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ON COMMON FIXED POINTS OF WEAKLY COMPATIBLE MAPPINGS SATISFYING 'GENERALIZED CONDITION (B)'

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Abstract

We prove the existence of common fixed points for two weakly compatible mappings satisfying a 'generalized condition (B)'. This result generalizes some theorems of Al-Thagafi and Shahzad [2] and Babu, Sandhya and Kameswari [3].

1 Introduction and preliminaries

In 1968, Kannan [10] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. Sessa [11] coined the term weakly commuting maps. Jungck [8] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps [9]

We now introduce almost contraction property to a pair of selfmaps as follows:

Definition 1.1. Let (X, d) be a metric space. A map $T : X \to X$ is called an *almost contraction* with respect to a mapping $f : X \to X$ if there exist a constant $\delta \in]0, 1[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta\ d(fx, fy) + L\ d(fy, Tx),$$

for all $x, y \in X$.

If we choose $f = I_X$, I_X is the identity map on X, we obtain the definition of *almost contraction,* the concept introduced by Berinde ([5], [6]).

This concept was introduced by Berinde as 'weak contraction' in [5]. But in [6], Berinde renamed 'weak contraction' as 'almost contraction' which is appropriate.

Berinde [5] proved the following two fixed point theorems for *almost contractions* in complete metric spaces.

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Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction. Then*

(1) $F(T) = \{x \in X : Tx = x\} \neq \emptyset$,

(2) for any $x_0 \in X$, the Picard iteration

$$x_{n+1} = Tx_n \ , n = 0, 1, 2, \cdots \tag{1.1}$$

converges to some $x^* \in F(T)$,

(3) the following estimates

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1) \text{ and } d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n)$$

hold, for $n = 1, 2, \cdots$.

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction for which there exist $\theta \in]0, 1[$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta d(x, y) + L_1 d(x, Tx) \quad \text{for all } x, y \in X.$$

Then

(1) T has a unique fixed point, i.e., $F(T) = \{x^*\}$,

(2) for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.1) converges to some $x^* \in F(T)$,

(3) a priori and posteriori error estimates

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1) \text{ and } d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n)$$

hold, for $n = 1, 2, \cdots$,

(4) the rate of convergence of Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.1) is given by

$$d(x_n, x^*) \leq \theta d(x_{n-1}, x^*)$$

for $n = 0, 1, 2, \cdots$.

It was shown in [5] that any strict contraction, the Kannan [10] and Zamfirescu [12] mappings, as well as a large class of quasi-contractions, are all almost contractions.

Let f and T be two selfmaps of a metric space (X, d). T is said to be f-*contraction* if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(fx, fy)$ for all $x, y \in E$.

In 2006, Al-Thagafi and Shahzad [2] proved the following theorem which is a generalization of many known results.

Theorem 1.4. (Al-Thagafi and Shahzad [2], Theorem 2.1). *Let E be a subset of a metric space (X,d), and f and T be selfmaps of E and $T(E) \subseteq f(E)$. Suppose that f and T are weakly compatible, T is f-contraction and $T(E)$ is complete. Then f and T have a unique common fixed point in E.*

Recently Babu, Sandhya and Kameswari [3] considered the class of mappings that satisfy 'condition (B)'.

Let (X,d) be a metric space. A map $T : X \to X$ is said to satisfy *'condition (B)'* if there exist a constant $\delta \in]0,1[$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for all $x, y \in X$.

They proved the following fixed point theorem.

Theorem 1.5. (Babu, Sandhya and Kameswari [3], Theorem 2.3). *Let (X,d) be a complete metric space and $T : X \to X$ be a map satisfying condition* (B). *Then T has a unique fixed point.*

Definition 1.6. A pair (f,T) of self-mappings on X is said to be *weakly compatible* if f and T commute at their coincidence point (i.e. $fTx = Tfx,\ x \in X$ whenever $fx = Tx$). A point $y \in X$ is called a *point of coincidence* of two self-mappings f and T on X if there exists a point $x \in X$such that $y = Tx = fx$.

The following lemma is Proposition 1.4 of [1].

