

## ON COMPACT COMPLEX PARALLELISABLE SOLVMANIFOLDS

Dedicated to the memory of Taira Honda

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(Received December 4, 1974)

### 1. Introduction

This paper deals with compact complex solvmanifolds. Our main purpose is to generalize the theory on the divisor group of a complex torus to these manifolds. By a solvmanifold we mean a homogeneous space of solvable Lie group. Let  $G$  be a simply connected complex solvable Lie group and  $\Gamma$  be a lattice of  $G$ , that is, a discrete subgroup of  $G$  such that  $G/\Gamma$  is compact. The de Rham cohomology group and the Dolbeault cohomology group of a compact complex manifold  $G/\Gamma$  play an important role in studying the divisor group of a complex manifold  $G/\Gamma$ . The de Rham cohomology group of a compact solvmanifold  $G/\Gamma$  has been discussed by Matsushima [7], Nomizu [10] and Mostow [8].

Let  $M$  be a compact connected complex manifold and  $H_{\mathbb{Z}}^{p,q}(M)$  denote the Dolbeault cohomology group of  $M$  of type  $(p, q)$ . Let  $\mathfrak{g}$  be a complex Lie algebra and  $I$  be the canonical complex structure of  $\mathfrak{g}$ . Then  $\mathfrak{g}^c = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ , where  $\mathfrak{g}^{\pm} = \{X \in \mathfrak{g}^c \mid IX = \pm \sqrt{-1}X\}$ . In section 2, we prove:

**Theorem 1.** *Let  $G$  be a simply connected complex nilpotent Lie group and  $\Gamma$  be a lattice of  $G$ . Then there is a canonical isomorphism*

$$H_{\mathbb{Z}}^{p,q}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \Lambda^p(\mathfrak{g}^+)^*$$

where  $H^q(\mathfrak{g}^-)$  denotes the Lie algebra cohomology group of  $\mathfrak{g}^-$  and  $(\mathfrak{g}^+)^*$  denotes the dual vector space of  $\mathfrak{g}^+$ .

Let  $G$  be a simply connected complex solvable Lie group and  $\Gamma$  be a lattice of  $G$  which has the following property:

(M)  *$Ad(G)$  and  $Ad(\Gamma)$  have the same Zariski closure in the group  $Aut(\mathfrak{g}^c)$ .*

This condition has been used by Mostow in his study of lattices of solvable

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1) This work was presented to the Graduate School of the University of Notre Dame in partial fulfillment of the requirement for the Ph. D. degree.

Lie group [8]. Denote by  $[G, G]$  the commutator group of  $G$  and let  $\pi: G \rightarrow G/[G, G]$  be the projection. Then  $\Gamma \cap [G, G]$  is a lattice of  $[G, G]$ , so that  $\pi(\Gamma)$  is a lattice of  $G/[G, G]$  and  $(G/\Gamma, \pi, (G/[G, G])/\pi(\Gamma), [G, G]/(\Gamma \cap [G, G]))$  is a holomorphic fiber bundle. Let  $T$  denote the complex torus  $(G/[G, G])/\pi(\Gamma)$ . In section 3, we study Chern classes of holomorphic line bundles over these compact complex solvmanifolds.

Let  $M$  and  $N$  be complex manifolds and  $\phi: M \rightarrow N$  be a surjective holomorphic map. For a divisor  $\tilde{D}$  on  $N$  let  $\phi^*(\tilde{D})$  denote the divisor on  $M$  defined by  $\phi_x^{-1}(\tilde{D}_{\phi(x)})$  for all  $x \in M$ . We call the divisor  $\phi^*(\tilde{D})$  on  $M$  the pull back of the divisor  $\tilde{D}$  on  $N$  [15]. In section 4, we prove:

**Theorem 2.** *Let  $G$  be a simply connected complex solvable Lie group and  $\Gamma$  be a lattice of  $G$ . Assume that  $\Gamma$  satisfies the condition (M) and that  $H_{q,1}^{0,1}(G/\Gamma) \cong H^1(\mathfrak{g}^-)$  canonically. Then, under the notation introduced above, for each positive divisor  $D$  on  $G/\Gamma$ , there exists a positive divisor  $\tilde{D}$  on the complex torus  $T$  such that the divisor  $D$  is the pull back of the divisor  $\tilde{D}$  on  $T$  by the projection  $\pi: G/\Gamma \rightarrow T$ , i.e.,  $D = \pi^*\tilde{D}$ .*

Note that our assumption in Theorem 2 is always satisfied if  $G$  is a simply connected complex nilpotent Lie group and  $\Gamma$  is a lattice of  $G$ .

If  $M$  is a compact connected complex manifold,  $K(M)$  will denote the field of all meromorphic functions on  $M$ .

**Corollary.** *Under the condition of Theorem 2, there is a canonical isomorphism*

$$\pi^*: K(T) \cong K(G/\Gamma).$$

*In particular, the transcendence degree of  $K(G/\Gamma)$  over  $\mathbb{C}$  is not larger than the complex dimension of the complex torus  $T$ .*

The author would like to express his deep appreciation to Professor Yozo Matsushima for his thoughtful guidance and encouragement given during the completion of this paper.

## 2. Dolbeault cohomology groups of compact complex nilmanifolds

Let  $M$  be a complex manifold and  $H_{p,q}^{\mathfrak{g}}(M)$  denote the Dolbeault cohomology of  $M$  of type  $(p, q)$ . Let  $G$  be a simply connected complex Lie group and  $\Gamma$  be a uniform lattice of  $G$ . Let  $\mathfrak{g}$  denote the Lie algebra of all right invariant vector fields on  $G$ ,  $I$  denote the complex structure of  $\mathfrak{g}$  and  $\mathfrak{g}^+$  (resp.  $\mathfrak{g}^-$ ) denote the vector space of the  $\sqrt{-1}$  (resp.  $-\sqrt{-1}$ ) eigenvectors of  $I$  in the complexification  $\mathfrak{g}^{\mathbb{C}}$  of  $\mathfrak{g}$ . We identify  $\mathfrak{g}^+$  to the Lie algebra of all right invariant holomorphic vector fields on  $G$  and the dual space  $(\mathfrak{g}^+)^*$  to the space of all right invariant holomorphic 1-forms on  $G$ . Moreover we may identify an element of

$\mathfrak{g}^+$  (resp.  $(\mathfrak{g}^+)^*$ ) to a holomorphic vector field (resp. a holomorphic 1-form) on  $G/\Gamma$ . Let  $\Lambda^p T^*(G/\Gamma)$  be the  $p$ -th exterior product bundle of the holomorphic cotangent bundle  $T^*(G/\Gamma)$  of  $G/\Gamma$ . Since  $G/\Gamma$  is a compact complex parallelisable manifold, the holomorphic vector bundle  $\Lambda^p T^*(G/\Gamma)$  on  $G/\Gamma$  is the trivial vector bundle  $G/\Gamma \times \Lambda^p(\mathfrak{g}^+)^*$ . Thus we have an isomorphism

$$(2.1) \quad H_{\mathfrak{g}^+}^{2,p}(G/\Gamma) \cong H_{\mathfrak{g}^+}^{0,p}(G/\Gamma) \otimes \Lambda^p(\mathfrak{g}^+)^* .$$

**Theorem 1.** *Let  $G$  be a simply connected complex nilpotent Lie group and  $\Gamma$  be a lattice of  $G$ . Then we have a canonical isomorphism*

$$H_{\mathfrak{g}^+}^{2,q}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \Lambda^p(\mathfrak{g}^+)^*$$

where  $H^q(\mathfrak{g}^-)$  denoted the  $q$ -th Lie algebra cohomology of with the trivial representation  $\rho_0: \mathfrak{g}^- \rightarrow \mathbb{C}$ .

We need some preparations to prove Theorem 1. Consider the descending central series  $\{C^k(G)\}$  of  $G$ , where  $C^k(G)=[G, C^{k-1}(G)]$  and  $C^0(G)=G$ . Since  $G$  is nilpotent, there is an integer  $m \in \mathbb{N}$  such that  $C^m(G) \neq (e)$  and  $C^{m+1}(G)=(e)$ . Let  $A$  denote the group  $C^m(G)$ . Then  $A$  is contained in the center  $Z(G)$  of  $G$ . Since  $G$  is a simply connected nilpotent Lie group and  $A$  is connected,  $A$  is a simply connected closed Lie subgroup. Let  $\Gamma$  be a lattice of  $G$ . Then  $A \cap \Gamma$  is a lattice of  $A$  ([11] p. 31 Corollary 1) and  $A\Gamma$  is closed in  $G$  ([11] p. 23 Theorem 1.13). Let  $\pi: G \rightarrow G/A$  be the canonical map. Then  $\pi(\Gamma)$  is a lattice of  $G/A$ . Since  $A/(A \cap \Gamma) \cong A\Gamma/\Gamma$  is a complex torus, we have a holomorphic principal fiber bundle  $(G/\Gamma, (G/A)/\pi(\Gamma), \pi, A/(A \cap \Gamma))$ .

Let  $C^\infty(G, \mathbb{C})$  be the vector space of all complex valued  $C^\infty$ -functions on  $G$ . Define the subspaces  $\underline{C}$  and  $\underline{C}'$  of  $C^\infty(G, \mathbb{C})$  by

$$\underline{C} = \{f \in C^\infty(G, \mathbb{C}) \mid f(g\gamma) = f(g) \quad \text{for all } \gamma \in \Gamma\}$$

and

$$\underline{C}' = \{f \in \underline{C} \mid f(ga) = f(g) \quad \text{for all } a \in A\} .$$

For a right invariant vector field  $X \in \mathfrak{g}$  and  $f \in C^\infty(G, \mathbb{C})$ , put

$$(Xf)(g) = \frac{d}{dt} f(a(t)g) \Big|_{t=0}$$

where  $a(t)$  is the one parameter subgroup corresponding to  $X$ . Then  $C^\infty(G, \mathbb{C})$  is a  $\mathfrak{g}$ -module, and hence  $\underline{C}$  and  $\underline{C}'$  are  $\mathfrak{g}^c$ -submodules of  $C^\infty(G, \mathbb{C})$ .

Let  $\mathfrak{a}$  be the Lie subalgebra of  $\mathfrak{g}$  corresponding to the complex Lie subgroup  $A$  of  $G$ . Then  $\mathfrak{a}^c$  has the decomposition  $\mathfrak{a}^c = \mathfrak{a}^+ \oplus \mathfrak{a}^-$  with respect to the complex structure  $I$ , and  $\underline{C}$  and  $\underline{C}'$  are  $\mathfrak{a}^-$ -modules. Let  $\{A^q(\mathfrak{a}^-, \underline{C}), d\}$  (resp.  $\{A^q(\mathfrak{a}^-, \underline{C}'), d\}$ ) denote the cochain complex of  $\mathfrak{a}^-$ -module  $\underline{C}$  (resp.  $\underline{C}'$ ) and  $H^*(\mathfrak{a}^-, \underline{C})$  (resp.  $H^*(\mathfrak{a}^-, \underline{C}')$ ) denote the Lie algebra cohomology of  $\mathfrak{a}^-$ -module

$\underline{\mathcal{C}}$  (resp.  $\underline{\mathcal{C}}'$ ). Since  $\mathfrak{a}^-$  is an ideal of  $\mathfrak{g}^-$ ,  $A^q(\mathfrak{a}^-, \underline{\mathcal{C}})$  (resp.  $A^q(\mathfrak{a}^-, \underline{\mathcal{C}}')$ ) is  $\mathfrak{g}^-$ -module by

$$(L_{\bar{X}}\omega(\bar{X}_1, \dots, \bar{X}_q) = \bar{X}(\omega(\bar{X}_1, \dots, \bar{X}_q)) - \sum_{j=1}^q \omega(\bar{X}_1, \dots, [\bar{X}, \bar{X}_j], \dots, \bar{X}_q)$$

where  $\bar{X} \in \mathfrak{g}^-$ ,  $\omega \in A^q(\mathfrak{a}^-, \underline{\mathcal{C}})$  (resp.  $\omega \in A^q(\mathfrak{a}^-, \underline{\mathcal{C}}')$ ) and  $\bar{X}_1, \dots, \bar{X}_q \in \mathfrak{a}^-$ . Moreover  $L_{\bar{X}} \circ d = d \circ L_{\bar{X}}$  for all  $\bar{X} \in \mathfrak{g}^-$ . Thus  $H^*(\mathfrak{a}^-, \underline{\mathcal{C}})$  and  $H^*(\mathfrak{a}^-, \underline{\mathcal{C}}')$  are  $\mathfrak{g}^-$ -modules.

**Proposition 2.1.** *The inclusion map  $\iota_0: \underline{\mathcal{C}}' \rightarrow \underline{\mathcal{C}}$  induces an isomorphism  $\iota_0^*$  of  $\mathfrak{g}^-$ -modules*

$$\iota_0^*: H^q(\mathfrak{a}^-, \underline{\mathcal{C}}') \rightarrow H^q(\mathfrak{a}^-, \underline{\mathcal{C}}).$$

This follows from Kodaira and Spencer [6] §2, but we shall give an elementary proof (cf. [11] VII §4).

