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ON COMMUTATIVE POWER-ASSOCIATIVE NILALGEBRAS OF LOW DIMENSION

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ABSTRACT. Commutative power-associative nilalgebras of dimension 4 and characteristic $\neq 2$ are shown to be nilpotent and all their isomorphism classes are determined.

The long-standing conjecture, originally due to A. A. Albert, that a commutative power-associative nilalgebra of finite dimension over a field is nilpotent, has recently been disproved by Suttles [4], who gave a counterexample of dimension 5. This dimension is generally the best possible, for we show here that every commutative power-associative nilalgebra of dimension 4 over a field of characteristic $\neq 2$ is nilpotent, and we determine the isomorphism classes of all such algebras. The proof is elementary and cannot be significantly simplified by the use of the general results of [1] and [2], which require, moreover, additional assumptions about the characteristic.

Throughout, A will denote a commutative power-associative nilalgebra of dimension 4 over a field F of characteristic $\neq 2$. The subspace of A generated by elements $u, v, w, \cdots$ will be denoted by $(u, v, w, \cdots)$. Since every $x \in A$ is nilpotent, the powers of x are linearly independent, so we must have $x^n = 0$ for $n \geq 5$; the least n such that $x^n = 0$ for all $x \in A$ is called the *nilindex*. Now the product of any two elements of A can be written as a linear combination of squares, for $xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2]$. Therefore, if the nilindex is 2, then every product vanishes. If it is 5, then $A = (x, x^2, x^3, x^4)$ for any x with $x^4 \neq 0$. These are trivial cases, in each of which there is, up to isomorphism, a unique, associative, algebra. Only the cases of nilindex 3 and 4 are of interest.

1. **Nilindex 3.** Linearizing the identity $(x^2)x = 0$ yields $2[(xy)z + (yz)x + (zx)y] = 0$ for all $x, y, z \in A$. Therefore, denoting right

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multiplication by x by R_x we have, for all $y, z \in A$,

(1) $$R_yR_z + R_zR_y = -R_{yz}.$$

Setting $y = z = x$ gives

(2) $$R_x^2 = -2R_{x^2}.$$

Setting $y = x$, $z = x^2$ in (1), and noting that $x^3 = 0$, gives $R_xR_{x^2} + R_{x^2}R_x = 0$, which with (2) implies that $R_x^3 = 0$. Now choose any $x \in A$ with $x^2 \neq 0$, and set $X = (x, x^2)$. This is carried into itself by R_x, which therefore operates on the two-dimensional quotient A/X. As R_x is nilpotent, we have $R_x^2(A/X) = 0$, so $(yx)x \in X$ for all $y \in A$, i.e., $(yx)x = \alpha x + \beta x^2$ for some α, $\beta \in F$. Since $R_x^3 = 0$, multiplying by x shows that $\alpha = 0$, after which using (2) and the fact that R_y is also nilpotent shows $\beta = 0$ also. Thus $yx^2 = 0$, and since every product is a linear combination of squares, this shows that the product of any three elements of A is zero. In particular, A is associative, so we have

Theorem 1. *A commutative power-associative nilalgebra A of nilindex 3 and of dimension 4 over a field F of characteristic $\neq 2$ is associative, and $A^3 = 0$.*

These algebras being associative, their classification is well known; cf. Kruse and Price [3, Chapter VI]: If $\dim A^2 = 1$, then one defines a symmetric bilinear form on the 3-dimensional space A/A^2 by choosing any x with $x^2 \neq 0$ and defining the product of the cosets of $u, v \in A$ to be α whenever $uv = \alpha x^2$; with respect to this form the length of x itself is clearly 1. The problem of classifying these algebras up to isomorphism is identical with that of classifying such forms with the additional condition that there be a vector of length 1. Unfortunately, this problem is completely solved only for certain special fields, e.g., the real and complex numbers. If $\dim A^2 = 2$, the only other possibility, then one chooses $x, y \in A$ such that x^2 and xy span A^2. Subtracting, if necessary, a multiple of x from y one can, moreover, so choose y such that $y^2 = \alpha x^2$ for some $\alpha \in F$. It is easy to check that if one takes any other x and y with these properties, then α is replaced by αc^2 for some $c \neq 0$. Therefore, denoting the multiplicative group of F by F^*, these algebras are parameterized by the elements of $F^*/(F^*)^2$ and 0.

