ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 30, nº 1 (1980), p. 129-157

<http://www.numdam.org/item?id=AIF_1980__30_1_129_0>

Ann. Inst. Fourier, Grenoble
30, 1 (1980), 129-157

ON COMPACT, HOMOGENEOUS SYMPLECTIC MANIFOLDS

by Ph. B. ZWART and W. M. BOOTHBY (*)

Dedicated to Professor S.S. Chern.
(On the occasion of the Chern Symposium, June 1979).

1. Introduction.

In this paper we study compact homogeneous spaces of Lie groups which have a symplectic structure which is invariant under the group action. Such objects have been studied quite extensively since their relation to the representations of nilpotent Lie groups was discovered by Kirillov (see e.g. Pukanszky [17]). Especially noteworthy in this respect is the work of Kostant [11] and Souriau [18], where many basic theorems of classification and characterization of homogeneous symplectic manifolds are obtained and applied to representation theory. In the present work the approach and the methods are somewhat different from those above. Almost no assumption is made about the Lie group G which acts transitively on the symplectic manifold M except that it is *connected.* In particular it is not assumed to be simply connected, to be semi-simple, nor to have any particular cohomology properties. We do however suppose that M is *compact,* which is not the case in much of the work cited above (and below); this assumption is essential to the methods used. Finally, it is not assumed that the action of G on M is effective, but merely *almost effective.* These properties, together with the existence on M of a G-invariant, closed, exterior two-form Ω_M of maximum rank, i.e. the invariant symplectic form,

(*) This paper is based in part upon work supported by the National Science Foundation under Grant No. ENG78-22166.

are sufficient, it turns out, to give a fairly complete description of G, M and the isotropy group K in terms of known results of Borel [3] (also Lichnerowicz [12], [13] and Matsushima [14]), and of the results obtained here in the solvable case.

Many of the basic results of this paper were obtained (with somewhat different proofs) in the Ph.D. dissertation [21] of one of the authors, Philip Zwart, written under the direction of the other author. For various reasons –an important one being that the computational complexity and length of some of the original proofs made it difficult to disengage any basic underlying principle– the results were never submitted to a mathematical journal for publication despite the fact that the authors, at least, both feel the results to be of interest. Research in various aspects of symplectic manifolds has, if anything, increased since then: a very nice summary of recent work may be found in the Alan Weinstein's notes [20] of the NSF-CBMS Regional Conference on Symplectic Manifolds at the University of North Carolina. There have also been some further recent interesting papers on the homogeneous case: B-Y Chu [6] and S. Sternberg [19]. With this activity in mind we hope that the present publication of these results is still timely.

In the present paper the results of Zwart [21] have not only been extended, but the proofs have been completely reworked in a number of ways. Although Section 2 is introductory and standard, most of Section 3 is new and contains some essential ideas for the later analysis. It is hoped that the ideas there can be applied to other instances of invariant forms on compact homogeneous spaces. In fact, a paper is in preparation applying similar techniques to the cosymplectic and contact cases. Section 4 applies the results to the general symplectic case and with Section 5 shows that the homogeneous manifold M can be split –as a homogeneous symplectic space– into a cartesian product of a compact homogeneous symplectic space of a compact semi-simple Lie group (a case already studied by Borel [3]) and a compact symplectic solvmanifold. The last two sections are devoted to a study of this latter case.

2. Notation and generalities.

Throughout this article we consider a connected Lie group, usually denoted G, acting transitively and almost effectively (on the left) on a connected manifold, say M. (All data are C^∞.) It is assumed that a fixed basepoint is chosen so that M is identified with G/K, the space of left cosets of K the isotropy group of the basepoint. The canonical projection of G onto G/K is denoted $\pi : G \longrightarrow G/K$ and the assumption of almost effective action is equivalent to the statement that K contains no connected normal subgroup of G or that the Lie algebra $\mathfrak{k}$ of K contains no ideal of $\mathfrak{g}$, the Lie algebra of G.

It is an important aspect of the results obtained that we make no assumption that $M = G/K$ is simply connected —or even has vanishing first Betti number. But as to G itself, let $F : \widetilde{G} \longrightarrow G$ be its universal covering group; since M may be naturally identified with $\widetilde{G}/\widetilde{K}$, where $\widetilde{K} = F^{-1}(K)$, there is no loss of generality in supposing that the group G is simply connected, even though we do not wish to assume M simply connected. Composing $F : \widetilde{G} \longrightarrow G$ with the action of G on M gives again a transitive, almost effective action.

We are interested in studying certain differential forms on M which are G-invariant. Let $\theta_M \in \Lambda^p(M)$ and $\theta = \pi^*\theta_M \in \Lambda^p(G)$, then we have the following (e.g. see [2]).

