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ON COMMON FIXED POINTS FOR SEVERAL CONTINUOUS AFFINE MAPPINGS

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It is known from Markov-Kakutani theorem that if T_j $(j=1,2,\cdots,J)$ are continuous affine commuting self-mappings on a compact convex subset of a locally convex space, then the intersection of the sets of fixed points of T_j $(j=1,2,\cdots,J)$ is nonempty. The object of this paper is to show a result which says more than the above theorem does, and actually our theorem shows in the case of $J=2$ that the set of fixed points of $\lambda T_1+(1-\lambda)T_2$ always coincides, for each λ $(0<\lambda<1)$, with the intersection of the sets of fixed points of T_1 and T_2.

1. **Introduction.** In this paper, we deal with a commuting family of continuous affine self-mappings on a compact convex subset of a locally convex space, and we give a result which seems to say more than Markov-Kakutani theorem itself does.

Let $F(T)$ denote the set of fixed points of a mapping T.

We have a following main theorem.

THEOREM. *Let K be a compact convex subset of locally convex space X, and let T_j $(j=1,2,\cdots,J)$ be continuous affine commuting self-mappings on K. Then $\bigcap_{j=1}^{J} F(T_j)$ is nonempty and equal to $F(\sum_{j=1}^{J}\alpha_j T_j)$ for any α_j $(j=1,2,\cdots,J)$ such that $\sum_{j=1}^{J}\alpha_j=1$, $0<\alpha_j<1$ $(j=1,2,\cdots,J)$.*

Before proving theorem, we first prove the following lemmas on which the proof of theorem is based.

LEMMA 1. *If T is a continuous affine self-mapping on a compact convex subset K of a locally convex space X, then*

(a) *for any $\varepsilon>0$, there exists an integer N such that $\varepsilon(K-K)=x_i-Tx_i$ for all x_0 in K and $i\geqq N$, where x_i is defined for each positive integer i,*

$$x_i=(1-\lambda)x_{i-1}+\lambda Tx_{i-1}\,,\quad (0<\lambda<1)\,,$$

(b) *a point of accumulation of $\{x_i\}_{i=0}^{\infty}$ is a fixed point of T.*

Proof. (a) Let I denote an identity mapping on K, then we have

$$\begin{aligned} x_i - Tx_i &= ((1-\lambda)I + \lambda T)^i x_0 - T((1-\lambda)I + \lambda T)^i x_0 \\ &= \sum_{h=1}^{i+1} ({}_iC_h(1-\lambda)^{i-h}\lambda^h - {}_iC_{h-1}(1-\lambda)^{i-h+1}\lambda^{h-1})T^h x_0 , \end{aligned}$$

where

$${}_iC_{-1} = {}_iC_{i+1} = 0 .$$

Put

$$L_h(i) = {}_iC_h(1-\lambda)^{i-h}\lambda^h - {}_iC_{h-1}(1-\lambda)^{i-h+1}\lambda^{h-1} \qquad \text{for} \quad 0 \leqq h \leqq i+1 .$$

It is clear that $L_h(i) \geqq 0$ if $0 \leqq h \leqq h_0$, and $L_h(i) < 0$ if $h_0 < h \leqq i+1$, where h_0 is an integer satisfying $h_0 \leqq (i+1)\lambda < h_0 + 1$. A simple calculation shows that

$$\sum_{h=1}^{h_0} L_h(i) = \sum_{h=h_0+1}^{i+1} |L_h(i)| = {}_iC_{h_0}(1-\lambda)^{i-h_0}\lambda^{h_0} .$$

Put $S(i) = {}_iC_{h_0}(1-\lambda)^{i-h_0}\lambda^{h_0}$. We have, then, by Stiring's formula that

$$\lim_{i\to\infty} S(i) = 0 . \tag{1}$$

Since K is convex, we see

$$\begin{aligned} x_i - Tx_i &= \sum_{h=1}^{i+1} L_h(i) T^h x_0 \\ &= S(i) \sum_{h=0}^{h_0} (L_h(i)/S(i)) T^h x_0 \\ &\quad - S(i) \sum_{h=h_0+1}^{i+1} (|L_h(i)|/S(i)) T^h x_0 \\ &\in S(i)(K-K) . \end{aligned} \tag{2}$$

From this and (1), (a) follows.