2. Nilindex 4. Choose any x with $x^3 \neq 0$ and set $X = (x, x^2, x^3)$. We claim $(x^2, x^3) = A^2$. It is sufficient to show that $y^2 \in (x^2, x^3)$ for all y, and we may further suppose $y \notin X$ and $y^2 \neq 0$, else the matter is trivial. Set $Y = (y, y^2, y^3)$. Then $X \cap Y$ is a proper subalgebra of X, hence must be contained in (x^2, x^3), and is a subalgebra of Y of dimension equal to dim $Y - 1$ (since dim $X = 3$), and therefore must contain y^2; thus $y^2 \in (x^2, x^3)$ as asserted. It follows that $A^2A^2 = 0$. Now y being arbitrary, we have $yx^2 \in A^2$, hence $yx^2 = ax^2 + bx^3$ for some $a, b \in F$. We claim $a = 0$. Otherwise, setting $z = (1/a)(y - bx)$, we have $zx^2 = x^2$; computing $[(z + x^2)^2(z + x^2)] \cdot (z + x^2)$, which must vanish, we find, using the fact that $A^2A^2 = 0$, that it is $2x^2$, a contradiction. If now we replace x by $x + x^2$, thereby replacing x^2 by $x^2 + 2x^3$ but leaving x^3 unchanged, it follows that yx^3 is also a multiple of x^3. In fact, $yx^3 = 0$, for if $yx^3 = dx^3$, then computing $[(y + x^3)^2(y + x^3)] \cdot (y + x^3)$, which must vanish, one gets $2d^3x^3$, so $d = 0$. We see now that replacing the original y by $y - bx$, for which we have $(y - bx)x^2 = 0$, one can so choose y such that $y \notin X$ and $yx^2 = yx^3 = 0$, so $yA^2 = 0$. The product of any four of the elements x, x^2, x^3, y vanishes, and as these span A, it follows that the product of any 4 elements of A vanishes, so $A^4 = 0$. Therefore, we have

Theorem 2. *If A is a commutative power-associative nilalgebra of nilindex 4 and of dimension 4 over a field F of characteristic $\neq 2$, then $A^4 = 0$ and there is $y \notin A^2$ such that $yA^2 = 0$.*

The y of the theorem is not unique, but as dim $A/A^2 = 2$, there cannot be, modulo A^2, two independent elements both annihilating A^2, so y^2 is determined up to multiplication by an element of $(F^*)^2$. As before, x will denote an element of A such that $x^3 \neq 0$; then clearly $y \notin (x, x^2, x^3)$ so $A = (x, x^2, x^3, y)$. We have the following possibilities:

1. We can so choose x and y such that $yx = 0$. If $y^2 = 0$ then A is unique, the direct sum of (y) and (x, x^2, x^3), and is associative. If $y^2 = \beta x^3$ with $\beta \neq 0$, setting $y' = y/\beta$ and $x' = x/\beta$ gives $y'^2 = x'^3$. This unique algebra is also associative. If $y^2 = \alpha x^2 + \beta x^3$ with $\alpha \neq 0$, replacing x by $x + (\beta/2\alpha)x^2$ shows we may assume $\beta = 0$. Clearly α is determined at most up to multiplication by an element of $(F^*)^2$, and it is easy to check that another choice of x and y replaces α by αc^2 for some $c \neq 0$, so we have a family of algebras, all nonassociative, parameterized by the elements of $F^*/(F^*)^2$.

2. We cannot so choose x and y such that $yx = 0$. Therefore we cannot have $yx = \beta x^3$ since $x(y - \beta x^2) = 0$. Choosing any x with $x^3 \neq 0$, we may assume that $yx = \gamma x^2 + \delta x^3$ with $\gamma \neq 0$. Replacing x by $\gamma x + (\delta/2)x^2$ shows we

may so choose x such that $yx = x^2$. If $y^2 = 0$ we have a unique nonassociative algebra. If $y^2 = \beta x^3$ with $\beta \neq 0$, replacing x by x/β and y by y/β shows we may assume $\beta = 1$ and we have a unique algebra, which is not associative. Finally, if $y^2 = \alpha x^2 + \beta x^3$ with $\alpha \neq 0$, then as $y(y - \alpha x) = \beta x^3$, we must have $(y - \alpha x)^3 = 0$ (else we would replace x by $y - \alpha x$); the left side is $\alpha^2(1 - \alpha)x^3$, so $\alpha = 1$. Replacing x by $x + (\beta/2)x^2$ and y by $y + \beta x^2$, shows we may assume $\beta = 0$, so we have a unique algebra given by $y^2 = yx = x^2$ and, as always, $yx^2 = yx^3 = 0$. It is not associative. This ends the classification. We have found one family of algebras parameterized by $F^*/(F^*)^2$, and 5 individual algebras of which precisely 2 are associative.

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