(2.1) *If* θ_M *is G-invariant, then*

(i) θ *is invariant under left translations, i.e.* $\theta \in \Lambda^p(\mathfrak{g})$,

(ii) $\mathrm{Ad}x^*\theta = \theta$ *for all* $x \in K$, *and*

(iii) $i_X\theta = 0$ *for all* $X \in \mathfrak{k}$.

Conversely, any form θ *on* G *satisfying (i), (ii) and (iii) is the image under* π^* *of a unique* G*-invariant form of* G/K, *which we call the induced form on* G/K.

We also remark that π^* is an isomorphism and θ is closed if and only if θ_M is closed. In fact we have the following useful formula (see, e.g. [10]) for the value of $d\theta$, θ a left-invariant p-form on G :

$$(2.2)\quad d\theta(X_0, X_1, \dots, X_p) = \frac{1}{p+1} \sum_{i<j} (-1)^{i+j} \theta([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p)$$

for all $X_0, X_1, \dots, X_p \in \mathfrak{g}$. Thus θ is closed if and only if the sum on the right is zero.

If θ is a left-invariant form on G, i.e. a form on $\mathfrak{g}$, we denote by H_θ –or (more usually) H– the closed subgroup

$$(2.3)\qquad H_\theta = \{x \in G \mid Adx^*\theta = \theta\}.$$

Clearly, in the circumstance that $\theta = \pi^*\theta_M$ discussed above, $H = H_\theta \supset K$ by (2.1) (ii), but they need not coincide. Even their Lie algebras may be distinct. In fact, if L_X denotes the Lie derivative with respect to $X \in \mathfrak{g}$,

$$(2.4)\qquad \mathfrak{h} = \{X \in \mathfrak{g} \mid L_X\theta = 0\} = \{X \in \mathfrak{g} \mid (ad\,X)^*\theta = 0\}$$

and

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid i_X\theta = 0\}.$$

If we assume further that θ is *closed*, then the relation $L_X\theta = di_X\theta + i_Xd\theta$ allows us to rewrite these characterizations as follows:

$$(2.5)\quad \mathfrak{h} = \{X \in \mathfrak{g} \mid di_X\theta = 0\} \text{ and } \mathfrak{k} = \{X \in \mathfrak{g} \mid i_X\theta = 0\};$$

thus $d\theta = 0$ implies $\mathfrak{h} \supset \mathfrak{k}$.

We shall be particularly interested in the case of a left invariant 2-form Ω on G. If Z is a fixed element of $\mathfrak{g}$, then $di_Z\Omega = 0$ if and only if $di_Z\Omega(X, Y) = 0$ for all $X, Y \in \mathfrak{g}$, i.e.

$$0 = di_Z\Omega(X, Y) = \Omega(Z, [X, Y]) \;\forall X, Y \in \mathfrak{g}.$$

It may happen that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ in which case we have for $\Omega \in \Lambda^2(\mathfrak{g})$ and *closed*:

$$(2.6)\quad \text{If } [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}, \text{ then } \mathfrak{h} = \{Z \in \mathfrak{g} \mid i_Z\Omega = 0\} = \mathfrak{k}.$$

Finally we consider the case of a left-invariant form θ on G when $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m$, i.e. its Lie algebra decomposes into a direct sum of ideals. An example is a semi-simple G and some $\theta \in \Lambda^p(\mathfrak{g})$. Let $p_i : \mathfrak{g} \longrightarrow \mathfrak{g}_i$ and $q_i : \mathfrak{g}_i \longrightarrow \mathfrak{g}$ be the corresponding projection and injection of Lie algebras. We let $\theta_i' = q_i^*\theta \in \Lambda(\mathfrak{g}_i)$ and $\theta_i = p_i^*\theta_i' \in \Lambda(\mathfrak{g})$. If $X = X_1 + \dots + X_m$ and $Y = Y_1 + \dots + Y_m$

in $\mathfrak{g}$ with $X_i, Y_i \in \mathfrak{g}_i$, we could write –somewhat ambiguously–

$$\theta_i'(X_i, Y_i) = \theta_i(X_i, Y_i)$$

if we make suitable identifications. We are particularly interested in the following case:

(2.7) DEFINITION. – *θ is said to be decomposable if $\theta = \theta_1 + \ldots + \theta_m$.*

(2.8) *Remark.* – It is easily seen that θ is decomposable if and only if it vanishes whenever two of its arguments are from different ideals $\mathfrak{g}_i, \mathfrak{g}_i$ $i \neq j$. For example if $\theta \in \Lambda^2(\mathfrak{g})$, θ is decomposable if and only if $\theta(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ whenever $i \neq j$.

