorem 2.4 are verified. Indeed, $\{0, 2\}$ is the set of common fixed points of A and B.

The mappings considered in the above example have two common fixed points which are 0 and 2. Mention that $\alpha(0, 2) = 0$, which is not greater than 1. So for the uniqueness, we need the following additional condition.

(U) For all $x, y \in CF(A, B)$, we have $\alpha(x, y) \geq 1$, where $CF(A, B)$ denotes the set of common fixed points of A and B.

Theorem 2.6. *Adding condition (U) to the hypotheses of Theorem 2.2 (resp. Theorem 2.4), we obtain that u is the unique common fixed point of A and B.*

Proof. We argue by contradiction, that is, there exist $u, v \in X$ such that $u = Au = Bu$ and $v = Av = Bv$ with $u \neq v$. By assumption (U), we have $\alpha(u, v) \geq 1$. First, assume that hypotheses of Theorem 2.2 hold. Since $d(u, v) = d(Au, Bv) > 0$, by (2.1), we have

$$\tau + F(d(u, v)) = \tau + F(d(Au, Bv)) \leq F(M(u, v)) = F(d(u, v)).$$

Then, by (F_1)

$$0 < d(u, v)) < d(u, v),$$

which is a contradiction. Hence $u = v$. Second, assume that hypotheses of Theorem 2.4 hold. Similarly, using (2.16), we get

$$\tau + F(d(u, v)) \leq F(\psi(M(u, v))) = F(\psi(d(u, v))).$$

Again, by (F_1)

$$0 < d(u, v)) < \psi(d(u, v)) \leq d(u, v),$$

which is a contradiction, so $u = v$.

□

In the following, we state some consequences and corollaries of our obtained common fixed point results.

Corollary 2.7. *Let (X, d) be a complete metric space and $A, B : X \to X$ be given continuous mappings. Suppose there exists $\tau > 0$ such that*

$$d(Ax, By) > 0 \Rightarrow \tau + F(d(Ax, By)) \leq F(M(x, y)) \tag{2.19}$$

for all $x, y \in X$, where $F \in \mathfrak{F}$ and $M(x, y)$ is defined by (2.2). Then, A and B have a unique common fixed point.

Proof. It suffices to take $\alpha(x, y) = 1$ in Theorem 2.2 and to apply Theorem 2.6.

□

Corollary 2.8. *Let (X, d) be a complete metric space and $A, B : X \to X$ be given continuous mappings. Suppose there exists $k \in (0, 1)$ such that for all $x, y \in X$*

$$d(Ax, By) \leq kM(x, y), \tag{2.20}$$

where $\psi \in \Psi$ and $M(x, y)$ defined by (2.2). Then, A and B have a unique common fixed point.

Proof. It suffices to take $F(t) = \ln(t)$ and $\tau = -\ln(k)$ in Corollary 2.7 for $Ax \neq Bx$. Note that (2.20) is also satisfied for all $x, y \in X$ with $Ax = Bx$. □

Another immediate corollary is

Corollary 2.9. *Let (X, d) be a complete metric space and $A, B : X \to X$ be given continuous mappings. Suppose there exists $k \in (0, 1)$ such that*

$$d(Ax, By) \leq kd(x, y), \tag{2.21}$$

for all $x, y \in X$. Then, A and B have a unique common fixed point.

Corollary 2.10 ([27], Theorem 2.1)**.** *Let (X, d) be a complete metric space and $T : X \to X$ be an $F-$contraction. Then T has a unique fixed point $x^\star \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to $x^\star$.*

Proof. Taking $A = B = T$ in Theorem 2.2, then by (F_1), if $d(Tx, Ty) > 0$ and $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$, we have $\tau + F(d(Tx, Ty)) \leq F(M(x, y))$. The proof is then concluded by Theorem 2.2. □

Corollary 2.11. *Let (X, d) be a complete metric space and $A, B : X \to X$ be given mappings. Suppose there exists $\tau > 0$ such that*

$$d(Ax, By) > 0 \Rightarrow \tau + F(d(Ax, By)) \leq F(\psi(M(x, y))), \tag{2.22}$$

for all $x, y \in X$, where $F \in \mathfrak{F}$, $\psi \in \Psi$ and $M(x, y)$ is defined by (2.2). Then A and B have a unique common fixed point.

Proof. It suffices to take $\alpha(x, y) = 1$ in Theorem 2.4 and to apply Theorem 2.6. □

Now, we present an example where we guarantee the uniqueness of the common fixed point.

