BULL. AUSTRAL. MATH. SOC. 54H25, 54E50
VOL. 10 (1974), 365-370.

On common fixed points of mappings

Kiyoshi Iséki

The object of this paper is to study common fixed points of mappings of a complete metric space into itself. The results obtained are generalizations of Ray Theorems.

Recently Ray [2] and Wong [4] proved some interesting theorems about common fixed points of mappings of a complete metric space in itself. In this note, we shall prove some theorems about common fixed points which are generalizations of results in Ray [2].

THEOREM 1. *Let* X *be a complete metric space,* T_n $(n = 1, 2, \ldots)$ *a sequence of mappings of* X *into itself. Suppose that there are non-negative numbers* α, β, γ *such that for* $x, y \in X$,

$$\rho\left(T_i(x), T_j(y)\right) \le \alpha\left(\rho\left(x, T_i(x)\right)+\rho\left(y, T_j(y)\right)\right) + \beta\left(\rho\left(x, T_j(y)\right)+\rho\left(y, T_i(x)\right)\right) + \gamma\rho(x, y) ,$$

where $2\alpha + 2\beta + \gamma < 1$. *Then the sequence of mappings* $\{T_n\}$ *has a unique common fixed point.*

Proof. Let $x_0 \in X$. Put

$$x_n = T_n\left(x_{n-1}\right) , \quad (n = 1, 2, \ldots) ;$$

then we have

Received 29 January 1974.

https://doi.org/10.1017/S0004972700041058 Published online by Cambridge University Press

$$\begin{aligned}\rho(x_1, x_2) &= \rho(T_1(x_0), T_2(x_1)) \\ &\le \alpha(\rho(x_0, x_1)+\rho(x_1, x_2)) + \beta(\rho(x_0, x_2)+\rho(x_1, x_1)) + \gamma\rho(x_0, x_1) \\ &= (\alpha+\gamma)\rho(x_0, x_1) + \alpha\rho(x_1, x_2) + \beta\rho(x_0, x_2) \\ &\le (\alpha+\gamma)\rho(x_0, x_1) + \alpha\rho(x_1, x_2) + \beta(\rho(x_0, x_1)+\rho(x_1, x_2)) .\end{aligned}$$

Hence

$$\rho(x_1, x_2) \le \frac{\alpha+\beta+\gamma}{1-\alpha-\beta} \rho(x_0, x_1) .$$

Similarly we have

$$\begin{aligned}\rho(x_2, x_3) &= \rho(T(x_1), T(x_2)) \\ &\le (\alpha+\gamma)\rho(x_1, x_2) + \alpha\rho(x_2, x_3) + \beta(\rho(x_1, x_2)+\rho(x_2, x_3)) .\end{aligned}$$

Therefore, we have

$$\rho(x_2, x_3) \le \frac{\alpha+\beta+\gamma}{1-\alpha-\beta} \rho(x_1, x_2) .$$

In general, we have

$$\rho(x_n, x_{n+1}) \le \left(\frac{\alpha+\beta+\gamma}{1-\alpha-\beta}\right)^n \rho(x_0, x_1) .$$

This means that the sequence $\{x_n\}$ is a Cauchy sequence. Hence, by the completeness of X, $\{x_n\}$ converges to some point x in X. For the point x,

$$\begin{aligned}\rho(x, T_n(x)) &\le \rho(x, x_{m+1}) + \rho(x_{m+1}, T_n(x)) \\ &= \rho(x, x_{m+1}) + \rho(T_{m+1}(x_m), T_n(x)) \\ &\le \rho(x, x_{m+1}) + \alpha(\rho(x_m, T_{m+1}(x_m))+\rho(x, T_n(x))) \\ &\qquad + \beta(\rho(x_m, T_n(x))+\rho(x, T_{m+1}(x_m))) + \gamma\rho(x_m, x) \\ &= \rho(x, x_{m+1}) + \alpha(\rho(x_m, x_{m+1})+\rho(x, T_n(x))) \\ &\qquad + \beta(\rho(x_m, T_n(x))+\rho(x, x_{m+1})) + \gamma\rho(x_m, x) .\end{aligned}$$

Letting $m \to \infty$, then we have

$$\rho(x, T_n(x)) \le (\alpha+\beta)\rho(x, T_n(x)) .$$

Therefore $\rho(x, T_n(x)) = 0$; that is, the point x is a common fixed

https://doi.org/10.1017/S0004972700041058 Published online by Cambridge University Press

point of all T_n .