The following facts, stated as lemmas, will be useful to us.

(2.9) LEMMA. – *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ is the Lie algebra of a Lie group G and $\theta \in \Lambda^p(\mathfrak{g})$ is decomposable, then the algebra $\mathfrak{h}$ of the subgroup $H(= H_\theta)$ has a compatible decomposition $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_m, \mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$, $i = 1, \ldots, m$. Moreover, if G decomposes in to a corresponding direct product $G = G_1 \times \ldots \times G_m$, then so does H, i.e. $H = H_1 \times \ldots \times H_m$ with $H_i = H \cap G_i$.*

Proof. – The Lie algebra statement follows from the decomposability for groups (we pass, if necessary, to the simply connected covering group of G). Therefore we check only the last statement.

If $x \in H \subset G$, then $x = x_1 \ldots x_m$, uniquely, with $x_i \in G_i$ and $x_i x_j = x_j x_i$ for $i \neq j$. The ideals $\mathfrak{g}_i$ are invariant under the adjoint action and in fact $Ad\, x_j$ is the identity on $\mathfrak{g}_i$ if $i \neq j$. We must show that each x_i is in H, i.e. that $Ad\, x_i^* \theta = \theta$. Since $\theta = \theta_1 + \ldots + \theta_m$, it is enough to see that $Ad\, x_i^* \theta_j = \theta_j$, $i, j = 1, \ldots, m$. Recall that $\theta_j(Z^{(1)}, \ldots, Z^{(p)}) = 0$ unless each argument is in $\mathfrak{g}_j$ and that

$$Ad\, x_i^* \theta_j(Z^{(1)}, \ldots, Z^{(p)}) = \theta_j(Ad\, x_i Z^{(1)}, \ldots, Ad\, x_i Z^{(p)}).$$

If $i \neq j$, then it is clear that $Ad\, x_i^* \theta_j = \theta_j$. Suppose $i = j$ and $Z^{(1)}, \ldots, Z^{(p)} \in \mathfrak{g}_i$. Then

$$\begin{aligned} Ad\, x_i^* \theta_i(Z^{(1)}, \ldots, Z^{(p)}) &= \theta_i(Ad\, x_i Z^{(1)}, \ldots, Ad\, x_i Z^{(p)}) \\ &= \theta_i(Ad\, x\, Z^{(1)}, \ldots, Ad\, x\, Z^{(p)}) = \theta(Ad\, x\, Z^{(1)}, \ldots, Ad\, x\, Z^{(p)} \\ &= Ad\, x^* \theta(Z^{(1)}, \ldots, Z^{(p)}) = \theta(Z^{(1)}, \ldots, Z^{(p)}) = \theta_i(Z^{(1)}, \ldots, Z^{(p)}), \end{aligned}$$

where we have used the fact that $\theta_j = 0$ on $Z^{(1)}, \ldots, Z^{(p)}$ unless $j = i$ and that $\mathrm{Ad}\, x_i \mathrm{Ad}\, x_j = \mathrm{Ad}\, x_j \mathrm{Ad}\, x_i$, etc. This completes the proof.

3. Preliminary lemmas.

In this section we collect some results which will be used in studying invariant forms on homogeneous spaces. Throughout G denotes a connected Lie group.

(3.1) LEMMA. – *Suppose* G *acts on a manifold* M *(not necessarily transitively) and that* K *is the isotropy subgroup of* $x_0 \in M$. *If* A *is any subgroup of* G *such that* $A(x_0)$, *the orbit of* x_0, *is closed in* M *and such that* K *normalizes* A, *then* $AK = KA$ *is a closed subgroup of* G.

Proof. – Let F denote the orbit $A(x_0)$ and define the subgroup $G_F = \{g \in G \mid g(F) = F\}$. If $\{g_n\}$ is a convergent sequence of elements of G_F with limit $\bar{g} \in G$ and if x is any element of F, then $\lim g_n x = \bar{g}x$ by continuity of the action of G on M. However $\{g_n x\} \subset F$ and F is closed, thus $\bar{g}x \in F$. Since x is arbitrary in F, $\bar{g}(F) = F$, therefore G_F is a closed subgroup of G.

On the other hand, $G_F = AK$. For if $ak \in AK$, and $x = a'x_0$ is any element of $F = A(x_0)$, then

$$(aka')x_0 = (aa''k)x_0 = (aa'')x_0 \in A(x_0).$$

This implies that $AK \subset G_F$. Further, if $g \in G_F$, then $gx_0 \in F$, i.e. $gx_0 = ax_0$ for some $a \in A$. It follows that $(a^{-1}g)x_0 = x_0$, i.e. $a^{-1}g \in K$ and $g \in AK$, or $G_F \subset AK$. Therefore $G_F = AK$, which completes the proof.