Let  $\{X_1, \dots, X_l\}$  be a basis of  $\mathfrak{a}^+$  and  $\{\omega_1, \dots, \omega_l\}$  be the dual basis. We regard  $\omega_j$  ( $j=1, \dots, l$ ) as the holomorphic invariant 1-forms on the complex torus  $A/(A \cap \Gamma)$ . Define an invariant hermitian metric  $h$  on  $A/(A \cap \Gamma)$  by  $h = \sum_{j=1}^l \omega_j \cdot \bar{\omega}_j$ . Let  $\Omega$  be the associated form of type (1, 1). Then

$$\Omega = \sqrt{-1} \sum_{j=1}^l \omega_j \wedge \bar{\omega}_j,$$

and  $\frac{1}{l!} \Omega^l$  defines a Haar measure  $da$  on  $A/A \cap \Gamma$ . We may assume that  $\int_{A/A \cap \Gamma} \frac{1}{l!} \Omega^l = 1$  by changing the choice of a basis of  $\mathfrak{a}^+$  if necessary. For  $f \in \underline{\mathcal{C}}$  and  $x \in G$ , let  $f_x(a) = f(xa)$  for  $a \in A$ . Then we can define a  $\mathfrak{g}^c$ -module homomorphism  $H: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}'$  by

$$H(f)(x) = \int_{A/A \cap \Gamma} f_x(a) \frac{\Omega^l}{l!} = \int_{A/A \cap \Gamma} f(xa) da.$$

Let  $Y_j = \frac{1}{2}(X_j + \bar{X}_j)$  and  $Y_{j+l} = \frac{\sqrt{-1}}{2}(X_j - \bar{X}_j)$  for  $j=1, \dots, l$ . Then  $\{Y_1, \dots, Y_{2l}\}$  is a basis of  $\mathfrak{a}$ . Let  $\{\theta_1, \dots, \theta_{2l}\}$  be its dual basis. Let  $A^r(\mathfrak{a}, \underline{\mathcal{C}})$  denote the vector space of all  $\underline{\mathcal{C}}$ -valued  $r$ -forms on  $A/A \cap \Gamma$ . Note that each element  $\omega \in A^r(\mathfrak{a}, \underline{\mathcal{C}})$  can be written uniquely as

$$\omega = \sum_{k_1 < \dots < k_r} f_{k_1 \dots k_r} \theta_{k_1} \wedge \dots \wedge \theta_{k_r} \quad \text{where } f_{k_1 \dots k_r} \in \underline{\mathcal{C}}.$$

For simplicity, let  $\theta_K = \theta_{k_1} \wedge \dots \wedge \theta_{k_r}$  and  $f_K = f_{k_1 \dots k_r}$  for  $K = (k_1, \dots, k_r)$  ( $1 \leq k_1 < \dots < k_r \leq 2l$ ). Then  $\omega = \sum_K f_K \theta_K$ .

Let  $A^{p,q}(\mathfrak{a}, \underline{\mathcal{C}})$  denote the vector space of all  $\underline{\mathcal{C}}$ -valued forms of type  $(p, q)$  on  $A/A \cap \Gamma$ . Each element  $\omega \in A^{p,q}(\mathfrak{a}, \underline{\mathcal{C}})$  can be written uniquely as

$$\omega = \sum_{I,J} f_{IJ} \omega_I \wedge \bar{\omega}_J$$

where  $I=(i_1, \dots, i_p)$  ( $1 \leq i_1 < \dots < i_p \leq l$ ),  $J=(j_1, \dots, j_q)$  ( $1 \leq j_1 < \dots < j_q \leq l$ ),  $f_{IJ} \in \underline{C}$ ,  $\omega_I = \omega_{i_1} \wedge \dots \wedge \omega_{i_p}$  and  $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \dots \wedge \bar{\omega}_{j_q}$ .

Define operators  $d: A^r(\alpha, \underline{C}) \rightarrow A^{r+1}(\alpha, \underline{C})$  by

$$d\omega = \sum_K \left( \sum_{j=1}^{2l} Y_j f_K \right) \theta_j \wedge \theta_K$$

for  $\omega = \sum_K f_K \theta_K \in A^r(\alpha, \underline{C})$ ,  $d': A^{p,q}(\alpha, \underline{C}) \rightarrow A^{p+1,q}(\alpha, \underline{C})$  by

$$d'\omega = \sum_{I,J}^l \left( \sum_{k=1}^l X_k f_{IJ} \right) \omega_k \wedge \omega_I \wedge \bar{\omega}_J$$

for  $\omega = \sum_{I,J} f_{IJ} \omega_I \wedge \bar{\omega}_J \in A^{p,q}(\alpha, \underline{C})$  and  $d'': A^{p,q}(\alpha, \underline{C}) \rightarrow A^{p,q+1}(\alpha, \underline{C})$  by

$$d''\omega = \sum_{I,J}^l \left( \sum_{k=1}^l \bar{X}_k f_{IJ} \right) \bar{\omega}_k \wedge \omega_I \wedge \bar{\omega}_J$$

for  $\omega = \sum_{I,J} f_{IJ} \omega_I \wedge \bar{\omega}_J \in A^{p,q}(\alpha, \underline{C})$ . Then  $d \circ d = d' \circ d' = d'' \circ d'' = 0$ .

Define  $\langle \omega, \eta \rangle \in \underline{C}'$  for  $\omega, \eta \in A^{p,q}(\alpha, \underline{C})$  by

$$\langle \omega, \eta \rangle(x) = \sum_{I,J} \int_{A/A \cap \Gamma} f_{IJ}(xa) \bar{g}_{IJ}(xa) da = \int_{A/A \cap \Gamma} \omega \wedge \bar{*}\eta,$$

where  $\omega = \sum_{I,J} f_{IJ} \omega_I \wedge \bar{\omega}_J$ ,  $\eta = \sum_{I,J} g_{IJ} \omega_I \wedge \bar{\omega}_J$  and  $*$  is the operation defined by the natural orientation of  $A/A \cap \Gamma$  and the metric  $h$  on  $A/A \cap \Gamma$ .

Let  $\tilde{f} \in C^\infty(G/A\Gamma, \underline{C})$  denote the function corresponding to  $f \in \underline{C}'$ . Define a hermitian inner product  $(, )$  on  $A^{p,q}(\alpha, \underline{C})$  by

$$(\omega, \eta) = \int_{G/A\Gamma} \langle \widetilde{\omega}, \widetilde{\eta} \rangle(x) dx$$

where  $dx$  denotes an invariant measure on  $G/A\Gamma$ .

Define  $(\omega, \eta) = 0$  if  $\omega \in A^{p,q}(\alpha, \underline{C})$ ,  $\eta \in A^{p',q'}(\alpha, \underline{C})$  for  $(p, q) \neq (p', q')$ . Since  $A^r(\alpha, \underline{C}) = \sum_{p+q=r} A^{p,q}(\alpha, \underline{C})$ , we have thus an hermitian inner product  $(, )$  on  $A^r(\alpha, \underline{C})$ .

Now define the adjoint operators  $\delta, \delta', \delta''$  of  $d, d', d''$  by  $\delta = -*d*$ ,  $\delta' = -*d'*$ ,  $\delta'' = -*d''*$  respectively. We then have

$$\begin{aligned} (d\omega, \eta) &= (\omega, \delta\eta) & \text{for } \omega \in A^r(\alpha, \underline{C}) & \text{ and } \eta \in A^{r+1}(\alpha, \underline{C}), \\ (d'\omega, \eta) &= (\omega, \delta'\eta) & \text{for } \omega \in A^{p,q}(\alpha, \underline{C}) & \text{ and } \eta \in A^{p+1,q}(\alpha, \underline{C}), \\ (d''\omega, \eta) &= (\omega, \delta''\eta) & \text{for } \omega \in A^{p,q}(\alpha, \underline{C}) & \text{ and } \eta \in A^{p,q+1}(\alpha, \underline{C}). \end{aligned}$$

with respect to the hermitian inner product  $(, )$ .

Define Laplacians  $\Delta, \square', \square''$  by

$$\Delta = d\delta + \delta d, \quad \square' = d'\delta' + \delta'd', \quad \square'' = d\delta'' + \delta''d''.$$

Then, by a direct computation we get

$$\Delta\omega = -\sum_K \left(\sum_{j=1}^{2l} Y_j^2 f_K\right)\theta_K$$

for  $\omega = \sum_K f_K \theta_K$ , and

$$\square'\omega = \square''\omega = -\sum_{I,J} \left(\sum_{j=1}^l X_j \bar{X}_j\right) f_I \bar{J} \omega_I \wedge \bar{\omega}_J$$

for  $\omega = \sum_{I,J} f_I \bar{J} \omega_I \wedge \bar{\omega}_J$ .

Since  $X_j \bar{X}_j f = (Y_j^2 + Y_{j+l}^2)f$  for each  $f \in \underline{C}$ , we see  $\Delta = \square' = \square''$ .

Since  $A$  is abelian and simply connected, we may identify  $A$  (resp. the lattice  $A \cap \Gamma$  of  $A$ ) with Euclidean space  $(\mathbf{R}^n, \langle, \rangle)$  (resp. a lattice  $D$  in  $\mathbf{R}^n$ ). For a fixed  $x \in G$  and  $f \in \underline{C}$ ,  $f_x$  can be regarded as a function on the torus  $\mathbf{R}^n/D$ . Consider the Fourier expansion of  $f_x$ ,

$$f_x(a) = f(xa) = \sum_{\alpha \in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle$$

where  $D' = \{\alpha \in \mathbf{R}^n \mid \langle\alpha, d\rangle \in \mathbf{Z} \text{ for any } d \in D\}$  and  $C_\alpha(x) = \int_{A/A \cap \Gamma} f(xa) \exp -2\pi\sqrt{-1}\langle\alpha, a\rangle da$  for  $\alpha \in D'$ . Note that  $H(f)(x) = C_0(x) = \int_{A/A \cap \Gamma} f(xa) da$ .

For  $Y \in \mathfrak{a}$ ,  $f \in \underline{C}$  and  $x \in G$ , we have

$$(Yf)(xa) = \frac{d}{dt} f(a(t)xa) \Big|_{t=0}$$

where  $a(t)$  is the one parameter subgroup corresponding to  $Y$ . Since  $A$  is contained in the center of  $G$ ,

$$\begin{aligned} (Yf)(xa) &= \frac{d}{dt} \Big|_{t=0} f(xa(t)a) \\ &= \frac{d}{dt} \Big|_{t=0} \left\{ \sum_{\alpha \in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}\langle\alpha, a(t)a\rangle \right\} \\ &= \frac{d}{dt} \Big|_{t=0} \left\{ \sum_{\alpha \in D'} C_\alpha(x) \exp 2\pi\sqrt{-1}(\langle\alpha, a\rangle + \langle\alpha, a(t)\rangle) \right\} \\ &= 2\pi\sqrt{-1} \sum_{\alpha \in D'} C_\alpha(x) \langle\alpha, Y\rangle \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle. \end{aligned}$$

Since  $\langle Y_j, Y_k \rangle = \frac{1}{4} \delta_{jk}$  for  $j, k = 1, \dots, 2l$ , it follows that  $4(\Delta f)(xa) = -4 \sum_{j=1}^{2l} (Y_j^2 f)(xa) = (2\pi)^2 \sum_{\alpha \in D'} C_\alpha(x) \|\alpha\|^2 \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle$  where  $\|\alpha\|^2 = \langle\alpha, \alpha\rangle$ .