Example 2.12. Recall to Example 2.5 and consider $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = |x - y|$. Consider the mappings $A, B : X \to X$ given by

$$Ax = \begin{cases} \frac{x}{3} & \text{if } x \in [0, 1] \\ x - \frac{2}{3} & \text{if } x > 1 \end{cases} \quad \text{and} \quad Bx = \begin{cases} 0 & \text{if } x \in [0, 1] \\ x^2 - 1 & \text{if } x > 1. \end{cases}$$

Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} \cosh(x^2 + y^2) & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(t) = \ln(t) + t$ and $\tau = \ln \frac{3}{2}$. It is clear that

(*i*) (A, B) is a generalized α-admissible pair;
(*ii*) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$ and $\alpha(Ax_0, x_0) \geq 1$.

Moreover, it is easy to show that the pair (A, B) is $\alpha - F-$contractive. Thus, all hypotheses of Theorem 2.2 are verified. Here, 0 is the unique common fixed points of A and B.

The investigation of existence of fixed points on metric spaces endowed with a partial order was initiated by Turinici [26] in 1986. Then, several interesting and valuable results appeared in this direction, for example see [2, 6, 17, 20, 23–25].

Definition 2.13. Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$x, y \in X,\ x \preceq y \Longrightarrow Tx \preceq Ty.$$

Definition 2.14. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 2.15. Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, d, \preceq)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$, for all k.

We have the following consequence from Theorem 2.4.

Corollary 2.16. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a nondecreasing mapping with respect to $\preceq$. Assume that there exists $\tau > 0$ such that*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\psi(M(x, y))) \tag{2.23}$$

for all $x, y \in X$ with $x \preceq y$, where $F \in \mathfrak{F}$, $\psi \in \Psi$ and $M(x, y)$ is defined by (2.2) (with $A = B = T$). Assume also that

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(ii) for a sequence $\{x_n\} \subset X$ with $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k \in \mathbb{N}$.

Then T has a fixed point in X.

Proof. Define $\alpha : X \times X \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ such that $\alpha(x, y) \geq 1$, then $x \preceq y$. The mapping T is nondecreasing with respect to $\preceq$, so $Tx \preceq Ty$. We get that $\alpha(Tx, Ty) \geq 1$. Hence T is $\alpha-$admissible. By condition (i), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Finally, by condition (ii), the sequence $\{x_n\}$ verifies hypothesis (H). Thus, all the hypotheses of Theorem 2.4 are satisfied, so T has a fixed point in X.

□

3. Applications

In this section, we state two applications, one on a dynamic programming and the second on an integral equation.

3.1. Application on a dynamic programming

In this subsection, we present an application on a dynamic programming. The existence of solutions of functional equations and system of functional equations arising in dynamic programming which have been studied by using various fixed point theorems. For more details, the reader can see [8–10]. In this paragraph, we will prove the existence of a common solution for classes of functional equations using Corollary 2.11. Here, we assume that U and V are Banach spaces, $W \subset U$ is a state space and $D \subset V$ is a decision space. It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well. In particular, we are interested in solving the following two functional equations arising in dynamic programming:

$$r(x) = \sup_{y \in D}\{f(x, y) + G(x, y, r(\tau(x, y)))\}, \quad x \in W, \tag{3.1}$$

$$r(x) = \sup_{y \in D}\{f(x,y) + Q(x,y,r(\tau(x,y)))\}, \quad x \in W, \tag{3.2}$$

where $\tau : W \times D \to W, f : W \times D \to \mathbb{R}$ and $G, Q : W \times D \times \mathbb{R} \to \mathbb{R}$. Here, we study the existence and uniqueness of $h_\star \in B(W)$ a common solution of the functional equations (3.1) and (3.2).

Let $B(W)$ denote the set of all bounded real-valued functions on W. We know that $B(W)$ endowed with the metric

$$d(h,k) = \sup_{x \in W} |h(x) - k(x)|, \quad h, k \in B(W), \tag{3.3}$$

is a complete metric space. Consider the mappings $A, B : B(W) \to B(W)$

$$A(h)(x) = \sup_{y \in D}\{f(x,y) + G(x,y,h(\tau(x,y)))\}, \quad x \in W, \tag{3.4}$$

$$B(h)(x) = \sup_{y \in D}\{f(x,y) + Q(x,y,h(\tau(x,y)))\}, \quad x \in W. \tag{3.5}$$

It's clear that, if f, G and Q are bounded, then the operators A and B are well-defined.

We shall prove the following theorem.

Theorem 3.1. *Let $0 < a < 1$. Suppose there exists $k \in (0, a)$ such that for every $(x,y) \in W \times D$ and $h_1, h_2 \in B(W)$, we have*

$$|G(x,y,h_1(\tau(x,y))) - Q(x,y,h_2(\tau(x,y)))| \le kM(h_1,h_2), \tag{3.6}$$

where

$$M(h_1,h_2) = \max\{d(h_1,h_2), d(h_1,Ah_2), d(h_2,Bh_2), \frac{d(h_1,Bh_2) + d(h_2,Ah_1)}{2}\}.$$

Then, A and B have a unique common fixed point in $B(W)$.