(3.2) LEMMA. – *Let* ρ, V *be a representation of the connected Lie group* A *on* V, *a finite dimensional vector space. Suppose that* $\rho(A)$ *is a unipotent subgroup of* $G\ell(V)$, *or equivalently that* $\rho_*(\mathfrak{a})$, *the image of the Lie algebra* $\mathfrak{a}$ *of* A *under the induced homomorphism, consists of nilpotent endomorphisms of* V. *Then for any* $v_0 \in V$, *the orbit* $\rho(A)(v_0)$ *is closed in* V.

Proof. – Let $A_1 = \rho(A)$, it is a connected, unipotent subgroup of $G\ell(V)$. According to § 3, page 50, of Pukansky [17], there is a basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$ of V such that the orbit of v_0 consists of the following subset

$$A_1(v_0) = \{w \in V \mid w = z_1 \mathbf{f}_1 + \ldots + z_d \mathbf{f}_d + P_{d+1}(z)\mathbf{f}_{d+1} + \ldots \\ \ldots + P_n(z)\mathbf{f}_n, P_i \text{ polynomials in } z = (z_1, \ldots, z_d) \in \mathbf{R}^d\}$$

(we have renumbered Pukansky's basis somewhat). Thus the orbit is the graph of a continuous mapping of $\mathbf{R}^d$ into $\mathbf{R}^{n-d}$, hence is closed.

(3.3) LEMMA. – *Let* ρ, V *be a finite dimensional representation of the connected Lie group* G *and let* v_0 *be a non-zero vector of* V *whose orbit is compact. If* A *is a connected, normal subgroup such that* $\rho(A)$ *is unipotent in* $G\ell(V)$, *then* v_0 *is fixed by* $\rho(A)$, *or equivalently,* $\rho_*(\mathfrak{a})v_0 = 0$.

Proof. – We denote by H_{v_0} the isotropy group of v_0 under the induced action of G on V. By our assumption G/H_{v_0} is compact and we must show that $A \subset H_{v_0}$. By the preceding lemma $\rho(A)(v_0)$ is closed and hence by (3.1) $AH_{v_0} = H_{v_0}A$ is a closed subgroup of G. Thus AH_{v_0}/H_{v_0} is compact and hence $A/A \cap H_{v_0}$ is compact. Let $A' = \rho(A)$, $G' = \rho(G)$ and $H'_{v_0} = \rho(H_{v_0})$. We know that these maps are onto and thus A' is a normal, unipotent subgroup of G'. By the previous two lemmas $AH_{v_0} = H_{v_0}A$ and $A'H'_{v_0} = H'_{v_0}$ are closed in G, G' respectively. G'/H'_{v_0} is compact so $A'H'_{v_0}/H'_{v_0} = A'/A' \cap H'_{v_0}$ is compact, in fact $A'/A' \cap H'_{v_0}$ is a compact nilmanifold. It follows that there is a basis $X'_1, \ldots, X'_r$ of $\mathfrak{a}'$ such that $\exp X'_i \in H'_{v_0}$ (Malcev [16]). Thus for any $n \in \mathbf{Z}$, $\exp n X'_i \in H_{v_0}$ for $i = 1, \ldots, r$. Now X'_i is a nilpotent endomorphism of V and therefore $\exp t X'_i$ is a polynomial in t, $P_i(t)$, with coefficients which are endomorphisms of V. We have seen that $P_i(n) \in H'_{v_0}$, i.e. $P_i(n)v_0 = v_0$, for all $n \in \mathbf{Z}$, but it must then be true that $P_i(t) = \exp t X'_i$ leaves v_0 fixed for all $t \in \mathbf{R}$. This means that $X'_i v_0 = 0$, $i = 1, \ldots, r$ and thus $\rho_*(\mathfrak{a}).v_0 = \mathfrak{a}'.v_0 = 0$. From this it follows that $\rho(A)v_0 = v_0$, or $A \subset H_{v_0}$ as claimed.

In the next section these lemmas will be applied to the adjoint representation of G on the space of left invariant p-forms on G.