Define an operator  $G: \underline{C} \rightarrow \underline{C}$  by

$$G(f)(xa) = \frac{1}{(2\pi^2)} \sum_{\alpha \in D' - \{0\}} \frac{C_\alpha(x)}{\|\alpha\|^2} \exp 2\pi\sqrt{-1}\langle\alpha, a\rangle$$

for  $x \in G$  and  $f \in \underline{C}$ . We can show that  $G(f)(xa) = G(f)(yb)$  if  $xa = yb$  where  $a, b \in A$  ([11] p. 118). Thus  $G(f) \in C^\infty(G, \underline{C})$ . We also have  $G(f)(x\gamma) = G(f)(x)$  for any  $\gamma \in \Gamma$ . Hence,  $G(f) \in \underline{C}$ . It is obvious that

$$4\Delta G(f) = 4G\Delta(f) = f \quad \text{if } H(f) = 0,$$

and  $G \circ H(f) = H \circ G(f) = 0$  for any  $f \in \underline{C}$ . Therefore

$$f = H(f) + 4\Delta G(f) = H(f) + 4G\Delta(f) \quad \text{for any } f \in \underline{C}.$$

Define  $H: A^{p,q}(\alpha, \underline{C}) \rightarrow A^{p,q}(\alpha, \underline{C}')$  and  $G: A^{p,q}(\alpha, \underline{C}) \rightarrow A^{p,q}(\alpha, \underline{C})$  by

$$H(\omega) = \sum_{I,J} H(f_{I\bar{J}})\omega_I \wedge \bar{\omega}_J \quad \text{for } \omega = \sum_{I,J} f_{I\bar{J}}\omega_I \wedge \bar{\omega}_J$$

and

$$G(\omega) = \sum_{I,J} G(f_{I\bar{J}})\omega_I \wedge \bar{\omega}_J \quad \text{for } \omega = \sum_{I,J} f_{I\bar{J}}\omega_I \wedge \bar{\omega}_J.$$

Then we have

$$\omega = H(\omega) + 4G\Delta(\omega) = H(\omega) + 4\Delta G(\omega)$$

and

$$\omega = H(\omega) + 4G\Box''(\omega) = H(\omega) + 4\Box''G(\omega).$$

Obviously  $d'' \circ H = d' \circ H = 0$ . Since  $\int_{A/A \cap \Gamma} (\bar{X}_j f)(x) da = \int_{A/A \cap \Gamma} (X_j f)(x) da = 0$  for  $j=1, \dots, l$  and  $f \in \underline{C}$ ,  $H \circ d'' = H \circ d' = 0$ . By the definition of  $H$ , it is obvious that  $* \circ H = H \circ *$ , so that  $\delta'' \circ H = H \circ \delta'' = 0$ .

Let  $A^*(\alpha, \underline{C}) = \sum_{p,q} A^{p,q}(\alpha, \underline{C})$ .

**Lemma 4.2.** *Let  $F: A^*(\alpha, \underline{C}) \rightarrow A^*(\alpha, \underline{C})$  be an additive operator which commutes with  $\Box''$ . Then  $F$  commutes with  $H$  and  $G$ . In particular,  $G$  commutes with  $d''$  and  $\delta''$ .*

*Proof.* See [15] Chapter IV lemma 3.

*Proof of Proposition 2.1.* Note that the cochain complex  $\{A^{0,q}(\alpha, \underline{C}), d''\}$  is exactly the cochain complex of  $\alpha^-$ -module  $\underline{C}$ . The inclusion map  $\iota_0: \underline{C}' \rightarrow \underline{C}$  induces a cochain map  $\iota_0^*: A^*(\alpha^-, \underline{C}') \rightarrow A^*(\alpha^-, \underline{C})$ . In particular, the following diagram commutes

$$\begin{array}{ccc} A^{0,q}(\alpha, \underline{C}') & \xrightarrow{\iota_0^*} & A^{0,q}(\alpha, \underline{C}) \\ \downarrow d'' & & \downarrow d'' \\ A^{0,q+1}(\alpha, \underline{C}') & \xrightarrow{\iota_0^*} & A^{0,q+1}(\alpha, \underline{C}). \end{array}$$

Since  $d''(\omega) = 0$  for any  $\omega \in A^{0,q}(\alpha, \underline{C}')$ ,  $H^q(\alpha^-, \underline{C}') = A^{0,q}(\alpha, \underline{C}')$ .

Let  $\iota_0^*: H^q(\alpha^-, \underline{C}') \rightarrow H^q(\alpha^-, \underline{C})$  denote the map induced from the cochain map  $\iota_0^*: A^*(\alpha^-, \underline{C}') \rightarrow A^*(\alpha^-, \underline{C})$ . Since  $H \circ d'' = d'' \circ H$ ,  $H: A^{0,q}(\alpha, \underline{C}) \rightarrow A^{0,q}(\alpha, \underline{C}')$  induces a linear map  $H: H^q(\alpha^-, \underline{C}) \rightarrow H^q(\alpha^-, \underline{C}')$ .

We claim that  $\iota_0^* \circ H = id$  and  $H \circ \iota_0^* = id$ . By definition  $H \circ \iota_0^*[\omega] = [\omega]$  for  $[\omega] \in H^q(\alpha^-, \underline{C}')$ . Since  $\omega = H(\omega) + 4G\Box''(\omega) = H(\omega) + 4Gd''\delta''\omega = H(\omega) + 4d''G\delta''\omega$  for any  $\omega \in A^{0,q}(\alpha, \underline{C})$  such that  $d''\omega = 0$ ,  $\iota_0^*H[\omega] = [\omega]$  for any  $[\omega] \in H^q(\alpha^-, \underline{C})$ . It is now obvious that  $\iota_0^*$  is a  $g^-$ -module homomorphism. q.e.d.

Proof of Theorem 1. Let  $A^{0,q}(G/\Gamma, \mathbf{C})$  be the space of all  $\mathbf{C}$ -valued  $C^\infty$ -differential forms on  $G/\Gamma$  of type  $(0, q)$ . Take a basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{g}^+$  and let  $\{\omega_1, \dots, \omega_n\}$  be the dual basis of  $(\mathfrak{g}^+)^*$ . We regard an element  $\omega \in (\mathfrak{g}^+)^*$  as a holomorphic 1-form on  $G/\Gamma$ . Then any element  $\omega \in A^{0,q}(G/\Gamma, \mathbf{C})$  can be written as  $\omega = \sum f_J \bar{\omega}_J$  where  $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \dots \wedge \bar{\omega}_{j_q}$ ,

$$J = (j_1, \dots, j_q) (1 \leq j_1 < \dots < j_q \leq n) \text{ and } f_J \in \underline{\mathbf{C}}.$$

The operator  $d'' : A^{0,q}(G/\Gamma, \mathbf{C}) \rightarrow A^{0,q+1}(G/\Gamma, \mathbf{C})$  can be written as

$$d''\omega = \sum_J \left( \sum_{k=1}^n \bar{X}_k f_J \right) \bar{\omega}_k \wedge \bar{\omega}_J + f_J d\bar{\omega}_J$$

for  $\omega = \sum_J f_J \bar{\omega}_J$ .

Therefore the Dolbeault cohomology group  $H_a^{0,q}(G/\Gamma)$  can be regarded as the Lie algebra cohomology  $H^q(\mathfrak{g}^-, \underline{\mathbf{C}})$  of  $\mathfrak{g}^-$ -module  $\underline{\mathbf{C}}$ .

$$(2.2) \quad H_a^{0,q}(G/\Gamma) \cong H^q(\mathfrak{g}^-, \underline{\mathbf{C}}).$$

Regarding  $\mathbf{C}$  as constant functions on  $G$ , we have the inclusion map  $\iota : \mathbf{C} \rightarrow \underline{\mathbf{C}}$  of  $\mathfrak{g}^-$ -modules. Now by (2.1), Theorem 1 is equivalent to assert that  $\iota$  induces an isomorphism on the cohomology groups

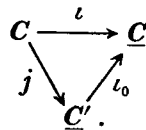
$$\iota^* : H^q(\mathfrak{g}^-) \rightarrow H^q(\mathfrak{g}^-, \underline{\mathbf{C}}).$$

We prove th the isomorphism  $\iota^* : H^q(\mathfrak{g}^-) \rightarrow H^q(\mathfrak{g}^-, \underline{\mathbf{C}})$  by the induction on the dimension of  $G/\Gamma$ . If  $G$  is abelian,  $G/\Gamma$  is a complex torus and our claim is well-known. As before, let  $A$  be the normal subgroup of  $G$  contained in the center of  $G$  and  $\mathfrak{a}$  be the ideal in  $\mathfrak{g}$  corresponding to  $A$ . Consider the Hochschild and Serre spectral sequences for  $\mathfrak{g}^-$ -modules  $\mathbf{C}$  and  $\underline{\mathbf{C}}$ , and a homomorphism of these spectral sequences induced by the inclusion map  $\iota : \mathbf{C} \rightarrow \underline{\mathbf{C}}$  [2];

$$E_2(\iota) : H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \mathbf{C})) \rightarrow H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{\mathbf{C}}))$$

for  $t, s=0, 1, 2, \dots$ .

Consider also the  $\mathfrak{g}^-$ -module  $\underline{\mathbf{C}}'$ . Then we have a commutative diagram of  $\mathfrak{g}^-$ -modules



This commutative diagram induces the corresponding commutative diagram of spectral sequences



$$\begin{array}{ccc}
 H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C})) & \xrightarrow{E_2(\iota)} & H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C})) \\
 E_2(j) \searrow & & \nearrow E_2(\iota_0) \\
 & & H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}')).
 \end{array}$$

By proposition 2.1, we have an isomorphism of  $\mathfrak{g}^-$ -modules  $\iota_0^*: H^s(\mathfrak{a}^-, \underline{C}') \rightarrow H^s(\mathfrak{a}^-, \underline{C})$ . Hence,

$$E_2(\iota_0): H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}')) \rightarrow H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}))$$

is an isomorphism.

We shall show that  $E_2(j)$  is an isomorphism. Since  $\mathfrak{a}^-$  is contained in the center of  $\mathfrak{g}^-$ ,  $\mathfrak{g}^-$  acts trivially on  $H^s(\mathfrak{a}^-, \underline{C}) = A^s(\mathfrak{a}^-, \underline{C})$ . Hence,

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C})) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{C}) \otimes H^s(\mathfrak{a}^-, \underline{C}).$$

Since  $\mathfrak{a}^-$  acts trivially on  $\underline{C}'$ ,  $H^s(\mathfrak{a}^-, \underline{C}') = A^s(\mathfrak{a}^-, \underline{C}')$ . Consider the action of  $\mathfrak{g}^-$  on  $H^s(\mathfrak{a}^-, \underline{C}')$ . For an  $s$ -cochain  $\omega = \sum_j f_j \bar{\omega}_j \in A^s(\mathfrak{a}^-, \underline{C}')$  and  $\bar{X} \in \mathfrak{g}^-$ ,  $L_{\bar{X}}\omega = \sum_j (\bar{X}f_j)\bar{\omega}_j$ , since  $\mathfrak{a}^-$  is contained in the center of  $\mathfrak{g}^-$ . Hence,  $H^s(\mathfrak{a}^-, \underline{C}')$  and  $\underline{C}' \otimes H^s(\mathfrak{a}^-, \underline{C})$  are isomorphic as  $\mathfrak{g}^-$ -modules. Hence, we have

$$\begin{aligned}
 H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}')) &\cong H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{C}' \otimes H^s(\mathfrak{a}^-, \underline{C})) \\
 &\cong H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{C}') \otimes H^s(\mathfrak{a}^-, \underline{C}).
 \end{aligned}$$

We now regard  $\underline{C}'$  as the vector space of all  $\underline{C}$ -valued  $C^\infty$ -functions on  $(G/A)/\pi(\Gamma)$ . It is easy to see that this identification is compatible with  $\mathfrak{g}^-/\mathfrak{a}^-$ -module structure. Thus we have

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, C^\infty((G/A)/\pi(\Gamma), \underline{C})) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{C}').$$

By the assumption of the induction, we get

$$H^t(\mathfrak{g}^-/\mathfrak{a}^-, C^\infty((G/A)/\pi(\Gamma), \underline{C})) = H^t(\mathfrak{g}^-/\mathfrak{a}^-, \underline{C}).$$

Hence, we have an isomorphism

$$E_2(j): H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C})) \rightarrow H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}')).$$

Thus  $E_2(\iota): H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C})) \rightarrow H^t(\mathfrak{g}^-/\mathfrak{a}^-, H^s(\mathfrak{a}^-, \underline{C}))$  is an isomorphism. By a theorem on spectral sequence ([13] Chapter 9, §1 Theorem 3), this implies an existence of an isomorphism

$$\iota^*: H^q(\mathfrak{g}^-, \underline{C}) \cong H^q(\mathfrak{g}^-, \underline{C}').$$

Combining this (2.1) and (2.2), we get

$$H_{\mathbb{Z}}^q(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \Lambda^q(\mathfrak{g}^+)^*. \quad \text{q.e.d.}$$

**Corollary 1** (Kodaira [9]). *Let  $r$  be the dimension of the vector space of all closed holomorphic 1-forms on a compact complex parallelisable nilmanifold  $G/\Gamma$ . Then  $\dim H_{\mathbb{C}}^{0,1}(G/\Gamma)=r$ .*

Proof. Let  $\omega$  be a closed holomorphic 1-form on  $G/\Gamma$ . Then  $\omega = \sum_{j=1}^n f_j \phi_j$  where  $(\phi_1, \dots, \phi_n)$  is a basis of  $(\mathfrak{g}^+)^*$  and  $f_j$  ( $j=1, \dots, n$ ) are holomorphic functions on  $G/\Gamma$ . Since  $G/\Gamma$  is compact,  $f_j$  are constant. Hence,  $\omega \in (\mathfrak{g}^+)^*$ . Moreover  $d\omega=0$  if and only if  $\omega([\mathfrak{g}^+, \mathfrak{g}^+])=(0)$ . Thus  $r = \dim(\mathfrak{g}^+ / [\mathfrak{g}^+, \mathfrak{g}^+])$ . Since  $\dim H^1(\mathfrak{g}^-) = \dim(\mathfrak{g}^- / [\mathfrak{g}^-, \mathfrak{g}^-]) = \dim(\mathfrak{g}^+ / [\mathfrak{g}^+, \mathfrak{g}^+])$ , we have  $r = \dim H_{\mathbb{C}}^{0,1}(G/\Gamma)$  by Theorem 1. q.e.d.

