response of the frequency estimation loop and simplified its design. The estimates were unbiased and ripple-free when the signal contained no noise and the parameters of the signal were constant.

A modified version of the algorithm provided improvements for situations in which the fundamental component of the signal could become small, or vanish for some periods of time. In this case, information from all components of the signal was used in the fundamental frequency estimation. Multiple signals with the same fundamental frequency were also combined to yield consistent estimation results despite changes in signal characteristics. In consequence, an advantage of the modified algorithm over the basic algorithm is that it is not necessary to know a priori which component is the most suitable to base the frequency estimation on. The algorithms were designed with real-time tracking applications in mind. They were simple in design and implementation, and effective in tracking time-varying parameters. The linear time-invariant approximations gave useful information about the dynamic behavior of the system, the tradeoff between convergence speed and noise sensitivity, and the selection of the design parameters.

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On Common Quadratic Lyapunov Functions for Pairs of Stable LTI Systems Whose System Matrices Are in Companion Form

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***Abstract*—In this note, the problem of determining necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for a pair of stable linear time-invariant systems whose system matrices A_1 and A_2 are in companion form is considered. It is shown that a necessary and sufficient condition for the existence of such a function is that the matrix product A_1A_2 does not have an eigenvalue that is real and negative. Examples are presented to illustrate the result.**

***Index Terms*—Quadratic stability, stability theory, switched linear systems.**

I. INTRODUCTION

In this note, we consider the problem of determining necessary and sufficient conditions for the existence of a positive–definite real symmetric matrix $P = P^T > 0, P \in \mathbb{R}^{n\times n}$ that simultaneously satisfies the matrix inequalities

$$A_1^T P + PA_1 < 0 \quad A_2^T P + PA_2 < 0 \tag{1}$$

where the matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ are Hurwitz (all of their eigenvalues are in the open-left half of the complex plane). When both inequalities are satisfied for some $P = P^T > 0$, then the scalar function $V(x) = x^T Px$ is said to be a *strong* common quadratic Lyapunov function (CQLF) for the dynamic systems $\Sigma_{A_1} : \dot{x} = A_1 x$ and $\Sigma_{A_2} : \dot{x} = A_2 x$. While a general analytic solution to this problem has yet to be obtained [1] (a number of numerical solutions have been obtained [2]), significant progress has been made for special classes of A_i matrices. In this note the case where A_1 and A_2 are in companion form is considered. It is shown that

1) a necessary and sufficient condition for the existence of a CQLF for the associated linear time-invariant (LTI) systems is that the matrix product A_1A_2 does not have any real negative eigenvalues;
2) the aformentioned condition may be interpreted as a time-domain formulation of the circle criterion.

The implications of the result for several problems in stability theory are then discussed.

Manuscript received July 25, 2002; revised November 7, 2002. Recommended by Associate Editor T. Iwasaki. This work was supported in part by the European Union funded research training network *Multi-Agent Control* under Grant HPRN-CT-1999-00107, in part by the Enterprise Ireland Grant SC/2000/084/Y, and in part by SFI under Grant 00/PI.1/C067. This work is the sole responsibility of the authors and does not reflect the opinion of the European Union.

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Digital Object Identifier 10.1109/TAC.2003.809795

II. Mathematical Preliminaries

The following results are useful in deriving the main result of this note. Lemma 2.1 is a well known result from linear algebra and Lemma 2.2 is a concise formulation of a result used in [3].

Lemma 2.1 [4]: Let $A \in \mathbb{R}^{n \times p}$, $B \in \mathbb{R}^{p \times n}$, and let I_n denote the identity matrix of dimension $n \times n$. Then

$$\det[I_n - AB] = \det[I_p - BA]. \tag{2}$$

Lemma 2.2 [3]: Let $A \in \mathbb{R}^{n \times n}$ be a companion matrix, and let $h, g \in \mathbb{R}^{n \times 1}$ such that $A - gh^T$ is also in companion form; namely $g^T = [0, \ldots, 1]$. Then, the numerator and denominator polynomials of the rational function

$$1 + Re\left\{h^T (j\omega I_n - A)^{-1} g\right\} = \frac{\Gamma(-\omega^2)}{|M(j\omega)|^2}$$

are given by

$$\begin{aligned} |M(j\omega)|^2 &= \det\left[\omega^2 I_n + A^2\right] \\ \Gamma(-\omega^2) &= \left(1 - h^T A\left(\omega^2 I_n + A^2\right)^{-1} g\right) \\ &\quad \times \det\left[\omega^2 I_n + A^2\right]. \end{aligned} \tag{3}$$