More precisely we will consider the following situation: let $\theta \in \Lambda^p(\mathfrak{g})$, i.e. a left invariant p-form on G, and let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_p$ be ideals of $\mathfrak{g}$, thus Ad G-invariant. We let $\theta_{\mathfrak{a}_1,\ldots,\mathfrak{a}_p}$ denote the restriction of θ to $\mathfrak{a}_1 \times \ldots \times \mathfrak{a}_p$ and denote by $\Sigma(\mathfrak{a}_1, \ldots, \mathfrak{a}_p)$ the subspace of $\Lambda^p(\mathfrak{g})$ consisting of all such restricted p-forms. Then G acts on $\Sigma(\mathfrak{a}_1, \ldots, \mathfrak{a}_p)$ as follows:

$$(Ad\, x)^* \theta_{\mathfrak{a}_1,\ldots,\mathfrak{a}_p}(X_1, \ldots, X_p) = \theta_{\mathfrak{a}_1,\ldots,\mathfrak{a}_p}(Ad(x) X_1, \ldots, Ad(x) X_p), \tag{3.4}$$

where $x \in G$, $X_i \in \mathfrak{a}_i$, $i = 1, \ldots, p$. The correspondence $x \longrightarrow (Ad\, x)^*$ is a representation ρ of G on the vector space $V = \Sigma(\mathfrak{a}_1, \ldots, \mathfrak{a}_p)$ in the terminology of the lemmas. Given $\theta \in \Lambda^p(\mathfrak{g})$, V_θ will denote the smallest $\rho(G)$-invariant subspace of V which contain θ. These conventions are used in the sequel.

4. Compact, homogeneous symplectic manifolds.

As a first application of the preceding ideas we consider a G-invariant 2-form Ω_M on a compact, connected homogeneous space $M = G/K$. Then $\Omega = \pi^* \Omega_M \in \Lambda^2(\mathfrak{g})$, and we define $H = H_\Omega$ as in (2.3), since $H \supset K$, G/H is also compact.

(4.1) LEMMA. – *Let* $\mathfrak{a}$, $\mathfrak{b}$ *be ideals of* $\mathfrak{g}$ *and let* $\mathfrak{n}$ *be its nilradical. Define* $H_{\mathfrak{a},\mathfrak{b}} = \{x \in G \mid (Ad\, x)^* \Omega_{\mathfrak{a},\mathfrak{b}} = \Omega_{\mathfrak{a},\mathfrak{b}}\}$. *Then the Lie algebra* $h_{\mathfrak{a},\mathfrak{b}}$ *of* $H_{\mathfrak{a},\mathfrak{b}}$ *contains* $\mathfrak{n}$, *i.e. if* $X \in \mathfrak{n}$ *then* $(ad\, X)^* \Omega_{\mathfrak{a},\mathfrak{b}} = 0$. *If in addition* $d\Omega = 0$, *then* $\Omega(\mathfrak{n}, [\mathfrak{a}, \mathfrak{b}]) = 0$.

Proof. – Since $H_{\mathfrak{a},\mathfrak{b}} \supset H \supset K$, $G/H_{\mathfrak{a},\mathfrak{b}}$ is compact. If $X \in \mathfrak{n}$ then $ad\, X$ is a nilpotent endomorphism of $\mathfrak{g}$. it follows that it induces a nilpotent endomorphism of $\Sigma(\mathfrak{a}, \mathfrak{b})$ and hence of $V_{\Omega_{\mathfrak{a},\mathfrak{b}}} \subset \Sigma(\mathfrak{a}, \mathfrak{b})$. The fact that $H_{\mathfrak{a},\mathfrak{b}}$ is the isotropy group of $\Omega_{\mathfrak{a},\mathfrak{b}}$ relative to the action of G on $V_{\Omega_{\mathfrak{a},\mathfrak{b}}}$ and $G/H_{\mathfrak{a},\mathfrak{b}}$ is compact give us via Lemma (3.3) the conclusion that N, the analytic subgroup corresponding to $\mathfrak{n}$ lies in $H_{\mathfrak{a},\mathfrak{b}}$ and that $ad\, \mathfrak{n}^* \Omega_{\mathfrak{a},\mathfrak{b}} = 0$. Let $Z \in \mathfrak{n}$, $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$, then $ad\, Z^*$ satisfies

$$0 = (ad\, Z^* \Omega_{\mathfrak{a},\mathfrak{b}})(X, Y) = \Omega([Z, X], Y) + \Omega(X, [Z, Y]).$$

If $d\Omega = 0$, this reduces to $\Omega(Z, [X, Y]) = 0$ by (2.2).

(4.2) DEFINITIONS. – *A 2-form* Ω_M *on a manifold* M *is a symplectic form if* $d\Omega = 0$ *and* Ω *has maximum rank, i.e.* Ω^n *never vanishes,* $2n = \dim M$. *If* $M = G/K$ *and* Ω_M *is* G-*invariant,* Ω_M *is a homogeneous symplectic form. In this case conditions (i)-(iii) of (2.1) are satisfied; moreover the condition of maximum rank is equivalent to:* $i_Z\Omega = 0$ *if and only if* $Z \in \mathfrak{k}$, *the Lie algebra of* K.