Let  $M$  be a compact connected complex manifold. Let  $b_r$  (resp.  $h^{p,q}$ ) denote  $\dim_{\mathbb{R}} H^r(M, \mathbb{R})$  (resp.  $\dim_{\mathbb{C}} H_{\mathbb{C}}^{p,q}(M)$ ).

**Corollary 2.** *If  $M$  is a compact complex parallelisable nilmanifold  $G/\Gamma$ ,*

$$\begin{aligned} b_{2k+1} &= 2(h^{0,2k+1} + h^{0,2k}h^{0,1} + \dots + h^{0,k+1}h^{0,k}) \\ b_{2k} &= 2(h^{0,2k} + h^{0,2k-1}h^{0,1} + \dots + h^{0,k+1}h^{0,k-1}) + (h^{0,k})^2 \end{aligned}$$

for  $2k+1, 2k \leq n = \dim_{\mathbb{C}} G$ .

Proof. By a theorem of Nomizu [10] (See [11] Corollary 7.28.),  $H^r(G/\Gamma, \mathbb{R}) \cong H^r(\mathfrak{g}, \mathbb{R})$ . Thus  $H^r(G/\Gamma, \mathbb{C}) \cong H^r(\mathfrak{g}, \mathbb{C}) \cong H^r(\mathfrak{g}^{\mathbb{C}})$ . Since  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$  and  $[\mathfrak{g}^+, \mathfrak{g}^-] = (0)$ ,  $H^r(\mathfrak{g}^{\mathbb{C}}) \cong \sum_{p+q=r} H^p(\mathfrak{g}^+) \otimes H^q(\mathfrak{g}^-)$ . Since  $\dim H^q(\mathfrak{g}^+) = \dim H^q(\mathfrak{g}^-) = h^{0,q}$  and  $\dim H^r(\mathfrak{g}^{\mathbb{C}}) = b_r$ ,  $b_r = \sum_{p+q=r} h^{0,p}h^{0,q}$ . q.e.d.

EXAMPLE. Let  $G$  be a nilpotent Lie group defined by

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$$

Let  $\Gamma$  be a lattice in  $G$ , for example,

$$\Gamma = \left\{ \begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{Z} + \sqrt{-1}\mathbb{Z} \right\}.$$

We can take a basis  $\{X_1, X_2, X_3\}$  of  $\mathfrak{g}^+$  such that

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = [X_1, X_3] = 0.$$

Then the dual basis  $\{\omega_1, \omega_2, \omega_3\}$  satisfies that

$$d\omega_3 = -\omega_1 \wedge \omega_2, \quad d\omega_1 = d\omega_2 = 0.$$

Now it follows easily from Theorem 1 that  $h^{0,1}=h^{0,2}=2$ . Note that  $h^{1,0}=3$ . By corollary 2, we get

$$b_0 = b_6 = 1, \quad b_1 = b_5 = 4, \quad b_2 = b_4 = 8 \quad \text{and} \quad b_3 = 10.$$

**3. Chern classes of holomorphic line bundles over a compact complex parallelisable solvmanifold**

Let  $G$  be a simply connected complex solvable Lie group and  $\Gamma$  be a lattice of  $G$ . We assume the following condition:

(M)  $Ad(G)$  and  $Ad(\Gamma)$  have the same Zariski closure in  $Aut(\mathfrak{g}^c)$ .

**Lemma 3.1.** *If  $G$  is non-abelian, we have  $\Gamma \cap [G, G] \neq \{e\}$ .*

Proof. Suppose that  $\Gamma \cap [G, G] = \{e\}$ . Since  $[\Gamma, \Gamma] \subset \Gamma \cap [G, G]$ ,  $\Gamma$  is abelian, so is  $Ad(\Gamma)$ . Since  $Ad(\Gamma)$  and  $Ad(G)$  have the same Zariski closure,  $\overline{Ad([\Gamma, \Gamma]^z)} = \overline{Ad(G), Ad(G)}^z = [\overline{Ad(G)}^z, \overline{Ad(G)}^z] = [\overline{Ad(\Gamma)}^z = \overline{Ad(\Gamma)}^z] = Ad([\Gamma, \Gamma]^z) = \{e\}$ , where  $\overline{X}^z$  denotes the Zariski closure of  $X$  in  $Aut(\mathfrak{g}^c)$ . Hence,  $[G, G]$  is contained in the center  $Z$  of  $G$ . Thus  $G$  is nilpotent. Since  $\Gamma$  is abelian,  $G$  is abelian [11]. This is a contradiction. q.e.d.

**Proposition 3.2.**  $\Gamma_1 = \Gamma \cap [G, G]$  is a lattice of  $[G, G]$ .

Proof. At first note the following:

If  $\mathfrak{m}$  is an ideal of  $\mathfrak{g}$  and  $\rho_1$  (resp.  $\rho_2$ ) is the representation on  $\mathfrak{m}^c$  (resp.  $\mathfrak{g}^c/\mathfrak{m}^c$ ) induced by the adjoint representation  $Ad: G \rightarrow Aut(\mathfrak{g}^c)$ ,  $\rho_1(G)$  and  $\rho_1(\Gamma)$  (resp.  $\rho_2(G)$  and  $\rho_2(\Gamma)$ ) have the same Zariski closure in  $Aut(\mathfrak{m}^c)$  (resp.  $Aut(\mathfrak{g}^c/\mathfrak{m}^c)$ ).

Now  $[G, G]$  is a simply connected nilpotent closed Lie subgroup of  $G$  and  $\Gamma_1$  is a discrete subgroup of  $[G, G]$ . Let  $H$  be the connected closed subgroup of  $[G, G]$  such that  $H/\Gamma_1$  is compact ([11] Proposition 2.5.). We claim that  $H$  is a normal subgroup of  $G$ . Let  $\exp: [\mathfrak{g}, \mathfrak{g}] \rightarrow [G, G]$  be the exponential map. Then  $\exp^{-1}(\Gamma_1) = \mathfrak{I}$  is a lattice in the Lie algebra  $\mathfrak{h}$  of  $H$  and  $\mathfrak{I} \otimes \mathbf{R} = \mathfrak{h}$  ([11] Theorem 2.12). Since  $\Gamma_1 = \Gamma \cap [G, G]$  is a normal subgroup of  $\Gamma$ ,  $\exp Ad(\gamma)L = \gamma(\exp L)\gamma^{-1} \in \Gamma_1$  for any  $L \in \mathfrak{I} = \exp^{-1}(\Gamma_1)$  and  $\gamma \in \Gamma$ . Hence,  $Ad(\gamma)\mathfrak{I} \subset \mathfrak{I}$  and  $Ad(\gamma)\mathfrak{h} = \mathfrak{h}$  for any  $\gamma \in \Gamma$ . Since  $Ad(G)$  and  $Ad(\Gamma)$  have the same Zariski closure in  $Aut(\mathfrak{g}^c)$ ,  $Ad(G)\mathfrak{h} = \mathfrak{h}$ . Hence,  $\mathfrak{h}$  is an ideal in  $\mathfrak{g}$ . Thus  $H$  is a normal subgroup of  $G$ .

Since  $H \subset [G, G]$  and  $\Gamma_1 \subset H$ ,  $H \cap \Gamma = H \cap [G, G] \cap \Gamma = H \cap \Gamma_1 = \Gamma_1$ . Thus  $H/H \cap \Gamma$  is compact and  $H \cdot \Gamma$  is closed in  $G$  ([11] Theorem 1.13). Hence,  $H \cdot \Gamma/H$  is a lattice of  $G/H$ . We claim that  $\Gamma H/H \cap [G/H, G/H] = \{e\}$ . Let  $a \in \Gamma H/H \cap [G/H, G/H]$ . Since  $[G/H, G/H] = [G, G]H/H = [G, G]/H$ ,  $a = \gamma H = g_1 H$  for some  $\gamma \in \Gamma$  and  $g_1 \in [G, G]$ , that is,  $\gamma = g_1 h$  for some  $h \in H$ . Since  $H \subset [G, G]$ ,  $\gamma \in [G, G] \cap \Gamma = \Gamma_1 \subset H$ . Hence,  $a = \gamma H = H$ .

Since  $Ad(G/H)$  and  $Ad(\Gamma H/H)$  have the same Zariski closure in  $Aut(\mathfrak{g}^c/\mathfrak{h}^c)$ ,  $G/H$  is abelian by Lemma 3.1. Hence  $H \supset [G, G]$ . Thus  $H = [G, G]$  and is  $\Gamma_1$  a lattice of  $[G, G]$ . q.e.d.

Since  $\Gamma \cap [G, G]$  is a lattice of  $[G, G]$ ,  $[G, G]\Gamma$  is closed in  $G$  ([11] Theorem 1.13.) and  $\pi(\Gamma) = \Gamma[G, G]/[G, G]$  is a lattice of  $G/[G, G]$ . Note that  $G/[G, G]\Gamma = (G/[G, G])/\pi(\Gamma)$  is a complex torus. Thus we have a holomorphic fiber bundle  $(G/\Gamma, \pi, (G/[G, G])/\pi(\Gamma), [G, G]/[G, G] \cap \Gamma)$ . Let  $T$  denote the complex torus  $G/[G, G]\Gamma$ .

Now we denote by  $A^{1,1}(G/\Gamma, \mathbf{R})$  the vector space of all real differential forms of type  $(1, 1)$  on  $G/\Gamma$ . Let  $H^{1,1}(G/\Gamma, \mathbf{R})$  be the vector space

$$\frac{\{\omega \in A^{1,1}(G/\Gamma, \mathbf{R}) \mid d\omega = 0\}}{\{\omega \in A^{1,1}(G/\Gamma, \mathbf{R}) \mid \omega = d\theta, \theta \text{ is a real 1-form}\}}.$$

We shall characterize  $H^{1,1}(G/\Gamma, \mathbf{R})$  in terms of the Lie algebra  $\mathfrak{g}$  of  $G$ .

**Proposition 3.3.** *Suppose that a lattice  $\Gamma$  of  $G$  satisfies the condition (M). Then, for any real closed form  $\alpha$  of type  $(1, 1)$  on  $G/\Gamma$ , there is a unique real right invariant closed form  $\beta \in \Lambda^2(\mathfrak{g}^*)$  of type  $(1, 1)$  on  $G$  such that  $\alpha = \beta + d\eta$  on  $G/\Gamma$  where  $\eta$  is a real 1-form on  $G/\Gamma$ .*

Proof. According to a theorem of Mostow ([8], [11]), for a given real closed 2-form  $\alpha$ , there is a real right invariant closed 2-form  $\beta \in \Lambda^2\mathfrak{g}^*$  such that

$$(3.1) \quad \alpha = \beta + d\gamma$$

where  $\gamma$  is a real 1-form on  $G/\Gamma$ . Let  $\beta = \beta^{0,2} + \beta^{1,1} + \beta^{2,0}$  where  $\beta^{p,q}$  is the component of  $\beta$  of type  $(p, q)$ . Since  $\beta$  is a real form,  $\beta^{2,0} = \bar{\beta}^{0,2}$  and  $\beta^{1,1}$  is a real form. Let  $\gamma = \gamma^{1,0} + \gamma^{0,1}$ ,  $\gamma^{1,0} = \bar{\gamma}^{0,1}$ . Taking a basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{g}^+$ , let  $\{\omega_1, \dots, \omega_n\}$  be its dual basis of  $(\mathfrak{g}^+)^*$ . We identify  $\omega_j$  ( $j=1, \dots, n$ ) as holomorphic 1-forms on  $G/\Gamma$ . We then have

$$\gamma^{0,1} = \sum_{j=1}^n f_j \bar{\omega}_j$$

where  $f_j \in C^\infty(G/\Gamma, \mathbf{C})$  for  $j=1, \dots, n$  and

$$\beta^{0,2} = \sum_{j < k} a_{jk} \bar{\omega}_j \wedge \bar{\omega}_k$$

where  $a_{jk} \in \mathbf{C}$ . Since  $\alpha$  is of type  $(1,1)$ , we get

$$(3.2) \quad \beta^{0,2} + d''\gamma^{0,1} = 0 \quad \text{and} \quad \alpha = \beta^{1,1} + d''\gamma^{1,0} + \bar{d}''\bar{\gamma}^{1,0}$$

by comparing the type of forms of both hands. We now have

$$\begin{aligned} d''\gamma^{0,1} &= d''\left(\sum_{j=1}^n f_j \bar{\omega}_j\right) = \sum_{j=1}^n (d''f_j \wedge \bar{\omega}_j + f_j d\bar{\omega}_j) \\ &= \sum_{j,k=1}^n \bar{X}_k f_j \bar{\omega}_k \wedge \bar{\omega}_j - \sum_{j=1}^n \sum_{k<l} f_j \bar{C}_{kl}^j \bar{\omega}_k \wedge \bar{\omega}_l \end{aligned}$$

where  $C_{kl}^j$  are the structure constant of Lie algebra  $\mathfrak{g}^+$  with respect to the basis  $\{X_1, \dots, X_n\}$ . By (3.2), we get the equalities

$$(3.3) \quad a_{kl} = \bar{X}_k f_l - \bar{X}_l f_k - \sum_{j=1}^n f_j \bar{C}_{kl}^j \quad \text{for } 1 \leq k < l \leq n.$$