.When the matrix A is Hurwitz, $\det[\omega^2 I_n + A^2] = \det[A]\det[\omega^2 A^{-1} + A] \neq 0 \, \forall \omega \in \mathbb{R}$.

Comment: A proof of Lemma 2.2 is given in the Appendix.

III. Main Result

Theorem 3.1: A necessary and sufficient condition for the existence of a common quadratic Lyapunov function for the LTI systems, $\Sigma_{A_1} : \dot{x} = A_1 x$, and $\Sigma_{A_2} : \dot{x} = A_2 x$, with $A_2 = A_1 - gh^T$, $A_1 \in \mathbb{R}^{n \times n}$, $h, g \in \mathbb{R}^{n \times 1}$, where A_1 and A_2 are companion matrices, is that the matrices A_1 and A_2 are Hurwitz, and that the matrix product $A_1 A_2$ does not have any negative real eigenvalues.

Proof: The circle criterion [5], [6] provides a necessary and sufficient condition for the existence of a common quadratic Lyapunov function for Σ_{A_1} and Σ_{A_2}; namely

$$1 + Re\left(h^T (j\omega I_n - A_1)^{-1} g\right) > 0 \qquad \forall \omega \in \mathbb{R}. \tag{4}$$

It is shown in this note that the condition that $A_1 A_2$ has no real negative eigenvalues is both necessary and sufficient for (4).

Necessity: From Lemma 2.2, the rational function (4) can be written as

$$1 + Re\left\{h^T (j\omega I_n - A_1)^{-1} g\right\} = \frac{\Gamma(-\omega^2)}{|M(j\omega)|^2}$$

where the denominator polynomial is strictly nonzero for all real ω and where the numerator polynomial is given by

$$\Gamma(-\omega^2) = \left(1 - h^T A_1 \left(\omega^2 I_n + A_1^2\right)^{-1} g\right) \det\left[\omega^2 I_n + A_1^2\right] > 0.$$

By applying Lemma 2.1, $\Gamma(-\omega^2)$ can be written

$$\begin{aligned} \Gamma(-\omega^2) &= \det\left[I_n - \left(\omega^2 I_n + A_1^2\right)^{-1} gh^T A_1\right] \det\left[\omega^2 I_n + A_1^2\right] \\ &= \det\left[\left(\omega^2 I_n + A_1^2\right)^{-1} \left(\omega^2 I_n + A_1^2 - gh^T A_1\right)\right] \\ &\quad \times \det\left[\omega^2 I_n + A_1^2\right] \\ &= \det\left[\omega^2 I_n + \left(A_1 - gh^T\right) A_1\right] \\ &= \det\left[\omega^2 I_n + A_2 A_1\right] > 0. \end{aligned}$$

Now, suppose that $A_2 A_1$ has a negative real eigenvalue. This implies that $\det[\omega^2 I_n + A_2 A_1] = 0$ for some $\omega^2 > 0$. Hence, it follows from the fact that the eigenvalues of $A_1 A_2$ and $A_2 A_1$ are identical, that a necessary condition for (4) is that the matrix product $A_1 A_2$ does not have a negative real eigenvalue.