(4.3) THEOREM. – *Let* $\mathfrak{a}$ *be any nilpotent ideal of* $\mathfrak{g}$, *then* $\Omega(\mathfrak{g}, [\mathfrak{a}, \mathfrak{a}]) = 0$ *and* $\Omega(\mathfrak{a}, [\mathfrak{a}, \mathfrak{g}]) = 0$. *In particular this holds for the nilradical* $\mathfrak{n}$.

Proof. – It is clearly sufficient to prove the assertion for $\mathfrak{a} = \mathfrak{n}$, the nilradical. Let $X \in \mathfrak{n}$, $Y \in \mathfrak{n}$ and $Z \in \mathfrak{g}$ with $\mathfrak{a} = \mathfrak{n}$ and $\mathfrak{b} = \mathfrak{g}$ in the preceding Lemma (4.1), then using $\Omega(\mathfrak{n}, [\mathfrak{a}, \mathfrak{b}]) = 0$ we obtain

$$0 = \Omega(X, [Y, Z]) \quad \text{and} \quad 0 = \Omega(Y, [X, Z]).$$

Because Ω is closed (2.2) yields

$$\Omega(X, [Y, Z]) = \Omega([X, Y], Z) + \Omega(Y, [X, Z]). \tag{4.4}$$

Thus $\Omega([X, Y], Z) = 0$ which implies $\Omega([\mathfrak{n}, \mathfrak{n}], \mathfrak{g}) = 0$. The lemma gives $\Omega(\mathfrak{n}, [\mathfrak{n}, \mathfrak{g}])$ as a special case (which we already used above).

(4.5) THEOREM. – *With the assumptions on* Ω *of the preceding theorem, let* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ *be a Levi decomposition of* $\mathfrak{g}$ *with* $\mathfrak{r}$ *the radical and* $\mathfrak{s}$ *a semi-simple subalgebra. Then*

$$\Omega(\mathfrak{g}, [\mathfrak{s}, \mathfrak{r}]) = 0 = \Omega(\mathfrak{s}, \mathfrak{r}).$$

Proof. – Since $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$, $[\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{s}$. If $\mathfrak{n}$ is the nilradical, then we have just proved that $\Omega(\mathfrak{n}, [\mathfrak{g}, \mathfrak{g}]) = 0$. It follows that $\Omega(\mathfrak{n}, \mathfrak{s}) = 0$. In particular $\Omega([\mathfrak{g}, \mathfrak{r}], \mathfrak{s}) = 0$, since $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$, see e.g. [8]; hence $\Omega([\mathfrak{s}, \mathfrak{r}], \mathfrak{s}) = 0$. Let $X, Y \in \mathfrak{s}$ and $Z \in \mathfrak{r}$ be arbitrarily chosen. We have just seen that $\Omega([X, Z], Y$ vanishes as does $\Omega([Y, Z], X)$. From this and $d\Omega = 0$ (see 4.4), follows $\Omega(Z, [X, Y]) = 0$, which give, using $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$, $\Omega(\mathfrak{r}, \mathfrak{s}) = 0$ as claimed.

Since $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, in order to verify that $\Omega(\mathfrak{g}, [\mathfrak{s}, \mathfrak{r}]) = 0$ it is now enough in view of the above to check that

$$\Omega(\mathfrak{r}\,,[\mathfrak{s}\,,\mathfrak{r}\,]) = \Omega([\mathfrak{s}\,,\mathfrak{r}\,],\mathfrak{r}\,) = 0\,.$$

We note that

$$[\mathfrak{s}\,,\mathfrak{r}] = [[\mathfrak{s}\,,\mathfrak{s}\,],\mathfrak{r}\,] \subset [[\mathfrak{s}\,,\mathfrak{r}\,],\mathfrak{s}\,] + [\mathfrak{s}\,,[\mathfrak{s}\,,\mathfrak{r}]] = [[\mathfrak{s}\,,\mathfrak{r}]\,,\mathfrak{s}\,]$$

by the Jacobi identity and $[\mathfrak{s}\,,\mathfrak{s}\,] = \mathfrak{s}$. Let $X_1, X_2 \in \mathfrak{s}$ and $Y_1, Y_2 \in \mathfrak{r}$ and use the fact that $d\Omega = 0$ to obtain

$$\Omega([X_1 Y_1], X_2], Y_2) = \Omega([[X_1 Y_1], Y_2], X_2) + \Omega([X_1 Y_1], [X_2 Y_2]).$$