Integrating (3.3) on  $G/\Gamma$ , we have

$$(3.4) \quad \int_{G/\Gamma} a_{kl} dg = \int_{G/\Gamma} (\bar{X}_k f_l) dg - \int_{G/\Gamma} (\bar{X}_l f_k) dg - \sum_{j=1}^n \int_{G/\Gamma} f_j \bar{C}_{kl}^j dg$$

where  $dg$  is an invariant measure on  $G/\Gamma$ . Since  $G$  is unimodular,  $\int_{G/\Gamma} (\bar{X}_k f_l) dg = \int_{G/\Gamma} (\bar{X}_l f_k) dg = 0$ , and we get

$$(3.5) \quad a_{kl} \int_{G/\Gamma} dg = - \sum_{j=1}^n \bar{C}_{kl}^j \int_{G/\Gamma} f_j dg.$$

Let  $b_j \in \mathbb{C}$  denote  $\int_{G/\Gamma} f_j dg / \int_{G/\Gamma} dg$ . Then (3.5) can be written as

$$\begin{aligned} (3.6) \quad a_{kl} &= - \sum_{j=1}^n b_j \bar{C}_{kl}^j. \\ \beta^{0,2} &= \sum_{k<l} a_{kl} \bar{\omega}_k \wedge \bar{\omega}_l = - \sum_{k<l} \sum_{j=1}^n b_j \bar{C}_{kl}^j \bar{\omega}_k \wedge \bar{\omega}_l \\ &= \sum_{j=1}^n b_j \left(- \sum_{k<l} \bar{C}_{kl}^j \bar{\omega}_k \wedge \bar{\omega}_l\right) = \sum_{j=1}^n b_j (d\bar{\omega}_j) = d\left(\sum_{j=1}^n b_j \bar{\omega}_j\right). \end{aligned}$$

Put  $\eta = \sum_{j=1}^n b_j \bar{\omega}_j$ . We then see that  $\eta$  is of type  $(0, 1)$ ,  $\beta^{0,2} = d\eta$  and  $\beta^{2,0} = d\bar{\eta}$ .

By (3.1), we get

$$\alpha = \beta^{1,1} + d(\eta + \bar{\eta}) + d\gamma = \beta^{1,1} + d\theta$$

where  $\theta = \eta + \bar{\eta} + \gamma$  is a real 1-form on  $G/\Gamma$ .

It remains to show the uniqueness of  $\beta^{1,1}$ . It is sufficient to see that if  $\beta^{1,1} = d\theta$ ,  $\theta$  is a real 1-form, then  $\beta^{1,1} = 0$ . Put  $\beta^{1,1} = \sum_{j,k=1}^n a_{jk} \omega_j \wedge \bar{\omega}_k$  and  $\theta = \theta^{0,1} + \bar{\theta}^{0,1}$  where  $\theta^{0,1} = \sum_{j=1}^n g_j \bar{\omega}_j$ ,  $g_j \in C^\infty(G/\Gamma, \mathbb{C})$  ( $j = 1, \dots, n$ ). Since  $d'\theta^{0,1} = \sum_{k,j=1}^n X_k g_j \omega_k \wedge \bar{\omega}_j$  and  $d''\bar{\theta}^{0,1} = d'\bar{\theta}^{0,1} = \sum_{k,j=1}^n \bar{X}_k \bar{g}_j \bar{\omega}_j \wedge \omega_k$ , we get

$$(3.7) \quad a_{jk} = X_j g_k - \bar{X}_k \bar{g}_j.$$

Integrating (3.7) on  $G/\Gamma$ , we have

$$a_{jk} \int_{G/\Gamma} dg = \int_{G/\Gamma} (X_j g_k) dg - \int_{G/\Gamma} (\bar{X}_k \bar{g}_j) dg = 0.$$

Hence,  $a_{jk}=0$  for  $j, k=1, \dots, n$  and  $\beta^{1,1}=0$ . q.e.d.

We now determine real closed right invariant forms of type (1, 1) on  $G/\Gamma$ . Take a basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{g}^+$  such that  $\{X_{r+1}, \dots, X_n\}$  is a basis of  $[\mathfrak{g}^+, \mathfrak{g}^+]$ . Let  $\{\omega_1, \dots, \omega_n\}$  be its dual basis of  $(\mathfrak{g}^+)^*$ .

**Proposition 3.4.** *Let  $\alpha$  be a right invariant real 2-form of type (1, 1) on  $G$ . Then  $d\alpha=0$  if and only if  $\alpha = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^r h_{jk} \omega_j \wedge \bar{\omega}_k$  where  $H=(h_{jk}) \in M(r, \mathbf{C})$  is a hermitian matrix, and  $r = \dim \mathfrak{g}^+ / [\mathfrak{g}^+, \mathfrak{g}^+]$ .*

Proof. Since  $\alpha$  is a right invariant form on  $G$ ,  $\alpha$  defines a bilinear form on  $\mathfrak{g}^+ \times \mathfrak{g}^-$ . Now  $d\alpha=0$  if and only if

$$\alpha([X, Y], \bar{Z}) = 0 \quad \text{and} \quad \alpha([\bar{X}, \bar{Y}], Z) = 0 \quad \text{for } X, Y, Z \in \mathfrak{g}^+,$$

since  $(d\alpha)(X, Y, Z) = -\alpha([X, Y], Z) + \alpha([X, Z], Y) - \alpha([Y, Z], X)$  for  $X, Y, Z \in \mathfrak{g}^c$  and since  $[\mathfrak{g}^+, \mathfrak{g}^-] = (0)$ . In particular, for a real form  $\alpha$  of type (1, 1), we get

$$(3.8) \quad d\alpha = 0 \text{ if and only if } \iota([X, Y])\alpha = 0 \quad \text{for } X, Y \in \mathfrak{g}^+.$$

Note that  $d\omega_j=0$  for  $j=1, \dots, r$ . Therefore, if  $\alpha = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^r h_{jk} \omega_j \wedge \bar{\omega}_k$  then  $d\alpha=0$ . Conversely, put  $\alpha = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^n h_{jk} \omega_j \wedge \bar{\omega}_k$ . If  $\alpha$  is closed, then  $\iota(X_j)\alpha=0$  for  $j=r+1, \dots, n$  by (3.8).

Since  $(\iota(X_j)\alpha)(\bar{X}_k) = \alpha(X_j, \bar{X}_k) = \frac{1}{2\sqrt{-1}} h_{jk}$  and  $H=(h_{jk})$  is a hermitian matrix, we have  $h_{jk}=0$  for  $j=r+1, \dots, n; k=1, \dots, n$  and  $j=1, \dots, n; k=r+1, \dots, n$ , so that  $\alpha = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^r h_{jk} \omega_j \wedge \bar{\omega}_k$ . q.e.d.

Consider a holomorphic line bundle  $L$  on  $G/\Gamma$ . Let  $C(L)$  denote the Chern class of  $L$ . Then we have  $C(L) \in H^{1,1}(G/\Gamma, \mathbf{R})$  ([15], Chapter V, n°4.).

**Proposition 3.5.** *Let  $G$  be a simply connected complex solvable Lie group and  $\Gamma$  be a lattice of  $G$  satisfying the condition (M) and such that  $H_{\text{inv}}^1(G/\Gamma) \cong H^1(\mathfrak{g}^-)$  (canonically). Let  $L$  be a holomorphic line bundle on  $G/\Gamma$ . Then there is a unique real invariant form  $\alpha \in \Lambda^2 \mathfrak{g}^*$  of type (1, 1) in  $C(L)$ , and this is a curvature form of a connection  $\eta$  of type (1, 0).*

Proof. It is easy to see that there is a real closed 2-form  $\beta$  of type (1, 1) in

$C(L)$  which is a curvature form of a connection  $\omega$  of type  $(1,0)$  ([15], Chapter V, n°4).

According to Proposition 3.3, we have  $\beta = \alpha + d\gamma$  where  $\gamma$  is a real 1-form on  $G/\Gamma$ . Decompose  $\gamma = \gamma^{1,0} + \gamma^{0,1}$  where  $\gamma^{1,0}$  (resp.  $\gamma^{0,1}$ ) is the component of type  $(1, 0)$  (resp.  $(0, 1)$ ) of  $\gamma$ . Then we have  $d''\gamma^{0,1} = 0$ , since  $\beta$  and  $\alpha$  are of type  $(1, 1)$ . By the assumption (2), there is a right invariant 1-form  $\theta$  of type  $(0, 1)$  such that  $\gamma^{0,1} - \theta = d''f$  where  $f \in C^\infty(G/\Gamma, \mathbb{C})$ .

We can write  $\theta = \sum_{j=1}^r a_j \bar{\omega}_j$ ,  $a_j \in \mathbb{C}$  ( $j=1, \dots, r$ ), where  $\{\omega_1, \dots, \omega_n\}$  is the same as before, since  $H^1(\mathfrak{g}^-) = (\mathfrak{g}^- / [\mathfrak{g}^-, \mathfrak{g}^-])^*$ . We then have  $d\theta = \sum_{j=1}^r a_j d\bar{\omega}_j = 0$ , so that  $\beta = \alpha + d'\gamma^{0,1} + d''\gamma^{1,0} = \alpha + d'\gamma^{0,1} + \overline{d''\gamma^{0,1}} = \alpha + d'(\theta + d''f) + \overline{d''(\theta + d''f)} = \alpha + d'd''(f - \bar{f})$ . Put  $\psi = d'(\bar{f} - f)$ . We then have  $\beta = \alpha + dd'(\bar{f} - f) = \alpha + d\psi$ . Since  $\beta$  is a curvature form of a connection of type  $(1, 0)$  by definition and  $\psi$  is of type  $(1, 0)$ ,  $\alpha$  is a curvature form of a connection  $\eta = \omega - \psi$  of type  $(1, 0)$ .  
 q.e.d.

From now on we always assume that  $G$  and  $\Gamma$  satisfies the assumptions of Proposition 3.5.

Consider a holomorphic line bundle  $L$  on  $G/\Gamma$ . We fix a (sufficiently fine) simple covering  $\{U_i\}$  on  $G/\Gamma$  and choose a connected component  $U_{i_0}$  of  $p^{-1}(U_i)$  for each  $i$ ,  $p: G \rightarrow G/\Gamma$  being the canonical map; let  $U_{i\gamma}$  denote the image of  $U_{i_0}$  under the right translation  $R_\gamma(g) = g\gamma$  for  $\gamma \in \Gamma$ . Then  $p^{-1}(U_i) = \bigcup_{\gamma \in \Gamma} U_{i\gamma}$  is a disjoint union and  $p$  maps each  $U_{i\gamma}$  biholomorphically to  $U_i$ .

We may consider a holomorphic line bundle  $L$  on  $G/\Gamma$  is given by a system of transition functions  $\{g_{jk}\}$  relative to the covering  $\{U_i\}$  of  $G/\Gamma$ . Let  $C(L)$  be the Chern class of  $L$  and  $\alpha$  be the unique real right invariant form of type  $(1, 1)$  in  $C(L)$ . By Proposition 3.5,  $\alpha$  is a curvature form of a connection  $\eta$  of type  $(1, 0)$ , so that there is an element  $\eta_j \in A^{1,0}(U_j)$  for each  $j$  satisfying  $\eta_k - \eta_j = \frac{\sqrt{-1}}{2\pi} d \log g_{jk}$  on  $U_j \cap U_k \neq \emptyset$  and  $\alpha = d\eta_j$  on  $U_j$ .