Sufficiency: Let $A_1 A_2$ (and hence $A_2 A_1$) have no negative real eigenvalues. Then, the polynomial $\det[\omega^2 I_n + A_2 A_1]$ is nonzero for $\omega \in \mathbb{R}$, or equivalently has no real roots for $\omega^2 \geq 0$. Hence, since both A_1 and A_2 are Hurwitz matrices

$$\det\left[\omega^2 I_n + A_2 A_1\right] > 0, \qquad \omega \in \mathbb{R}.$$

Since the matrix A_1 is Hurwitz, $\det(\omega^2 I_n + A_1^2) \neq 0$ for all $\omega^2 \geq 0$. Therefore, it follows that

$$\begin{aligned} &\det\left[\left(\omega^2 I_n + A_1^2\right)^{-1}\right] \\ &\times \det\left[\omega^2 I_n + A_2 A_1\right] \det\left[\omega^2 I_n + A_1^2\right] > 0 \\ &\Rightarrow \det\left[\left(\omega^2 I_n + A_1^2\right)^{-1}\right] \\ &\times \det\left[\omega^2 I_n + \left(A_1 - gh^T\right) A_1\right] \det\left[\omega^2 I_n + A_1^2\right] > 0 \\ &\Rightarrow \det\left[\left(\omega^2 I_n + A_1^2\right)^{-1}\right] \\ &\times \det\left[\omega^2 I_n + A_1^2 - gh^T A_1\right] \det\left[\omega^2 I_n + A_1^2\right] > 0 \\ &\Rightarrow \det\left[I_n - \left(\omega^2 I_n + A_1^2\right)^{-1} gh^T A_1\right] \\ &\times \det\left[\omega^2 I_n + A_1^2\right] > 0. \end{aligned}$$

By applying Lemma 2.1, it follows that

$$\begin{aligned} &\left(1 - h^T A_1 \left(\omega^2 I_n + A_1^2\right)^{-1} g\right) \det\left[\omega^2 I_n + A_1^2\right] > 0 \\ &\Rightarrow \Gamma(-\omega^2) > 0. \quad \mathbf{Q.E.D.} \end{aligned}$$

Comment: The circle criterion provides a convenient frequency domain test for the existence of a CQLF for systems of the form $\dot{x} = (A_1 - k(t)gh^T)x$ where $k_1 \leq k(t) \leq k_2$. Theorem 3.1 may be interpreted as providing a time-domain formulation of the circle criterion with $k_1 = 0$ and $k_2 = 1$.

Comment: Megretski [7, problem 30(1)]) notes the importance of relating the multiplication operation in the time domain and the frequency domain. This problem is also considered in [8]. Theorem 3.1 may provide insight into this relationship.

Comment: Theorem 3.1 provides a coordinate free condition for the existence of a CQLF; the matrices A_1 and A_2 need only be simultaneously similar to companion matrices [9].

IV. Implications of Main Result

The implications of the main result are given in 1)–3), as shown later. Before proceeding, we note the following results.

Lemma 4.1 [10], [11]: Consider the LTI systems

$$\begin{aligned} \Sigma_A &: \dot{x} = Ax \\ \Sigma_{A^{-1}} &: \dot{x} = A^{-1}x \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$ is Hurwitz. Then, any quadratic Lyapunov function for Σ_A is also a quadratic Lyapunov function for $\Sigma_{A^{-1}}$.

Lemma 4.2 [12]: Let $V(x)$ be a strong CQLF for the stable LTI systems Σ_{A_1}, Σ_{A_2}. From Lemma 4.1, $V(x)$ is a Lyapunov function for the LTI systems $\Sigma_{A_1 + \lambda A_2}$, $\Sigma_{A_1 + \lambda A_2^{-1}}$, $\lambda \geq 0$. It follows that the matrix pencils $A_1 + \lambda A_2$ and $A_1 + \lambda A_2^{-1}$ are both Hurwitz for all $\lambda \in [0, \infty)$. Thus, the nonsingularity of these two pencils is a necessary condition for the existence of a CQLF for the systems Σ_{A_1}, Σ_{A_2}.