(b) Let p be a point of accumulation of $\{x_i\}_{i=0}^{\infty}$. Then there exists a subsequence $\{x_{i(k)}\}_{k=0}^{\infty}$ which converges to p. Since T is continuous, for any convex neighborhood U of 0 in X, we can choose an integer N_1 such that

$$p - x_{i(k)} \in U/3 \quad \text{and} \quad Tx_{i(k)} - Tp \in U/3 \tag{3}$$

for all $k \geqq N_1$. Since $K - K$ is compact, because of (a), we can take an integer N_2 such that $S(i(k))(K-K) \subset U/3$ for all $k \geqq N_2$. From this and (3), it follows that, if $k \geqq \max\{N_1, N_2\}$,

$$\begin{aligned} p - Tp = (p - x_{i(k)}) + (x_{i(k)} - Tx_{i(k)} + (Tx_{i(k)} - Tp) &\in (U/3) \\ + (U/3) + (U/3) &= U , \end{aligned}$$

which implies that p is a fixed point of T.

LEMMA 2. *Under the same assumption of Lemma* 1, *for any convex neighborhood* U *of* 0, *there exists a number* N *such that for any* $i \geqq N$, $z_i \in F(T)$ *can be chosen such that* $x_i - z_i \in U$ *for any* x *in* K, *where* x_i *is the one defined in Lemma* 1 (a).

Proof. Since K is compact and T is continuous, for any convex neighborhood U of 0, we can take a convex neighborhood V of 0 such that $\{x + U\} \cap F(T) \neq \emptyset$ for any x in K such that $x - Tx$ in V. If we take a number N such that $S(i)(K-K) \subset V$ for all $i \geqq N$, it is clear from (2) that, for any $i \geqq N$, $x_i - Tx_i$ belongs to V for all x in K. This implies tnat, for any $i \geqq N$, z_i can be chosen in $\{x_i + U\} \cap F(T)$ for all x in K.

Proof of Theorem. Without loss of generality, we can take $J = 2$. Put $\alpha_1 = \lambda$ and $\alpha_2 = 1 - \lambda$. It is clear that $F(T_1) \cap F(T_2) \subset F(\lambda T_1 + (1 - \lambda)T_2)$. Hence we shall show that $F(T_1) \cap F(T_2) \supset F(\lambda T_1 + (1 - \lambda)T_2)$. Take any point p in $F(\lambda T_1 + (1 - \lambda)T_2)$, which is nonempty by Lemma 1 (b). Set $A = \lambda T_1 + (1 - \lambda)I$ and $B = (1 - \lambda)T_2 + \lambda I$. Then we have

$$(4) \qquad p = \left(\frac{A + B}{2}\right)p = \left(\frac{A + B}{2}\right)^i p \qquad \text{for all} \quad i\,.$$

By Lemma 2, for any convex neighborhood U of 0, there exists a number N satisfying that, we can take $z_i \in F(T_1)$ such that $A^i B^i p - z_i \in U/2$, for all $i \geqq N$, and if $0 \leqq i \leqq N$, we define $z_i = z_N$. Put $w_n = \sum_{i=0}^{n} 2^{-n}{}_nC_i z_i$. Since T_1 is affine, w_n belongs to $F(T_1)$. By the commutativity of T_1 and T_2, we see

$$\begin{aligned}\left(\frac{A + B}{2}\right)^n p - w_n &= \sum_{i=0}^{n} 2^{-n}{}_nC_i(A^i B^{n-i} p - z_i) \\ &= \sum_{i=0}^{n} 2^{-n}{}_nC_i(A^i B^{n-i} p - z_i) \\ &= \sum_{i=0}^{N-1} 2^{-n}{}_nC_i(A^i B^{n-i} p - z_i) + \sum_{i=1}^{n} 2^{-n}{}_nC_i(A^i B^{n-i} p - z_i) \\ &\in \Big(\sum_{i=0}^{N-1} 2^{-n}{}_nC_i\Big)(K - K) + \Big(\sum_{i=N}^{n} 2^{-n}{}_nC_i\Big)U/2\,.\end{aligned}$$

If we take n such that $(\sum_{i=0}^{N-1} 2^{-n}{}_nC_i)(K - K) \subset U/2$, this implies, by (4), that

$$p - w_n = \left(\frac{A + B}{2}\right)^n p - w_n \in U\,.$$

Since $w_n \in F(T_1)$, it follows that p belongs to $F(T_1)$. In the same way, we see that p belongs to $F(T_2)$. Therefore $F(T_1) \cap F(T_2) \supset F(\lambda T_1 + (1 - \lambda)T_2)$. This completes the proof of theorem.

From the finite intersection property, we have the following corollary.

COROLLARY (Markov-Kakutani). *Let K be a compact convex subset of a locally convex space. Let F be a commuting family of continuous affine self-mappings on K. Then there exists a point p in K such that $Tp = p$ for each T in F.*

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