**Proposition 3.6.** *Identify  $\mathfrak{g}^+$  to the complex Lie algebra  $(\mathfrak{g}, I)$ . Then we can take a basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{g}^+$  such that a map  $\psi: \mathfrak{g}^+ \rightarrow G$  defined by*

$$\psi\left(\sum_{i=1}^n z_i X_i\right) = (\exp z_1 X_1) \cdots (\exp z_n X_n)$$

*is biholomorphic. In particular,  $G$  is biholomorphic to  $\mathbb{C}^n$ . Moreover  $G$  has a system of coordinates  $(z_1, \dots, z_n)$  such that, for  $j=1, \dots, r$ ,  $z_j(gg') = z_j(g) + z_j(g')$  for any  $g, g' \in G$ , where  $r = \dim \mathfrak{g}^+ / [\mathfrak{g}^+, \mathfrak{g}^+]$ .*

**Proof.** We prove this proposition by induction on the dimension  $n$  of  $\mathfrak{g}^+$ . Assume that it has been proved for all dimensions  $< n$ . Since  $\mathfrak{g}^+$  is solva-

ble, it has an abelian ideal  $\mathfrak{a}^+$  of dimension  $> 0$ . Let  $A$  be the connected complex abelian subgroup of  $G$  whose Lie algebra is  $\mathfrak{a}^+$ ;  $A$  is simply connected and  $G/A$  is a simply connected complex solvable Lie group of complex dimension  $< n$ . Applying our proposition to  $G/A$ , we get a basis  $\{X_1^*, \dots, X_m^*\}$  of  ${}^+\mathfrak{g}/\mathfrak{a}^+$  such that a map  $\psi^*: \mathfrak{g}^+/\mathfrak{a}^+ \rightarrow G/A$  defined by

$$\psi^*\left(\sum_{i=1}^m z_i X_i^*\right) = (\exp z_1 X_1^*) \cdots (\exp z_m X_m^*)$$

is biholomorphic. Take elements  $X_1, \dots, X_m \in \mathfrak{g}^+$  such that  $\pi_*(X_i) = X_i^*$  where  $\pi_*: \mathfrak{g}^+ \rightarrow \mathfrak{g}^+/\mathfrak{a}^+$  is a projection. Choose also a basis  $\{X_{m+1}, \dots, X_n\}$  of  $\mathfrak{a}^+$ . Then every element of  $A$  can be written uniquely in the form  $(\exp z_{m+1} X_{m+1}) \cdots (\exp z_n X_n)$ . Let  $g$  be any element of  $G$  and  $g^* = \pi(g)$  where  $\pi: G \rightarrow G/A$  is a projection. Then we can write uniquely  $g^*$  in the form  $(\exp z_1 X_1^*) \cdots (\exp z_m X_m^*)$ . Hence, we have  $g = (\exp z_1 X_1) \cdots (\exp z_m X_m) a$  ( $a \in A$ ) and  $a$  can be written in the form  $(\exp z_{m+1} X_{m+1}) \cdots (\exp z_n X_n)$ , which proves that  $g$  is in the form  $(\exp z_1 X_1) (\exp z_2 X_2) \cdots (\exp z_n X_n)$ . Moreover  $z_1, \dots, z_m$  are uniquely determined by  $\pi(g)$  (and a fortiori by  $g$ ); hence  $a$  is determined by  $g$  and  $z_{m+1}, \dots, z_n$  are uniquely determined by  $g$ . Since  $\exp$  is holomorphic,  $z_j$  ( $j=1, \dots, n$ ) are holomorphic functions on  $G$  and  $\psi: \mathfrak{g}^+ \rightarrow G$  is biholomorphic.

Since we can choose a basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{g}^+$  in such a way that  $\{X_{r+1}, \dots, X_n\}$  is a basis of  $[\mathfrak{g}^+, \mathfrak{g}^+]$  and  $\psi: \mathfrak{g}^+ \rightarrow G$  is biholomorphic, the last assertion follows from the Campbell-Hausdorff formula ([4] p. 170). q.e.d.

We may assume that  $\omega_j = dz_j$  for  $j=1, \dots, r$  by changing a basis of  $\mathfrak{g}^+$  if necessary. Then by Proposition 3.4, we get

$$\alpha = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^r h_{jk} \omega_j \wedge \bar{\omega}_k = \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^r h_{jk} dz_j \wedge d\bar{z}_k$$

where  $(h_{jk})$  is a hermitian matrix.

#### 4. Divisors on a compact complex parallelisable solvmanifold

Let  $M$  and  $N$  be complex manifolds and  $\Phi: M \rightarrow N$  be a surjective holomorphic map. For a divisor  $\tilde{D}$  on  $N$ ,  $\Phi^*(\tilde{D})$  denotes the divisor on  $M$  defined by  $\Phi_x^{-1}(\tilde{D}_{\Phi(x)})$  for all  $x \in M$  ([15] Appendice n°7). We call this divisor  $\Phi^*(\tilde{D})$  on  $M$  the pull back of the divisor  $\tilde{D}$  on  $N$ . In this section we prove the following theorem.

**Theorem 2.** *Let  $G$  be a simply connected complex solvable Lie group. Let  $\Gamma$  be a lattice of  $G$ . Assume that  $\Gamma$  satisfies the condition (M) and that  $H_x^{0,1}(G/\Gamma) \cong H^1(\mathfrak{g}^-)$  (canonically). Then, for each positive divisor  $D$  on  $G/\Gamma$ , there exists a positive divisor  $\tilde{D}$  on the complex torus  $T$  such that the divisor  $D$  is the pull back of the divisor  $\tilde{D}$  on  $T$  by the projection  $\pi: G/\Gamma \rightarrow T$ , i.e.,  $D = \pi^* \tilde{D}$ .*



If  $G$  is nilpotent, the condition  $(M)$  is always satisfied ([11] Theorem 2.1). Moreover, by Theorem 1 in the section 2,  $H_{2i}^{0,i}(G/\Gamma) \cong H^i(\mathfrak{g}^-)$ . Thus we get:

**Corollary.** *Let  $G$  be a simply connected complex nilpotent Lie group and  $\Gamma$  be a lattice of  $G$ . Then the conclusion of Theorem 2 holds.*

Let  $D$  denote a positive divisor on  $G/\Gamma$ . Take a representative  $\{(U_i, f_i)\}$  of  $D$ , where  $f_i: U_i \rightarrow \mathbb{C}$  is a holomorphic function. Let  $L = \{D\}$  denote the holomorphic line bundle corresponding to the divisor  $D$ . ([15] Chapter V, n° 6). Let  $\{g_{j,k}\}$  denote the system of transition functions of  $L = \{D\}$  with respect to  $\{(U_i, f_i)\}$ . We then have  $f_j = g_{j,k} f_k$  on  $U_j \cap U_k \neq \emptyset$  by definition.

Let  $M$  be a complex manifold,  $\tilde{M}$  be the universal covering of  $M$  and  $p: \tilde{M} \rightarrow M$  be the covering map. Let  $\Pi$  denote the fundamental group  $\pi_1(M)$  of  $M$ .

A map  $j: \Pi \times \tilde{M} \rightarrow \mathbb{C}^*$  is said to be an automorphic factor if

- (1) the function  $z \rightarrow j(\sigma, z)$  is holomorphic for any  $\sigma \in \Pi$ , and
- (2)  $j(\sigma\tau, z) = j(\sigma, \tau(z)) \cdot j(\tau, z)$  for any  $\sigma, \tau \in \Pi$  and any  $z \in \tilde{M}$ .

Let  $f$  be a holomorphic function on  $\tilde{M}$  which is not identically zero.  $f$  is said to be automorphic of type  $j$  if

$$f(\sigma(z)) = j(\sigma, z)f(z) \quad \text{for } z \in \tilde{M} \text{ and } \sigma \in \Pi.$$

**Proposition 4.1.** *Let  $D$  be a positive divisor of  $G/\Gamma$ . Then  $D$  is the divisor of a holomorphic automorphic function  $\theta$  on  $G$ , for which the automorphic factor  $j(\gamma, g): \Gamma \times G \rightarrow \mathbb{C}^*$  is given by*

$$j(\gamma, g) = \exp 2\pi\sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} z_k(g) \bar{z}_l(\gamma) + C(\gamma) \right),$$

where  $H = (h_{j,k})$  is a hermitian matrix determined by the form  $\alpha$  in the Chern class  $C(L) = C(\{D\})$ :

$$\alpha = \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} dz_k \wedge d\bar{z}_l,$$

and  $C(\gamma) \in \mathbb{C}$  is a constant depending only on  $\gamma \in \Gamma$ .

Proof. Let us define  $\varphi_{i\gamma}(g)$  for  $g \in U_{i\gamma}$  by

$$\varphi_{i\gamma}(g) = \eta_i(p(g)) + \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} \bar{z}_l(g\gamma^{-1}) dz_k,$$

where  $\eta_i$  is the component of the connection introduced before. Then  $\varphi_{i\gamma}$  is an element of  $A^{1,0}(U_{i\gamma})$  satisfying  $d\varphi_{i\gamma} = 0$ . Since  $U_{i\gamma}$  is simply connected, there is a holomorphic function  $\psi_{i\gamma}$  satisfying  $d\psi_{i\gamma} = \varphi_{i\gamma}$ . Define  $\theta_{i\gamma}(g)$  for  $g \in U_{i\gamma}$  by

$$\theta_{i\gamma}(g) = f_i(p(g)) \exp 2\pi\sqrt{-1}(\psi_{i\gamma}(g)).$$

We then have

$$\theta_{i\gamma}(g) = \theta_{j\delta}(g) \exp 2\pi\sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} z_k(g) \bar{z}_l(\gamma\delta^{-1}) + C_{i\gamma, j\delta} \right)$$

on  $U_{i\gamma} \cap U_{j\delta}$ , where  $C_{i\gamma, j\delta} \in \mathbf{C}$  is a constant. Applying Proposition 3.6, we get

$$\frac{\sqrt{-1}}{2} d \log g_{ij}(p(g)) + \varphi_{i\gamma}(g) - \varphi_{j\delta}(g) = \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} (\bar{z}_l(\delta) - \bar{z}_l(\gamma)) dz_k.$$

Put  $a_{i\gamma, j\delta} = \exp 2\pi\sqrt{-1} C_{i\gamma, j\delta}$ .  $\{a_{i\gamma, j\delta}\}$  satisfies relations

$$(4.1) \quad a_{i\gamma, j\delta} \cdot a_{j\delta, k\nu} = a_{i\gamma, k\nu} \quad \text{on } U_{i\gamma} \cap U_{j\delta} \cap U_{k\nu} \neq \emptyset,$$

since

$$\begin{aligned} a_{i\gamma, j\delta} &= \exp 2\pi\sqrt{-1} C_{i\gamma, j\delta} \\ &= g_{ij}^{-1}(p(g)) \exp 2\pi\sqrt{-1} \left\{ (\psi_{i\gamma} - \psi_{j\delta}) + \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} (\bar{z}_l(\gamma) - \bar{z}_l(\delta)) \right\}. \end{aligned}$$

By the principal of monodromy ([15], Chapter V, n°1), there is a system of constant functions  $\{b_{i\gamma}\}$  such that

$$a_{i\gamma, j\delta} = b_{i\gamma}^{-1} \cdot b_{j\delta},$$

since  $G$  is simply connected and  $\{U_i\}$  is an open covering of  $G$ . We define a holomorphic function  $\theta$  on  $G$  by

$$\theta(g) = \theta_{i\gamma}(g) \exp 2\pi\sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} z_k(g) \bar{z}_l(\gamma) + b_{i\gamma} \right)$$

on  $g \in U_{i\gamma}$ .

We can see easily that  $\theta$  is well defined and  $\theta$  is different from zero.

Note that

$$\theta_{i\gamma}(g\gamma) = \theta_{i_0}(g) \exp 2\pi\sqrt{-1} d_{i\gamma}$$

for  $g \in U_{i_0}$ , where  $d_{i\gamma}$  is a constant. In fact, we have

$$d(R_\gamma^* \psi_{i\gamma}) - d\psi_{i_0} = R_\gamma^* \varphi_{i\gamma} - \varphi_{i_0} = 0 \quad \text{on } U_{i_0},$$

and

$$\psi_{i\gamma}(g\gamma) - \psi_{i_0}(g) = d_{i\gamma} \quad \text{on } U_{i_0}.$$

We now show that

$$\theta(g\gamma) = \theta(g) \cdot \exp 2\pi\sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{j,k=1}^r h_{jk} z_j(g) \bar{z}_k(\gamma) + C(\gamma) \right)$$

for  $g \in G$  and  $\gamma \in \Gamma$ , where  $C(\gamma)$  is a constant. For  $g \in U_{i_0}$ , we have

$$\begin{aligned} \theta(g\gamma) &= \theta_{i\gamma}(g\gamma) \cdot \exp 2\pi\sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} z_k(g\gamma) \bar{z}_l(\gamma) + b_{i\gamma} \right) \\ &= \theta_{i_0}(g) \cdot \exp 2\pi\sqrt{-1} \left\{ d_{i\gamma} + \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} z_k(g\gamma) \bar{z}_l(\gamma) + b_{i\gamma} \right\}. \end{aligned}$$

Since  $\theta(g) = \theta_{i_0}(g) \exp 2\pi\sqrt{-1} b_{i_0}$  on  $U_{i_0}$ , and since  $z_k(g\gamma) = z_k(g) + z_k(\gamma)$  by Proposition 3.6,

$$\theta(g\gamma) = \theta(g) \exp 2\pi\sqrt{-1} \left\{ \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} z_k(g) \bar{z}_l(\gamma) + C_i(\gamma) \right\}$$

for  $g \in U_{i_0}$ , where  $C_i(\gamma)$  is a constant. Since  $\theta(g\gamma)$  and

$$\theta(g) \exp 2\pi\sqrt{-1} \left\{ \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} z_k(g) \bar{z}_l(\gamma) + C_i(\gamma) \right\}$$

are holomorphic functions on  $G$ , we have

$$\theta(g\gamma) = \theta(g) \exp 2\pi\sqrt{-1} \left( \frac{1}{2\sqrt{-1}} \sum_{k,l=1}^r h_{kl} z_k(g) \bar{z}_l(\gamma) + C(\gamma) \right)$$

for  $g \in G$  and  $\gamma \in \Gamma$ . By the definition of  $\theta$ , we have  $p^*D = \text{div}(\theta)$ . q.e.d.