1) Necessity of the circle criterion for the existence of a CQLF: While sufficiency of the circle criterion for the existence of a CQLF for Σ_{A_1} and Σ_{A_2} was shown in [5] (using direct arguments from Lyapunov stability theory), necessity was first established by [6] using an indirect argument. Necessity follows immediately from Lemma 4.2 as follows. Let (4) be false. Then, from the proof of Theorem 3.1, $\det[\omega^2 I_n + A_2 A_1] = \det[A_2]\det[\omega^2 A_2^{-1} + A_1] = 0$ for some $\omega \in \mathbb{R}$. Since A_2 is Hurwitz, it follows that $\det[\omega^2 A_2^{-1} + A_1] = 0$ for some $\omega \in \mathbb{R}$. Hence, the pencil $\omega^2 A_2^{-1} + A_1$ is not Hurwitz for some $\omega^2 > 0$. From Lemma 4.2, a CQLF cannot exist for Σ_{A_1} and Σ_{A_2} and (4) is both necessary and sufficient for the existence of a CQLF.
2) Pencils of matrices: The existence of a CQLF for Σ_{A_1} and Σ_{A_2}, namely that $A_1 A_2$ has no negative eigenvalues, is sufficient to guarantee the Hurwitz stability of each of the following pencils:

$$A_1 + \lambda A_2,\ A_1^{-1} + \lambda A_2,\ A_1^{-1} + \lambda A_2^{-1},\ A_1 + \lambda A_2^{-1}$$

for all $\lambda \geq 0$. It consequently follows from item 1) that the condition that the product $A_1 A_2$ has no negative real eigenvalues is both necessary and sufficient for the simultaneous Hurwitz stability of the above matrix pencils. We also note that the existence of a CQLF implies the Hurwitz stability of all convex combinations of the previous matrices and their inverses.
3) The stability of switching systems: The existence of a strong CQLF for Σ_{A_1} and Σ_{A_2} is sufficient to guarantee the exponential stability of the switching system

$$\dot{x} = A(t)x,\ A(t) \in \{A_1, A_2\}. \tag{5}$$

Hence, when A_1, A_2 are Hurwitz and are simultaneously similar to companion matrices, a sufficient condition for the exponential stability of (5) is that the matrix product $A_1 A_2$ does not have any negative real eigenvalues. It follows from Lemma 4.1 that this condition is also sufficient for the exponential stability of

$$\dot{x} = A(t)x,\ A(t) \in \{A_1^{-1}, A_2\}. \tag{6}$$

The following theorem provides insights into the system (6).

Theorem 4.1 [13], [14]: Let $A_i \in \mathbb{R}^{n\times n}$, $i = \{1, 2\}$, be general Hurwitz matrices. A sufficient condition for the existence of an unstable switching sequence for the system

$$\dot{x} = A(t)x,\ A(t) \in \{A_1, A_2\}$$

is that the matrix pencil $A_1 + \lambda A_2$ has an eigenvalue with a positive real part for some positive λ.

Suppose now that a CQLF does not exist; the matrix $A_1 A_2$ has at least one negative real eigenvalue, and the pencil $A_1^{-1} + \lambda A_2$ is singular, and hence not Hurwitz, for some positive λ. It follows from Theorem 4.1 that an unstable, or a marginally stable, switching sequence exists for (6). Hence, for systems of this form, a necessary and sufficient condition for exponential stability for arbitrary switching sequences is that a CQLF exists for $\Sigma_{A_1^{-1}}$ and Σ_{A_2} (the matrix pencil $A_1 A_2$ has no negative real eigenvalues).

V. Examples

In this section, we present two examples to illustrate the main features of our result.

Example 1 (No CQLF): Consider the dynamic systems Σ_{A_1} and Σ_{A_2} with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -1 \end{bmatrix}.$$

The matrix product $A_1 A_2$ is given by

$$A_1 A_2 = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & -1 \\ 6 & 8 & 1 \end{bmatrix}.$$

A CQLF cannot exist as the eigenvalues of $A_1 A_2$ are given by $\lambda_i = \{1, -2, -1\}$, $i \in \{1, 2, 3\}$.