Lemma 1.7. *Let X be a non-empty set and the mappings $f, T : X \to X$ have a unique point of coincidence v in X. If the pair (f,T) is weakly compatible, then f and T have a unique common fixed point.*

Definition 1.8. Let (X, d) be a metric space, f and T be self-mappings on X, with $T(X) \subset f(X)$, and $x_0 \in X$. Choose a point x_1 in X such that $fx_1 = Tx_0$. This can be done since $T(X) \subset f(X)$. Continuing this process having chosen $x_1, \cdots, x_k$, we choose x_{k+1} in X such that

$$fx_{k+1} = Tx_k, \quad k = 0, 1, 2, \cdots .$$

The sequence $\{fx_n\}$ is called a *T-sequence* with initial point x_0.

We now introduce a generalization of 'condition (B)' for a pair of self maps.

Definition 1.9. A selfmap T on a metric space X is said to satisfy '*generalized condition* (B)' associated with a selfmap f of X if there exists $\delta \in]0,1[$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta M(x,y) + L\min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\} \quad (1.2)$$

for all $x, y \in X$, where

$$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty) + d(fy,Tx)}{2}\}.$$

If $f = I_X$, then we say that T satisfies 'generalized condition (B)'.

Here we observe that 'condition (B)' implies 'generalized condition (B)'. But its converse need not be true.

Example 1.10. Let $X = \{0, \frac{1}{2}, 1\}$ with the usual metric. We define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{2} & if\ x \in \{0, \frac{1}{2}\}, \\ 0 & if\ x = 1. \end{cases}$$

Then T satisfies generalized condition (B) with $\delta = \frac{1}{2}$ and $L = 0$. But T does not satisfy condition (B), for by taking $x = \frac{1}{2}$ and $y = 1$; condition (B) fails to hold for any $\delta \in]0, 1[$ and any $L \geq 0$.

Recently, Berinde established the following fixed point result.

Theorem 1.11. (Berinde [6], Theorem 3.4). *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exist $\alpha \in]0, 1[$ and some $L \geq 0$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \alpha M(x, y) + L\ \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then

(1) T has a unique fixed point, i.e., $F(T) = \{x^*\}$;

(2) for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.1) covnerges to some $x^* \in F(T)$

(3) the priori estimate

$$d(x_n, x^*) \leq \frac{\alpha^n}{(1-\alpha)^2} d(x_0, x_1)$$

holds, for $n = 1, 2, \cdots,$

(4) the rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \leq \theta\ d(x_{n-1}, x^*)$$

for $n = 0, 1, 2, \cdots$.

In this paper, we prove a result on the existence of points of coincidence for two maps satisfying generalized condition (B). We apply this result to obtain common fixed points of two weakly compatible selfmaps, which is the main result of this paper (Theorem 2.2). Our result generalizes some theorems of Al-Thagafi and Shahzad [2] and Babu, Sandhya and Kameswari [3].

2 Common fixed point theorems

First, we establish a result on the existence of points of coincidence and then we apply this result to obtain common fixed points for two self mappings of weakly compatible maps.

Theorem 2.1. *Let (X, d) be a metric space. Let $f, T : X \to X$ be such that $T(X) \subseteq f(X)$. Assume that T satisfies generalized condition (B) associated with f. If either $f(X)$ or $T(X)$ is a complete subspace of X, then f and T have a unique point of coincidence.*

Proof. Let x_0 be an arbitrary point in X and let $\{fx_n\}$ be a $T-$sequence with initial point x_0. Now,

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{d(fx_n, fx_{n-1}), d(fx_n, Tx_n), d(fx_{n-1}, Tx_{n-1}), \\ &\qquad \frac{d(fx_n, Tx_{n-1}) + d(fx_{n-1}, Tx_n)}{2}\} \\ &= \max\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n), \\ &\qquad \frac{d(fx_n, fx_n) + d(fx_{n-1}, fx_{n+1})}{2}\} \\ &= \max\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1})}{2}\}. \end{aligned}$$

Thus by taking x_n for x and x_{n-1} for y in the inequality (1.2), it follows that

$$\begin{aligned} d(Tx_n, Tx_{n-1}) \leq \delta \max\{&d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), \\ &\frac{d(fx_{n-1}, fx_{n+1})}{2}\} + L \min\{d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n), \\ &d(fx_n, fx_n), d(fx_{n-1}, fx_{n+1})\} \end{aligned}$$

which further gives that

$$d(fx_n, fx_{n+1}) \leq \delta \max\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1})}{2}\}.$$

Now if $\max\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1})}{2}\} = d(fx_n, fx_{n-1})$, then

$$d(fx_n, fx_{n+1}) \leq \delta d(fx_{n-1}, fx_n).$$

If $\max\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1})}{2}\} = d(fx_n, fx_{n+1})$, then

$$d(fx_n, fx_{n+1}) \leq \delta d(fx_n, fx_{n+1})$$

which implies that $d(fx_n, fx_{n+1}) = 0$ and hence $fx_n = fx_{n+1} = Tx_n$ and the result follows.