To show that x is a unique common fixed point of all T_n , we consider a point y in X such that $T_n(y) = y$ for every n . Then we have

$$\rho(x, y) = \rho\big(T_n(x), T_n(y)\big)$$
$$\le \alpha\big(\rho(x, T_n(x))+\rho(y, T_n(y))\big) + \beta\big(\rho(x, T_n(y))+\rho(y, T_n(x))\big)$$
$$+ \gamma\rho(x, y) = (2\beta+\gamma)\rho(x, y) .$$

Hence $\rho(x, y) = 0$; that is, $x = y$. This completes the proof of Theorem 1.

THEOREM 2. *Let $\{T_n\}$ be a sequence of mappings of a complete metric space X into itself. Let x_n be a fixed point of T_n $(n = 1, 2, \ldots)$, and suppose that T_n converges uniformly to T_0 . If T_0 satisfies the condition*

(1) $$\rho\big(T_0(x), T_0(y)\big) \le \alpha\big(\rho(x, T_0(x))+\rho(y, T_0(y))\big)$$
$$+ \beta\big(\rho(x, T_0(y))+\rho(y, T_0(x))\big) + \gamma\rho(x, y) ,$$

where α, β, γ are non-negative and $2\alpha + 2\beta + \gamma < 1$, then $\{x_n\}$ converges to the fixed point x_0 of T_0 .

Under condition (1), T_0 has a unique fixed point by a result of Ćirić [1] (quoted from Rus [3], p. 21).

Proof. Let $\varepsilon > 0$ be given; then there is a natural number N such that

(2) $$\rho\big(T_n(x), T_0(x)\big) < \varepsilon$$

for all $x \in X$ and $N \le n$. Hence

https://doi.org/10.1017/S0004972700041058 Published online by Cambridge University Press

$$\begin{aligned}\rho(x_n, x_0) &= \rho(T_n(x_n), T_0(x_0)) \\ &\le \rho(T_n(x_n), T_0(x_n)) + \rho(T_0(x_n), T_0(x_0)) \\ &\le \rho(T_n(x_n), T_0(x_n)) + \alpha(\rho(x_n, T_0(x_n))+\rho(x_0, T_0(x_0))) \\ &\qquad + \beta(\rho(x_n, T_0(x_0))+\rho(x_0, T_0(x_n))) + \gamma\rho(x_n, x_0) \\ &\le \rho(T_n(x_n), T_0(x_n)) + (\alpha+\beta)\rho(T_n(x_n), T_0(x_n)) \\ &\qquad + (\alpha+\beta)(\rho(x_0, x_n)+\rho(T_n(x_n), T_0(x_n))) + \gamma\rho(x_n, x_0) \\ &= (1+2(\alpha+\beta))\rho(T_n(x_n), T_0(x_n)) + ((\alpha+\beta)+\gamma)\rho(x_n, x_0) .\end{aligned}$$

Hence

$$(1-(\alpha+\beta+\gamma))\rho(x_n, x_0) \le (1+2(\alpha+\beta))\rho(T_n(x_n), T_0(x_n)) .$$

From the hypotheses, $2(\alpha+\beta) + \gamma < 1$. Hence, for $n \ge N$, we have

$$\rho(x_n, x_0) \le \frac{1+2(\alpha+\beta)}{1-(\alpha+\beta+\gamma)} \varepsilon ,$$

which shows that $\{x_n\}$ converges to x_0 . We complete the proof.

THEOREM 3. *Let* T_n $(n = 1, 2, \dots)$ *be a sequence of mappings with fixed point* x_n *of a metric space* X *into itself. Suppose that*

$$(3) \quad \rho(T_n(x), T_n(y)) \le \alpha(\rho(x, T_n(x))+\rho(y, T_n(y))) + \beta(\rho(x, T_n(y))+\rho(y, T_n(x))) + \gamma\rho(x, y) ,$$

where α, β, γ *are non-negative and* $2\alpha + 2\beta + \gamma < 1$. *If* $\{T_n\}$ *converges to a mapping* T_0 *, and* x_0 *is an accumulation point of* $\{x_n\}$ *, then* x_0 *is a fixed point of* T_0 .