Example 2 (CQLF exists): Consider the dynamic systems Σ_{A_1} and Σ_{A_2} with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -3 \end{bmatrix}.$$

By Theorem 1, a CQLF exists as the eigenvalues of $A_1 A_2$ are given by $\lambda_i = \{6.6264, -0.3132 + 0.2298i, -0.3132 - 0.2298i\}$, $i \in \{1, 2, 3\}$. Using the MATLAB LMI toolbox, the following matrix is obtained:

$$P = \begin{bmatrix} 0.5570 & 0.4236 & 0.8062 \\ 0.4236 & 1.1359 & 0.9203 \\ 0.8062 & 0.9203 & 2.3521 \end{bmatrix}.$$

P is positive definite and satisfies the simultaneous Lyapunov inequalities (1).

VI. Concluding Remarks

In this note, necessary and sufficient conditions for the existence of a CQLF for a pair of dynamic systems whose system matrices are in companion form are derived. These conditions may be viewed as a time-domain formulation of the circle criterion and provide a computationally simple test for verifying the quadratic stability of a class of switched linear systems.

Appendix

Proof of Lemma 2.2: We use the following representation of the transfer function $G(s) = h^T(sI_n - A_1)^{-1}g$ (see [4, Sec. A.13])

$$G(s) = h^T(sI_n - A_1)^{-1}g = \frac{\det\left[sI_n - A_1 + gh^T\right] - \det[sI_n - A_1]}{\det[sI_n - A_1]}.$$

Hence, $1 + Re(h^T(j\omega I_n - A_1)^{-1}g)$, can be written as shown in the first group of equations at the top of the next page. By assumption, the matrices A_1 and $A_1 + gh^T$ are companion matrices. Hence, the matrix gh^T has nonzero elements only in the last row. It follows that the second group of equations shown at the top of the next page holds. Hence, from Lemma 2.1, the last group of equations shown at the top of the next page holds. Q.E.D

Acknowledgment

The first author would like to thank Dr. F. O'Cairbre, Mr. O. Mason, and especially Dr. P. Curran for this contributions to this work. Neither the European Union or Enterprise Ireland is responsible for any use of data appearing in this publication.

$$1 + Re\left(h^T(j\omega I_n - A_1)^{-1}g\right) = 1 + Re\left(\frac{\det\left[j\omega I_n - A_1 + gh^T\right] - \det[j\omega I_n - A_1]}{\det[j\omega I_n - A_1]}\right)$$
$$= Re\left(1 + \frac{\left(\det\left[j\omega I_n - A_1 + gh^T\right] - \det[j\omega I_n - A_1]\right)\det[-j\omega I_n - A_1]}{\det[j\omega I_n - A_1]\det[-j\omega I_n - A_1]}\right)$$
$$= Re\left(\frac{\det\left[\omega^2 I_n + A_1^2 - gh^T A_1 - j\omega gh^T\right]}{\det\left[\omega^2 I_n + A_1^2\right]}\right).$$

$$Re\left(\frac{\det\left[\omega^2 I_n + A_1^2 - gh^T A_1 - j\omega gh^T\right]}{\det\left[\omega^2 I_n + A_1^2\right]}\right) = \frac{\det\left[\omega^2 I_n + A_1^2 - gh^T A_1\right]}{\det\left[\omega^2 I_n + A_1^2\right]}$$
$$= \frac{\det\left[\omega^2 I_n + A_1^2\right]\det\left[I_n - \left(\omega^2 I_n + A_1^2\right)^{-1} gh^T A_1\right]}{\det\left[\omega^2 I_n + A_1^2\right]}.$$

$$Re\left(\frac{\det\left[\omega^2 I_n + A_1^2 - gh^T A_1 - j\omega gh^T\right]}{\det\left(\omega^2 I_n + A_1^2\right)}\right) = \frac{\det\left[\omega^2 I_n + A_1^2\right]\left(1 - h^T A_1\left(\omega^2 I_n + A_1^2\right)^{-1} g\right)}{\det\left[\omega^2 I_n + A_1^2\right]}$$
$$= \frac{\Gamma(-\omega^2)}{|M(j\omega)|^2}.$$

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