The second term on the right vanishes since $\Omega([\mathfrak{g}\,,\mathfrak{r}\,], [\mathfrak{g}\,,\mathfrak{r}\,]) = 0$ as we see by noting that $[\mathfrak{g}\,,\mathfrak{r}\,] \subset \mathfrak{n}$ and applying Lemma (4.1) with $\mathfrak{a} = \mathfrak{g}$, $\mathfrak{b} = \mathfrak{r}$. The first term is zero since $\Omega(\mathfrak{r}\,,\mathfrak{s}\,) = 0$, thus we conclude that $\Omega([[\mathfrak{s}\,,\mathfrak{r}\,],\mathfrak{s}\,],\mathfrak{r}\,) = 0$. Combining this with $[\mathfrak{s}\,,\mathfrak{r}\,] \subset [[\mathfrak{s}\,,\mathfrak{r}\,],\mathfrak{s}\,] + [\mathfrak{s}\,,[\mathfrak{s}\,,\mathfrak{r}\,]]$ gives $\Omega(\mathfrak{n}\,,[\mathfrak{s}\,,\mathfrak{r}\,]) = 0$, which completes the proof.

(4.6) NOTATION. – *For convenience a CHS-space will denote a compact, homogeneous symplectic space* $M = G/K$ *(as above) on which we always assume* G *acts almost effectively. This last is equivalent to:*

(4.7) K *contains no connected normal subgroup of* G, *or* $\mathfrak{k}$ *contains no ideal of* $\mathfrak{g}$ *except* (0).

The fact that Ω_M has maximum rank means that $\dim G/K$ is even say $2n$, and $\Omega_M^n \neq 0$. Combining this with (2.1) iii gives:

(4.8) $\qquad X \in \mathfrak{k}$ *if and only if* $i_X \Omega = 0$.

We also see that:

(4.9) $\qquad \mathfrak{a}$ *an ideal of* $\mathfrak{g}$ *and* $\Omega(\mathfrak{a}\,,\mathfrak{g}\,) = 0$ *implies* $\mathfrak{a} = (0)$.

For by (4.8) $\mathfrak{a} \subset \mathfrak{k}$, but $\mathfrak{k}$ contains no ideal of $\mathfrak{g}$ other than (0).

Using this remark is helpful in proving:

(4.10) THEOREM. – *Let* $M = G/K$ *be a CHS-space and* $\Omega = \pi^*\Omega_M$. *Then every nilpotent ideal of* $\mathfrak{g}$ *is abelian.*

Proof. – If $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, then by (4.3) $\Omega(\mathfrak{g}\,,[\mathfrak{n}\,,\mathfrak{n}]) = 0$. Since $[\mathfrak{n}\,,\mathfrak{n}]$ is an ideal, (4.9) implies $[\mathfrak{n}\,,\mathfrak{n}] = 0$. Thus $\mathfrak{n}$ is abelian. Since it contains all nilpotent ideals, the statement follows.

Now let $\mathfrak{g} = \mathfrak{s} + \mathfrak{r}$ be a Levi decomposition as before with $\mathfrak{r}$ the radical (maximal solvable ideal) and $\mathfrak{s}$ a semi-simple subalgebra complementary to it. In general $\mathfrak{s}$ is neither an ideal nor unique. Here, however we have for compact, homogeneous symplectic manifolds:

(4.11) THEOREM. – *In the Levi decomposition $\mathfrak{s}$ is an ideal and hence unique.*

Proof. – Since $\mathfrak{s}$ is a subalgebra, to see that $\mathfrak{s}$ is an ideal we need only show that $[\mathfrak{s}, \mathfrak{r}] = 0$. For this, by virtue of the remark (4.9) it is enough to see that it is an ideal since according to Theorem (4.5), $\Omega(\mathfrak{g}, [\mathfrak{s}, \mathfrak{r}]) = 0$.

We note that $[\mathfrak{s}, \mathfrak{r}] \subset [\mathfrak{g}, \mathfrak{r}]$ and the latter is a nilpotent, hence abelian ideal since $[\mathfrak{g}, \mathfrak{r}]$ lies in the nilradical (see Jacobson [8]). Consider $[\mathfrak{g}, [\mathfrak{s}, \mathfrak{r}]] \subset [\mathfrak{s}, [\mathfrak{s}, \mathfrak{r}]] + [\mathfrak{r}, [\mathfrak{s}, \mathfrak{r}]]$. Since $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{r}$, the first subspace already lies in $[\mathfrak{s}, \mathfrak{r}]$ and we must show only that $[\mathfrak{r}, [\mathfrak{s}, \mathfrak{r}]] \subset [\mathfrak{s}, \mathfrak{r}]$. We use $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ and write $[\mathfrak{r}, [\mathfrak{s}, \mathfrak{r}]] = [\mathfrak{r}, [[\mathfrak{s}, \mathfrak{s}], \mathfrak{r}]]$. By the Jacobi identity:

$$[[\mathfrak{s}, \mathfrak{s}], \mathfrak{r}] \subset [[\mathfrak{s}, \mathfrak{r}], \mathfrak{s}] + [\mathfrak{s}[\mathfrak{s}, \mathfrak{r}]] = [\mathfrak{s}[\mathfrak{s}, \mathfrak{r}]].$$