Proof. Since x_0 is an accumulation point of the set $\{x_n\}$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to x_0 :

$$\rho(x_0, T_0(x_0)) \le \rho\left(x_0, T_{n_i}(x_{n_i})\right) + \rho\left(T_{n_i}(x_{n_i}), T_{n_i}(x_0)\right) + \rho\left(T_{n_i}(x_0), T_0(x_0)\right) .$$

Let $\varepsilon > 0$; then there is a natural number N such that

https://doi.org/10.1017/S0004972700041058 Published online by Cambridge University Press

$$\rho\left(x_0, x_{n_i}\right) < \varepsilon ,$$

$$\rho\left(T_{n_i}(x_0), T_0(x_0)\right) < \varepsilon ,$$

for $N \le n_i$. Hence for $N \le n_i$, we have

(4) $$\rho(x_0, T_0(x_0)) < 2\varepsilon + \rho\left(T_{n_i}\left(x_{n_i}\right), T_{n_i}(x_0)\right) .$$

To estimate $\rho\left(T_{n_i}\left(x_{n_i}\right), T_{n_i}(x_0)\right)$, we use the condition (3). Then

$$\rho\left(T_{n_i}\left(x_{n_i}\right), T_{n_i}(x_0)\right) \le \alpha\left(\rho\left(x_{n_i}, T_{n_i}\left(x_{n_i}\right)\right)+\rho\left(x_0, T_{n_i}(x_0)\right)\right)$$
$$+ \beta\left(\rho\left(x_0, T_{n_i}\left(x_{n_i}\right)\right)+\rho\left(x_{n_i}, T_{n_i}(x_0)\right)\right) + \gamma\rho\left(x_{n_i}, x_0\right) .$$

For $N \le n_i$, we have

$$\rho\left(T_{n_i}\left(x_{n_i}\right), T_{n_i}(x_0)\right) \le \alpha\rho\left(x_0, T_{n_i}(x_0)\right) + (\beta+\gamma)\varepsilon + \beta\rho\left(x_{n_i}, T_{n_i}(x_0)\right) .$$

Hence

(5) $$(1-\beta)\rho\left(x_{n_i}, T_{n_i}(x_0)\right) \le \alpha\rho\left(x_0, T_{n_i}(x_0)\right) + (\beta+\gamma)\varepsilon .$$

Next consider $\rho\left(x_0, T_{n_i}(x_0)\right)$; then

$$\rho\left(x_0, T_{n_i}(x_0)\right) \le \rho\left(x_0, x_{n_i}\right) + \rho\left(x_{n_i}, T_{n_i}(x_0)\right) .$$

For $N \le n_i$, we have

(6) $$\rho\left(x_0, T_{n_i}(x_0)\right) \le \varepsilon + \rho\left(x_{n_i}, T_{n_i}(x_0)\right) .$$

(5) and (6) imply

$$(1-\beta)\rho\left(x_0, T_{n_i}(x_0)\right) \le (1-\beta)\varepsilon + \alpha\rho\left(x_0, T_{n_i}(x_0)\right) + (\beta+\gamma)\varepsilon .$$

Hence

$$\rho\left(x_0,\ T_{n_i}(x_0)\right) \le \frac{1+\gamma}{1-\alpha-\beta}\varepsilon . \tag{7}$$

From (4), (5) and (7), we have

$$\rho(x_0,\ T_0(x_0)) \le 2\varepsilon + \frac{1}{1-\beta}\left\{\alpha\rho\left(x_0,\ T_{n_i}(x_0)\right)+(\beta+\gamma)\varepsilon\right\}$$
$$\le \left[2+\frac{1}{1-\beta}\left\{\frac{\alpha(1+\gamma)}{1-\alpha-\beta}+(\beta+\gamma)\right\}\right]\varepsilon .$$

This shows that x_0 is a fixed point of T_0. We complete the proof.

References

[1] Ljubomir B. Ćirić, "Generalized contractions and fixed-point theorems", *Publ. Inst. Math. Beograd, N.S.* 12 (26) (1971), 19-26.

[2] Baidyanath Ray, "Some results on fixed points and their continuity", *Colloq. Math.* 27 (1973), 41-48.

[3] Ioan A. Rus, *Teoria punctului fix.* II *Teoria punctului fix in analiza functională* (Universitatea „Babeş-Bolyai" din Cluj, Cluj, 1973).

[4] Chi Song Wong, "Common fixed points of two mappings", *Pacific J. Math.* 48 (1973), 299-312.

Department of Mathematics,
Kobe University,
Rokko, Kobe,
Japan.