Finally, $\max\{d(fx_n, fx_{n-1}), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1})}{2}\} = \frac{d(fx_{n-1}, fx_{n+1})}{2}$ gives that

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq \frac{\delta}{2} d(fx_{n-1}, fx_{n+1}) \\ &\leq \frac{\delta}{2} d(fx_{n-1}, fx_n) + \frac{\delta}{2} d(fx_n, fx_{n+1}) \end{aligned}$$

which implies that

$$d(fx_n, fx_{n+1}) \leq \frac{\delta}{2-\delta} d(fx_{n-1}, fx_n).$$

So

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq \delta d(fx_{n-1}, fx_n) \\ &\leq \ldots \leq \delta^n d(fx_0, fx_1). \end{aligned}$$

Now, for any positive integers m and n with $m > n$, we have

$$\begin{aligned} d(fx_m, fx_n) &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \cdots + d(fx_{m-1}, fx_m) \\ &\leq [\delta^n + \delta^{n+1} + \cdots + \delta^{m-1})] d(fx_0, fx_1) \\ &\leq \frac{\delta^n}{1-\delta} d(fx_0, fx_1), \end{aligned}$$

which implies that $\{fx_n\}$ is a Cauchy sequence. If $f(X)$ is a complete subspace of X, there exists a $p \in f(X)$ such that $fx_n \to p$. Hence we can find u^* in X such that $fu^* = p$. Now,

$$\begin{aligned} d(p, Tu^*) &\leq d(p, fx_{n+1}) + d(fx_{n+1}, Tu^*) \\ &= d(p, fx_{n+1}) + d(Tx_n, Tu^*) \\ &\leq d(p, fx_{n+1}) + \delta \max\{d(fx_n, fu^*), d(fx_n, Tx_n), d(fu^*, Tu^*), \\ &\quad \frac{d(fx_n, Tu^*) + d(fu^*, Tx_n)}{2}\} \\ &\quad + L \min\{d(fx_n, fx_{n+1}), d(fu^*, Tu^*), d(fx_n, Tu^*), d(fu^*, fx_{n+1})\} \end{aligned}$$

which on taking limit as $n \to \infty$ gives that

$$\begin{aligned} d(p, Tu^*) &\leq \delta \max\{d(p,p), d(p,p), d(p, Tu^*), \frac{d(p, Tu^*) + d(p,p)}{2}\} \\ &\quad + L\ \min\{d(p,p), d(p, Tu^*), d(p, Tu^*), d(p,p)\} \end{aligned}$$

which further implies

$$d(p, Tu^*) \leq \delta d(p, Tu^*).$$

Hence $d(p, Tu^*) = 0$ and $fu^* = p = Tu^*$.

Now, if $T(X)$ is complete, then there exists a $q \in T(X)$ such that $Tx_n \to q$ as $n \to \infty$. Since $T(X) \subset f(X)$, we have $q \in f(X)$ and $fx_n \to q$ as $n \to \infty$. Now from the above discussion, q is a point of coincidence.

Uniqueness of point of coincidence:

Assume that there exist points $p,\ p^*$ in X such that $p = fu = Tu$ and $p^* = fu^* = Tu^*$, for some u, u^* in X. Now

$$\begin{aligned} M(u,u^*) &= \max\{d(fu,fu^*), d(fu,Tu), d(fu^*,Tu^*), \frac{d(fu,Tu^*)+d(fu^*,Tu)}{2}\} \\ &= \max\{d(fu,fu^*), d(fu,fu), d(fu^*,fu^*), \frac{d(fu,fu^*)+d(fu^*,fu)}{2}\} \\ &= d(fu,fu^*) \end{aligned}$$

and from the inequality (1.2) we have

$$\begin{aligned} d(p,p^*) &= d(Tu,Tu^*) \\ &\leqslant \delta d(fu,fu^*) + L\min\{d(fu,Tu), d(fu^*,Tu^*), d(fu,Tu^*), d(fu^*,Tu)\} \\ &= \delta d(fu,fu^*) + L\min\{d(fu,fu), d(fu^*,fu^*), d(fu,fu^*)\}. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} d(p,p^*) &\le \delta\ d(fu,fu^*) \\ &= \delta\ d(p,p^*), \end{aligned}$$

we deduce that $p = p^*$. □

Theorem 2.2. *Let (X,d) be a metric space. Let $f,T : X \to X$ be such that $T(X) \subseteq f(X)$. Assume that T satisfies generalized condition (B) associated with f. If either $f(X)$ or $T(X)$ is a complete subspace of X, then f and T have a unique common fixed point in X provided that the pair (f,T) is weakly compatible.*

Proof. By Theorem 2.1, f and T have a unique point of coincidence. Since the pair (f,T) is weakly compatible, by Lemma 1.4, f and T have a unique common fixed point. □

Corollary 2.3. *Let (X,d) be a metric space. Let $f,T : X \to X$ be such that $T(X) \subseteq f(X)$. Assume that there exist $\delta \in]0,1[$ and $L \ge 0$ such that*

$$d(Tx,Ty) \le \delta m(x,y) + L\min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\} \quad (2.2)$$

for all $x,y \in X$, where

$$m(x,y) = \max\{d(fx,fy), \frac{1}{2}[d(fx,Tx)+d(fy,Ty)], \frac{1}{2}[d(fy,Tx)+d(fx,Ty)]\}.$$

If either $f(X)$ or $T(X)$ is a complete subspace of X, then f and T have a point of coincidence. Moreover, f and T have a unique common fixed point provided that the pair (f,T) is weakly compatible.

Proof. As the inequality (2.2) is a special case of (1.2), the result follows from Theorem 2.2. □

The following example is in support of Theorem 2.2.

Example 2.4. Let $X = [0, 1)$ with usual metric. Define $T, f : X \to X$ as

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x < \frac{2}{3} \\ \frac{2}{3} & \text{if } \frac{2}{3} \le x < 1 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} \frac{5}{6} & \text{if } 0 \le x < \frac{2}{3} \\ \frac{4}{3} - x & \text{if } \frac{2}{3} \le x < 1. \end{cases}$$

We observe that $T(X) \subset f(X)$ and the pair (f, T) is weakly compatible on X. Also, f and T satisfy the inequality (1.2) with $\delta = \frac{1}{2}$ and $L = 0$. Hence f and T satisfy all hypotheses of Theorem 2.2 and $\frac{2}{3}$ is the unique common fixed point of f and T.

But, when $x \in [0, \frac{2}{3})$ and $y = \frac{2}{3}$, we have $d(Tx, Ty) = \frac{1}{6}$; and $d(fx, fy) = \frac{1}{6}$ so that for any $\alpha \in [0, 1)$, T fails to be an f-contraction. Hence Theorem 1.4 is not applicable.

This example shows that Theorem 2.2 is a generalization of Theorem 1.4.

By choosing $f = I_X$ in Theorem 2.2, we have the following corollary.

Corollary 2.5. *Let (X, d) be a metric space. Let $T : X \to X$ satisfies generalized condition (B). If $T(X)$ is a complete subspace of X, then T has a unique fixed point.*

Remark 2.6. Theorem 1.5 follows as a corollary to Corollary 2.5. In fact, Example 1.10 shows that Corollary 2.5 is a generalization of Theorem 1.5.

Now, we have the following result on the continuity in the set of common fixed points. Let $F(f, T)$ denote the set of all common fixed points of f and T.