From now on, let  $e$  denote  $\exp 2\pi\sqrt{-1}$  and  $H(g_1, g_2) = \sum_{k,l=1}^r h_{kl} z_k(g_1) \bar{z}_l(g_2)$ .

Then  $j(\gamma, g) = e \left( \frac{1}{2\sqrt{-1}} H(g, \gamma) + C(\gamma) \right)$  for  $g \in G$  and  $\gamma \in \Gamma$ .

Since  $j(\gamma_1 \gamma_2, g) = j(\gamma_1, g) j(\gamma_2, g\gamma_1)$ , we get

$$C(\gamma_1 \gamma_2) \equiv C(\gamma_1) + C(\gamma_2) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2) \pmod{1}.$$

In particular,  $C(e) \in \mathbf{Z}$  and

$$C(\gamma^{-1}) \equiv -C(\gamma) + \frac{1}{2\sqrt{-1}} H(\gamma, \gamma) \quad \text{for } \gamma \in \Gamma.$$

**Lemma 4.2.**  $C(\gamma) \in \mathbf{R}$  for  $\gamma \in [\Gamma, \Gamma]$ .

Proof. Since  $[\Gamma, \Gamma] \subset [G, G]$ ,  $H(g, \gamma) = 0$  for  $\gamma \in [\Gamma, \Gamma]$  and  $g \in G$ . It is enough to show that  $C(\gamma) \in \mathbf{R}$  for  $\gamma = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$ ,  $\gamma_1, \gamma_2 \in \Gamma$ . In this case,

$$\begin{aligned} C(\gamma) &\equiv C(\gamma_1 \gamma_2) + C(\gamma_1^{-1} \gamma_2^{-1}) + \frac{1}{2\sqrt{-1}} H(\gamma_1 \gamma_2, \gamma_1^{-1} \gamma_2^{-1}) \\ &\equiv C(\gamma_1) + C(\gamma_2) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2) + C(\gamma_1^{-1}) + C(\gamma_2^{-1}) + \frac{1}{2\sqrt{-1}} H(\gamma_1, \gamma_2) \\ &\quad + \frac{1}{2\sqrt{-1}} \{ -H(\gamma_1, \gamma_1) - H(\gamma_2, \gamma_2) - H(\gamma_2, \gamma_1) - H(\gamma_1, \gamma_2) \} \end{aligned}$$

$$\equiv \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2) - H(\gamma_2, \gamma_1)) = \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2) - \overline{H(\gamma_1, \gamma_2)}) \in \mathbf{R}.$$

q.e.d.

**Proposition 4.3.**  $[\Gamma, \Gamma]$  is a lattice of  $[G, G]$  and  $[\Gamma, \Gamma]$  is a subgroup of finite index of  $\Gamma \cap [G, G]$ .

*Proof.* It follows from Proposition 3.2 that  $\Gamma[G, G]/[G, G]$  is a lattice of  $G/[G, G]$ . Since  $G/[G, G]$  is a vector group of dimension  $2r = \dim_{\mathbf{R}} \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ ,  $\Gamma[G, G]/[G, G] = \Gamma/\Gamma \cap [G, G]$  is a free abelian group of rank  $2r$ . On the other hand, since  $G$  is simply connected,  $\pi_1(G/\Gamma) = \Gamma$  and is  $\Gamma$  finitely generated. It follows that  $H_1(G/\Gamma, \mathbf{Z}) \cong \Gamma/[\Gamma, \Gamma]$ . Since  $\dim H^1(G/\Gamma, \mathbf{R}) = \dim H^1(\mathfrak{g}, \mathbf{R}) = \dim_{\mathbf{R}} \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 2r$  by a theorem of Mostow (cf. [8], [11] Corollary 7.29.),  $\Gamma/[\Gamma, \Gamma]$  is then direct sum of a free abelian group of rank  $2r$  and a finite group. The group  $(\Gamma \cap [G, G])/[\Gamma, \Gamma]$  is finite, because  $\Gamma/\Gamma \cap [G, G] \approx (\Gamma/[\Gamma, \Gamma])/(\Gamma \cap [G, G]/[\Gamma, \Gamma])$  is a free abelian group of rank  $2r$ . Since  $[G, G]/\Gamma \cap [G, G]$  is compact by Proposition 3.2,  $[G, G]/[\Gamma, \Gamma]$  is compact q.e.d.

**Proposition 4.4.**  $C(\gamma) \in \mathbf{Z}$  for  $\gamma \in [\Gamma, \Gamma]$ .

*Proof.* Let  $\theta$  be a holomorphic automorphic function on  $G$  of type  $j(\gamma, g)$ . We then have

$$\theta(g\gamma_1) = \theta(g)e(C(\gamma_1)) \quad \text{for } g \in G \text{ and } \gamma_1 \in [\Gamma, \Gamma].$$

Since  $\theta$  is not identically zero, there is a point  $g_0 \in G$  such that  $\theta(g_0) \neq 0$ .

Define a holomorphic function  $F: [G, G] \rightarrow \mathbf{C}$  by  $F(g_1) = \theta(g_0 g_1)$ . Then  $F$  is different from zero and satisfies  $F(g_1 \gamma_1) = \theta(g_0 g_1 \gamma_1) = \theta(g_0 g_1) e(C(\gamma_1)) = F(g_1) e(C(\gamma_1))$  for  $g_1 \in [G, G]$  and  $\gamma_1 \in [\Gamma, \Gamma]$  and  $F(e) \neq 0$ .

Let  $f: [G, G] \rightarrow \mathbf{R}$  denote  $C^\infty$ -function  $|F(g_1)|$ . Then  $f(g_1 \gamma_1) = f(g_1)$  for  $\gamma_1 \in [\Gamma, \Gamma]$  since  $C(\gamma_1) \in \mathbf{R}$  by Lemma 4.2.

We also denote by  $f$  the function on  $[G, G]/[\Gamma, \Gamma]$  induced by  $f: [G, G] \rightarrow \mathbf{R}$ . Since  $[\Gamma, \Gamma]$  is a lattice of  $[G, G]$ ,  $[G, G]/[\Gamma, \Gamma]$  is a compact complex manifold. Hence,  $f: [G, G]/[\Gamma, \Gamma] \rightarrow \mathbf{R}$  is bounded:

$$|F(g_1)| = f(g_1) = f(p(g_1)) \leq c$$

for some constant  $c > 0$ .

Since  $[G, G]$  is biholomorphic onto  $\mathbf{C}^m$ , a holomorphic bounded function  $F: [G, G] \rightarrow \mathbf{C}$  is constant. Since  $F(\gamma_1) = F(e) e(C(\gamma_1))$ ,  $C(\gamma_1) \in \mathbf{Z}$ . q.e.d.

Let  $A(g_1, g_2) = \frac{1}{2\sqrt{-1}}(H(g_1, g_2) - \overline{H(g_1, g_2)})$ . We then get

$$\begin{aligned} A(\gamma_1, \gamma_2) &= \frac{1}{2\sqrt{-1}}(H(\gamma_1, \gamma_2) - \overline{H(\gamma_1, \gamma_2)}) \\ &\equiv C(\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}) \equiv 0 \pmod{1}. \end{aligned}$$

We put  $d(\gamma) = C(\gamma) - \frac{1}{4\sqrt{-1}}H(\gamma, \gamma)$  for  $\gamma \in \Gamma$ . We have then

$$d(\gamma\delta) \equiv \frac{1}{2}A(\gamma, \delta) + d(\gamma) + d(\delta) \pmod{1} \text{ for } \gamma, \delta \in \Gamma.$$

Let  $\rho(\gamma)$  be the imaginary part of  $d(\gamma)$ . We see that  $\rho(\gamma\delta) = \rho(\gamma) + \rho(\delta)$  for  $\gamma, \delta \in \Gamma$ , that is,  $\rho: \Gamma \rightarrow \mathbf{R}$  is a homomorphism. It is clear that  $\text{Ker } \rho \supset [\Gamma, \Gamma]$ . Moreover we have  $\text{Ker } \rho \supset \Gamma \cap [G, G]$ , since  $[\Gamma, \Gamma]$  is a subgroup of finite index of  $\Gamma \cap [G, G]$ . Hence  $\rho$  induces a homomorphism  $\rho: \Gamma/\Gamma \cap [G, G] \rightarrow \mathbf{R}$ .

Since  $\pi(\Gamma) = \Gamma \cdot [G, G]/[G, G] \approx \Gamma/\Gamma \cap [G, G]$  and  $\pi(\Gamma)$  is a lattice of  $G/[G, G]$ ,  $\tilde{\rho}$  can be extended to a homomorphism from  $G/[G, G]$  to  $\mathbf{R}$ , so that  $\rho: \Gamma \rightarrow \mathbf{R}$  can be extended to a homomorphism  $\rho: G \rightarrow \mathbf{R}$ .

Consider now the biholomorphic map  $\Phi: G \rightarrow \mathbf{C}^n$  given by  $\Phi(g) = (z_1(g), \dots, z_n(g))$ . Let  $z_j(g) = x_j(g) + \sqrt{-1}y_j(g)$  for  $j = 1, \dots, r$ . Note that  $\Phi: G \rightarrow \mathbf{C}^n$  induces a map from  $G/[G, G]$  onto  $\mathbf{C}^r$  given by  $\pi(g) \rightarrow (z_1(g), \dots, z_r(g))$ . We can write  $\rho: G \rightarrow \mathbf{R}$  as

$$\rho(g) = \sum_{j=1}^r a_j x_j(g) + \sum_{j=1}^r b_j y_j(g)$$

for  $g \in G$ , where  $a_j, b_j \in \mathbf{R}, j = 1, \dots, r$ .

Define  $l: G \rightarrow \mathbf{C}$  by

$$l(g) = \sqrt{-1} \cdot \sum_{j=1}^r a_j z_j(g) + \sum_{j=1}^r b_j \bar{z}_j(g).$$

We have  $\text{Im } l(g) = \rho(g)$  and  $d(\gamma) - l(\gamma) \in \mathbf{R}$  for  $\gamma \in \Gamma$ .

Note that  $l: G \rightarrow \mathbf{C}$  is a holomorphic homomorphism.

Since we can regard  $A(g_1, g_2)$  as an alternating form on a vector group  $G/[G, G]$  such that  $A(g_1, g_2)$  takes integers on the lattice  $\pi(\Gamma)$ , there is a  $\mathbf{R}$ -bilinear form  $B$  which is  $\mathbf{Z}$ -valued on the lattice  $\pi(\Gamma)$  and  $A(g_1, g_2) = B(g_1, g_2) - B(g_2, g_1)$  ([15] Chapter VI, n°2).

Define  $\chi: \Gamma \rightarrow \{z \in \mathbf{C} \mid |z| = 1\}$  by

$$\chi(\gamma) = e\left(d(\gamma) - l(\gamma) - \frac{1}{2}B(\gamma, \gamma)\right).$$

$\chi$  is a character of  $\Gamma$ , since  $A(\gamma_1, \gamma_2) \in \mathbf{Z}$  for  $\gamma_1, \gamma_2 \in \Gamma$ . Put

$$\psi(\gamma) = \chi(\gamma)e\left(\frac{1}{2}B(\gamma, \gamma)\right) \text{ for } \gamma \in \Gamma.$$

We get

$$j(\gamma, g) = e\left(\frac{1}{2\sqrt{-1}}H(g, \gamma) + \frac{1}{4\sqrt{-1}}H(\gamma, \gamma) + l(\gamma)\right) \cdot \psi(\gamma)$$

for  $\gamma \in \Gamma$  and  $g \in G$ .

Since  $l(g): G \rightarrow \mathbf{C}$  is a holomorphic map which satisfies  $l(g\gamma) = l(g) + l(\gamma)$  for  $g \in G$  and  $\gamma \in \Gamma$ ,  $j(\gamma, g)$  is equivalent to the automorphic factor

$$e\left(\frac{1}{2\sqrt{-1}}H(g, \gamma) + \frac{1}{4\sqrt{-1}}H(\gamma, \gamma)\right)\psi(\gamma).$$

We need the following proposition to show that  $\psi|_{\Gamma \cap [G, G]} = id$ .