Proof. Let $\lambda > 0$ be an arbitrary positive real number, $x \in W$, $h_1, h_2 \in B(W)$. Then by (3.4) and (3.5), there exist $y_1, y_2 \in D$ such that

$$A(h_1)(x) < f(x,y_1) + G(x,y_1,h_1(\tau(x,y_1))) + \lambda, \tag{3.7}$$

$$B(h_2)(x) < f(x,y_2) + Q(x,y_2,k(\tau(x,y_2))) + \lambda, \tag{3.8}$$

$$A(h_1)(x) \ge f(x,y_2) + G(x,y_2,h_1(\tau(x,y_2))), \tag{3.9}$$

and

$$B(h_2)(x) \ge f(x,y_1) + Q(x,y_1,h_2(\tau(x,y_1))). \tag{3.10}$$

From (3.7) and (3.10), it follows that

$$\begin{aligned} A(h_1)(x) - B(h_2)(x) &\le G(x,y_1,h_1(\tau(x,y_1))) - Q(x,y_1,h_2(\tau(x,y_1))) + \lambda \\ &\le |G(x,y_1,h_1(\tau(x,y_1))) - Q(x,y_1,h_2(\tau(x,y_1)))| + \lambda \\ &\le kM(h_1,h_2) + \lambda. \end{aligned}$$

Similarly, from (3.8) and (3.9), we obtain

$$B(h_2)(x) - A(h_1)(x) \le kM(h_1,h_2) + \lambda.$$

Consequently, we deduce that

$$|A(h_1)(x) - B(h_2)(x)| \le kM(h_1,h_2) + \lambda. \tag{3.11}$$

Since the inequality (3.11) is true for any $x \in W$, we get

$$d(A(h_1), B(h_2)) \leq kM(h_1, h_2) + \lambda. \tag{3.12}$$

Finally, λ is arbitrary, so

$$d(A(h_1), B(h_2)) \leq kM(h_1, h_2), \tag{3.13}$$

that is, (2.22) holds by taking $\tau = -\ln(\frac{k}{a})$, $\psi(t) = at$ and $F(t) = \ln(t)$. Applying Corollary 2.11, the mappings A and B have a unique common fixed point, that is, the functional equations (3.1) and (3.2) have a unique common solution $h_\star \in B(W)$.

□

3.2. Application on an integral equation

In this subsection, we apply the result given by Corollary 2.11 to study the existence of a solution to a class of nonlinear integral equations.

For instance, we consider the nonlinear integral equations

$$x(t) = g(t) + \int_0^1 K_1(t, s, x(s))ds, \quad t \in [0, 1], \tag{3.14}$$

and

$$x(t) = g(t) + \int_0^1 K_2(t, s, x(s))ds, \quad t \in [0, 1], \tag{3.15}$$

where K_1, $K_2 : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous mappings. Let $X = C([0, 1], \mathbb{R})$ be the set of all continuous real-valued functions defined on $[0, 1]$. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = ||x - y|| = \sup\{|x(t) - y(t)| : t \in [0, 1]\}.$$

It is well known that (X, d) is a complete metric space. Now, we prove the following result.

Theorem 3.2. *Suppose the following hypotheses hold:*
Let $0 < a < 1$. Suppose there exist $k \in (0, 1)$ and $\beta : X \times X \to [0, \infty)$ such that for all $x, y \in X$ and $s \in [0, 1]$, we have

$$0 \leq |K_1(t, s, x(s)) - K_2(t, s, y(s))| \leq \beta(t, s)|y(s) - x(s)|, \tag{3.16}$$

and

$$\sup_{t\in[0,1]} \int_0^1 \beta(t, s)\, ds = k.$$

Then, the integral equations (3.14) and (3.15) have a unique common solution $x^\star$ in X.

Proof. For $x \in X$ and $t \in [0, 1]$, define the mappings

$$Ax(t) = g(t) + \int_0^1 K_1(t, r, x(r))dr \quad \text{and} \quad Bx(t) = g(t) + \int_0^1 K_2(t, r, x(r))dr.$$

Thus, by condition (3.16)

$$\begin{aligned} |Ax(t) - By(t)| &\leq \int_0^1 |K_1(t, s, x(s)) - K_2(t, s, y(s))|\, ds \\ &\leq \int_0^1 \beta(t, s)(|x(s) - y(s)|)\, ds \\ &\leq k||x - y||. \end{aligned}$$

We deduce that for all $x, y \in X$

$$d(Ax, By) \leq kM(x, y). \tag{3.17}$$

Again, as Theorem 3.1, (2.22) holds by taking $\tau = -\ln(\frac{k}{a})$, $\psi(t) = at$ and $F(t) = \ln(t)$. Corollary 2.11 is applicable and so the mappings A and B have a unique common fixed point, that is, the functional equations (3.14) and (3.15) have a unique common solution $x^{\star} \in X$.

□

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