Theorem 2.7. *Let $(X,\ d)$ be a metric space. Assume that $T : X \to X$ satisfies generalized condition (B) associated with a selfmap f on X. If $F(T, f) \ne \emptyset$, then T is continuous at $p \in F(f, T)$ whenever f is continuous at p.*

Proof. $p \in F(f, T)$. Let $\{z_n\}$ be any sequence in X converging to p. Then by taking $y := z_n$ and $x := p$ in (1.2), we get

$$\begin{aligned} d(Tp, Tz_n) \quad \le \quad & \delta M(p, z_n) + L \min\{d(fp, Tp), d(fz_n, Tz_n), \\ & d(fp, Tz_n), d(fz_n, Tp)\}, \ n = 1, 2, \cdots \end{aligned}$$

where

$$M(p, z_n) = \max\{d(fp, fz_n), d(fp, Tp), d(fz_n, Tz_n), \frac{d(fp, Tz_n) + d(fz_n, Tp)}{2}\}$$

which, in view of $Tp = fp$, is equivalent to

$$d(Tp, Tz_n) \le \delta \ \max\{d(Tp, fz_n), d(fz_n, Tz_n), \frac{d(Tp, Tz_n) + d(fz_n, Tp)}{2}\},$$

$n = 1, 2, \cdots$. Now, by letting $n \to \infty$ we get $Tz_n \to Tp$ as $n \to \infty$ whenever f is continuous at p and $0 < \delta < 1$ □

3 Discussion

Following the similar arguments to those given in the proof of Theorem 2.2, we can prove the following theorem.

Theorem 3.1. *Let (X, d) be a metric space. Let $f, T : X \to X$ be such that $T(X) \subseteq f(X)$. Assume that there exist a constant $\delta \in \,]0, \frac{1}{2}[$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta\ m(x, y) + L \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\} \quad (3.1)$$

for all $x, y \in X$, where

$$m(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}.$$

If either $f(X)$ or $T(X)$ is a complete subspace of X, then f and T have a unique point of coincidence. Moreover, f and T have a unique common fixed point provided that the pair (f, T) is weakly compatible.

Now the following question is natural:
Open problem 1. Is Theorem 3.1 valid for $\frac{1}{2} \leq \delta < 1$?

If this open problem is solved affirmatively, then Theorem 3.1 together with the solution of open problem 1 extends Theorem 1.11 (Theorem 3.4 of Berinde [6]) to a pair of selfmaps.

Berinde [6] introduced the concept of Ćirić almost contraction, that is, a mapping for which there exist a constant $\alpha \in [0, 1[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \alpha M(x, y) + Ld(y, Tx), \text{ for all } x, y \in X,$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Berinde proved the following two fixed point theorems for this class of mappings in complete metric spaces.

Theorem 3.2. (Berinde [6], Theorem 3.2). *Let (X, d) be a complete metric space and $T : X \to X$ a Ćirić almost contraction. Then*

(1) $F(T) = \{x \in X : Tx = x\} \neq \emptyset$,

(2) for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=1}^{\infty}$ defined by (1.1) converges to some $x^* \in F(T)$,

(3) the following estimate

$$d(x_n, x^*) \leq \frac{\alpha^n}{(1 - \alpha)^2} d(x_0, x_1)$$

holds, for $n = 1, 2, \cdots$.

Theorem 3.3. (Berinde [6], Theorem 3.3). *Let (X, d) be a complete metric space and $T : X \to X$ a Ćirić almost contraction. If there exist $\theta \in]0, 1[$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta d(x, y) + L_1 d(x, Tx), \quad \text{for all } x, y \in X.$$

Then

(1) T has a unique fixed point, i.e., $F(T) = \{x^*\}$,

(2) for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=1}^{\infty}$ defined by (1.1) converges to some $x^* \in F(T)$,

(3) the a priori error estimate (3) of Theorem 3.2 holds,

(4) the rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \leq \theta d(x_{n-1}, x^*)$$

for $n = 0, 1, 2, \cdots$.

But the following example shows that for a pair of selfmaps f and T of a complete metric space X, even if T is an almost contraction with respect to f and both f and T are continuous on X, the maps f and T may not have a common fixed point in X.

Example 3.4. Let $X = R$, R be the real line with the usual metric. We define mappings $f, T : X \to X$ by $Tx = \frac{x+1}{4}$ and $fx = \frac{x}{2}$, $x \in X$.

Then, with $\delta = \frac{1}{2}$ and for any $L \geq 0$, T is an almost contraction with respect to f. But f and T have no common fixed points.

Thus the following question is possible:
Open problem 2. Under what additional assumptions, *either* on f and T *or* on the domain of f and T, the maps f and T have common fixed points?

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