**Proposition 4.5.** *Let  $\theta$  be a holomorphic automorphic function on  $G$  of type*

$$j(\gamma, g) = e\left(\frac{1}{2\sqrt{-1}}H(g, \gamma) + \frac{1}{4\sqrt{-1}}H(\gamma, \gamma)\right) \cdot \psi(\gamma).$$

*Then the hermitian form  $H = (h_{jk})$  is non-negative. Moreover  $\theta(g \cdot g_0) = \theta(g)$  for  $g \in G$ , if  $g_0 \in G$  satisfies  $H(g_0, g_0) = 0$ .*

*Proof.* Let  $f: G \rightarrow \mathbf{R}$  denote the function defined by

$$f(g) = |\theta(g)|^2 e\left(\frac{-1}{2\sqrt{-1}}H(g, g)\right) = |\theta(g)|^2 \exp(-\pi H(g, g)).$$

We have  $f(g\gamma) = f(g)$  for  $\gamma \in \Gamma$ , so that  $f$  induces a function  $F: G/\Gamma \rightarrow \mathbf{R}$ . Since  $G/\Gamma$  is compact, there is a constant  $c > 0$  such that  $0 \leq F(p(g)) \leq c$  for  $g \in G$ . Therefore we get

$$f(g) = |\theta(g)|^2 \exp(-\pi H(g, g)) \leq c \quad \text{for } g \in G.$$

Thus we have

$$|\theta(g)|^2 \leq c \exp \pi H(g, g) \quad \text{for } g \in G.$$

Suppose that  $H(g_1, g_1) < 0$  for some  $g_1 \in G$ . Define  $g(\tau) \in G$  ( $\tau \in \mathbf{C}$ ) by

$$g(\tau) = \Phi^{-1}(\tau z_1(g_1) + z_1(g), \dots, \tau z_n(g_1) + z_n(g)).$$

Then we have  $g(0) = g$  and

$$|\theta(g(\tau))|^2 \leq c \exp \pi H(g(\tau), g(\tau)).$$

Put  $\rho = H(g(\tau), g(\tau))$ .

$$\begin{aligned} \rho &= \sum_{j,k=1}^r h_{jk}(\tau z_j(g_1) + z_j(g)) \cdot \overline{(\tau z_k(g_1) + z_k(g))} \\ &= |\tau|^2 \cdot \sum_{j,k=1}^r h_{jk} z_j(g_1) \bar{z}_k(g_1) + 2 \operatorname{Re}(\tau H(g_1, g)) + H(g, g) \\ &= |\tau|^2 H(g_1, g_1) + 2 \operatorname{Re}(\tau H(g_1, g)) + H(g, g). \end{aligned}$$

For any  $\varepsilon > 0$ , there is  $R > 0$  such that  $\pi \rho \leq \log \varepsilon$  for every  $\tau$  satisfying  $|\tau| \geq R$ .

Fix  $g_1, g \in G$ , and we have

$$|\theta(g(\tau))|^2 \leq c\varepsilon \quad \text{for } |\tau| \geq R.$$

Therefore  $\theta(g(\tau))$  is a bounded holomorphic function on  $\mathbf{C}$ . Hence  $\theta(g(\tau))$  is constant with respect to  $\tau \in \mathbf{C}$ . Tending  $\varepsilon \rightarrow 0$ , we get  $|\theta(g(\tau))|^2 = 0$ . In particular,

$$|\theta(g)|^2 = |\theta(g(0))|^2 = 0.$$

Hence  $\theta \equiv 0$  on  $G$ , since  $g$  can be any element of  $G$ . This is a contradiction. Therefore  $H = (h_{j\bar{k}})$  is a non-negative hermitian form.

Take an element  $g_0 \in G$  satisfying  $H(g_0, g_0) = 0$ . Then we have  $H(g, g_0) = 0$  for any  $g \in G$  since  $H(g, g) \geq 0$  for any  $g \in G$ . Put

$$g_0(\tau) = \Phi^{-1}(\tau z_1(g_0), \dots, \tau z_n(g_0)) \in G$$

for  $\tau \in \mathbf{C}$ . Then we have

$$\begin{aligned} |\theta(g \cdot g_0(\tau))|^2 &\leq c \cdot \exp \tau H(g \cdot g_0(\tau), g \cdot g_0(\tau)) \\ &= c \cdot \exp \pi(H(g, g) + 2 \operatorname{Re} \tau H(g, g_0) + |\tau|^2 H(g_0, g_0)) \\ &= c \cdot \exp \pi H(g, g). \end{aligned}$$

This shows that  $\theta(g \cdot g_0(\tau))$  is a bounded holomorphic function with respect to  $\tau \in \mathbf{C}$ . Hence  $\theta(g \cdot g_0(\tau))$  is constant with respect to  $\tau \in \mathbf{C}$ . In particular,  $\theta(g) = \theta(g \cdot g_0(0)) = \theta(g \cdot g_0(1)) = \theta(g \cdot g_0)$ . q.e.d.

Take an element  $g_1 \in G$  satisfying  $\theta(g_1) \neq 0$ . Since  $H(g_0, g_0) = 0$  for  $g_0 \in [G, G]$ ,  $\theta(g g_0) = \theta(g)$  for  $g \in G$ . In particular,  $\theta(g \cdot \gamma) = \theta(g)$  for  $\gamma \in \Gamma \cap [G, G]$ . Put  $g = g_1 \gamma^{-1}$ . Then  $0 \neq \theta(g_1) = \theta(g \cdot \gamma) = \theta(g)$ . Since

$$\theta(g \cdot \gamma) = \theta(g) \cdot e \left( \frac{1}{2\sqrt{-1}} H(g, \gamma) + \frac{1}{4\sqrt{-1}} H(\gamma, \gamma) \right) \psi(\gamma) = \theta(g) \psi(\gamma),$$

$\psi(\gamma) = 1$ , for  $\gamma \in \Gamma \cap [G, G]$ . Note that  $B(g, g) = 0$  for  $g \in [G, G]$ . Hence,  $\chi: \Gamma \rightarrow \{z \in \mathbf{C} \mid |z| = 1\}$  satisfies that

$$\chi|_{\Gamma \cap [G, G]} \equiv 1.$$

Since  $\pi(\Gamma) \cong \Gamma / \Gamma \cap [G, G]$ ,  $\chi$  induces a character

$$\tilde{\chi}: \pi(\Gamma) \rightarrow \{z \in \mathbf{C} \mid |z| = 1\}.$$

Let  $\Theta: G/[G, G] \rightarrow \mathbf{C}$  denote the holomorphic function on  $G/[G, G]$  induced by  $\theta: G \rightarrow \mathbf{C}$  and  $\tilde{j}: \pi(\Gamma) \times G/[G, G] \rightarrow \mathbf{C}^*$  the automorphic factor induced by  $j: \Gamma \times G \rightarrow \mathbf{C}^*$ .

Denote  $\tilde{D}$  the divisor on  $(G/[G, G])/\pi(\Gamma)$  denfied by the holomorphic automorphic function  $\Theta$  on  $G/[G, G]$ . We then get  $D = \pi^* \tilde{D}$ . Therefore we have proved Theorem 2.

Let  $D$  be a divisor on  $G/\Gamma$ . Then there exist positive divisors  $D^+, D^-$  on  $G/\Gamma$  such that  $D^+$  and  $D^-$  are relatively prime and  $D=D^+-D^-$  ([15], Appendix n°6). By Theorem 2, there are holomorphic theta functions  $\Theta_1, \Theta_2$  on the complex torus  $T$  such that  $D^+=\pi^*(\text{div } \Theta_1)$  and  $D^-=\pi^*(\text{div } \Theta_2)$ . Since  $\pi: G/\Gamma \rightarrow T$  is onto holomorphic,

$$\begin{aligned} D &= D^+ - D^- = \pi^*(\text{div } \Theta_1) - \pi^*(\text{div } \Theta_2) \\ &= \pi^* \text{div} \left( \frac{\Theta_1}{\Theta_2} \right). \end{aligned}$$

Note that  $\frac{\Theta_1}{\Theta_2}$  is a meromorphic theta function on the complex torus  $T$ .

It is easy to see that if the divisor  $D=0$  the corresponding automorphic function  $\theta$  is trivial.

Take a meromorphic function  $\psi$  on  $G/\Gamma$ . Let  $D = \text{div}(\psi)$ . Since  $D = \pi^* \text{div} \left( \frac{\Theta_1}{\Theta_2} \right)$ , we get that  $\psi = \frac{\Theta_1 \circ \pi}{\Theta_2 \circ \pi}$ . Since  $\psi(g\gamma) = \psi(\gamma)$  for  $g \in G, \gamma \in \Gamma$ ,  $\frac{\Theta_1 \circ \pi(g\gamma)}{\Theta_2 \circ \pi(g\gamma)} = \frac{\Theta_1 \circ \pi(g)}{\Theta_2 \circ \pi(g)}$ , hence  $\frac{\Theta_1}{\Theta_2}$  is a meromorphic function on  $T$ . Thus we get that if  $\psi$  is a meromorphic function on  $G/\Gamma$ , there is a meromorphic function  $\tilde{\psi}$  on the torus  $T$  such that  $\psi = \pi^* \tilde{\psi}$ .

Let  $K(G/\Gamma)$  (resp.  $K(T)$ ) denote the field of all meromorphic functions on  $G/\Gamma$  (resp. on  $T$ ).

We now get the following corollary of Theorem 2.

**Corollary** *Under the assumptions of Theorem 2, there is a canonical isomorphism  $\pi^*: K(T) \rightarrow K(G/\Gamma)$ . In particular, the transcendence degree of  $K(G/\Gamma)$  over  $C$  is not more than the complex dimension of complex torus  $T$ .*

### 5. Remarks and examples of compact complex parallelisable nilmanifolds

**Proposition 5.1.** *Let  $M$  be a compact complex parallelisable manifold of complex dimension 2. Then  $M$  is a complex torus.*

*Proof.* By a theorem of Wang [14],  $M=G/\Gamma$  where  $G$  is a simply connected complex Lie group of dimension 2 and  $\Gamma$  is a lattice of  $G$ . Let  $\{X_1, X_2\}$  be a basis of  $\mathfrak{g}^+$  and  $\{\omega_1, \omega_2\}$  be the dual basis of  $(\mathfrak{g}^+)^*$ . We may consider  $\omega_1, \omega_2$  as holomorphic 1-forms on  $G/\Gamma$ . Since  $G/\Gamma$  is 2 dimensional,  $\omega_1, \omega_2$  are  $d$ -closed;  $d\omega_1 = d\omega_2 = 0$ . Thus  $[X_1, X_2] = 0$ . Hence,  $G$  is abelian and  $G/\Gamma$  is a complex torus. q.e.d.

Now we shall give some examples of compact complex parallelisable nilmanifolds.



(1) Let  $G$  be a simply connected complex nilpotent Lie group defined by

$$G = \left\{ \left( \begin{array}{cccc} 1 & z_{12} & \dots & z_{1n} \\ & 1 & \dots & z_{2n} \\ & & \ddots & \vdots \\ & & & 1 & z_{n-1n} \\ 0 & & & & 1 \end{array} \right) \mid z_{ij} \in \mathbf{C}, i < j \right\}$$

and  $\Gamma$  be a lattice of  $G$  defined by

$$\Gamma = \left\{ \left( \begin{array}{cccc} 1 & a_{12} & \dots & a_{1n} \\ & 1 & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & 1 & a_{n-1n} \\ 0 & & & & 1 \end{array} \right) \mid a_{ij} \in \mathbf{Z} + \sqrt{-1}\mathbf{Z}, i < j \right\}.$$

Then  $G/\Gamma$  is a compact complex parallelisable nilmanifold. In this case, we see that the transcendence degree of  $K(G/\Gamma)$  over  $\mathbf{C}$  is  $n-1$ .

(2) Let  $G$  be a simply connected complex nilpotent Lie group defined by

$$G = \left\{ \left( \begin{array}{cccc} 1 & z_1 & z_2 & \dots & z_{n-1} & w \\ & 1 & 0 & \dots & 0 & y_{n-1} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & 1 & \dots & 0 \\ & & & & 1 & 0 \\ 0 & & & & & 1 & y_1 \\ & & & & & & 1 \end{array} \right) \mid \begin{array}{l} z_j, y_j, w \in \mathbf{C} \\ j = 1, 2, \dots, n-1 \end{array} \right\}$$

for  $n \geq 2$ , and  $\Gamma$  be a lattice of  $G$  defined by

$$\Gamma = \left\{ \left( \begin{array}{cccc} 1 & a_1 & a_2 & \dots & a_{n-1} & c \\ & 1 & 0 & \dots & 0 & b_{n-1} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & 1 & \dots & 0 \\ & & & & 1 & 0 \\ 0 & & & & & 1 & b_1 \\ & & & & & & 1 \end{array} \right) \mid \begin{array}{l} a_j, b_j, c \in \mathbf{Z} + \sqrt{-1}\mathbf{Z} \\ j = 1, 2, \dots, n-1 \end{array} \right\}.$$

Then  $G/\Gamma$  is a compact complex parallelisable nilmanifold. In this case, we see that the transcendence degree of  $K(G/\Gamma)$  over  $C$  is  $2(n-1)$ .

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