Hence:

$$[\mathfrak{r}[\mathfrak{s}, \mathfrak{r}]] \subset [\mathfrak{r}, [\mathfrak{s}[\mathfrak{s}, \mathfrak{r}]]] \subset [[\mathfrak{r}, \mathfrak{s}], [\mathfrak{s}, \mathfrak{r}]] + [\mathfrak{s}[\mathfrak{r}[\mathfrak{s}, \mathfrak{r}]]].$$

On the right the first term vanishes since $[\mathfrak{g}, \mathfrak{r}]$ is abelian and the second lies in $[\mathfrak{s}, \mathfrak{r}]$ since $[\mathfrak{r}, [\mathfrak{s}\, \mathfrak{r}]] \subset \mathfrak{r}$, an ideal. This completes the proof that $\mathfrak{s}$ is an ideal, from which uniqueness follows also.

This result gives us the means to decompose $G/K = M$ into a cartesian product of similar symplectic manifolds.

5. Decomposition of CHS-spaces.

We consider now a CHS-space $M = G/K$ which is such that $G = G_1 \times \ldots \times G_m$, with corresponding Lie algebra decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$. The form $\Omega \in \Lambda^2(\mathfrak{g})$, as previously, is defined by $\Omega = \pi^*\Omega_M$. According to Definition (2.7) and Remark (2.8), Ω is *decomposable* if $\Omega(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for $i \neq j$ in which case $\Omega = \Omega_1 + \ldots + \Omega_m$.

(5.1) *Example.* – According to Theorems (4.5) and (4.11), if $G = \mathfrak{s} \oplus \mathfrak{r}$ is the Levi decomposition of $\mathfrak{g}$, then $\mathfrak{s}$, as well as $\mathfrak{r}$, is an ideal and $\Omega(\mathfrak{s}, \mathfrak{r}) = 0$. If G is simply connected, $G = S \times R$, i.e. the group has a corresponding direct product decomposition.

A further example will result from the next Lemma.

(5.2) LEMMA. – *If* $\mathfrak{a}$, $\mathfrak{b}$ *are ideals of* $\mathfrak{g}$ *with* $\mathfrak{a}$ *semi-simple and* $\mathfrak{a} \cap \mathfrak{b} = (0)$, *then for any closed 2-form* Ω *on* $\mathfrak{g}$ *we have* $\Omega(\mathfrak{a}, \mathfrak{b}) = 0$.

Proof. – $d\Omega = 0$ is equivalent to

$$\Omega(Z, [X, Y]) = \Omega([Z, X], Y) + \Omega(X, [Z, Y]) \tag{5.3}$$

for all $X, Y, Z \in \mathfrak{g}$, see (4.4). Let $X, Y \in \mathfrak{a}$ and $Z \in \mathfrak{b}$, then $\mathfrak{a} \cap \mathfrak{b} = 0$ implies $[\mathfrak{a}, \mathfrak{b}] = 0$, so the right side vanishes and we see that $\Omega(Z, [\mathfrak{a}, \mathfrak{a}]) = 0$. However, Z is arbitrary in $\mathfrak{b}$ and since $\mathfrak{a}$ is semi-simple $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$, thus $\Omega(\mathfrak{b}, \mathfrak{a}) = 0$ so the conclusion holds.

(5.4) COROLLARY. – *If* Ω *is a closed 2-form on a semi-simple Lie algebra,* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$, $\mathfrak{g}_i$ *simple ideals for* $i = 1, \ldots, m$, *then* Ω *is decomposable:* $\Omega = \Omega_1 + \ldots + \Omega_m$ *and* $\Omega(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ *for* $i \neq j$.

This corollary is an immediate consequence of the lemma and the criterion for decomposability of Ω given in (2.8). Again, simple connectedness of the Lie group G whose Lie algebra is $\mathfrak{g}$ is sufficient to guarantee a direct product decomposition of the group corresponding to that of $\mathfrak{g}$.

We must now consider to what extent K and H decompose into direct products compatible with the decomposition of G. We also will make more precise the relation between K and H. As we shall see, this is easier in the semi-simple case.

(5.5) LEMMA. – *Let* Ω *be a closed left-invariant 2-form on the connected Lie group* G. *Then* $M = G/H$ *is a symplectic homogeneous space with symplectic form* Ω_M *induced by* Ω *if and only if* $\mathfrak{h} = \{Z \in \mathfrak{g} \mid i_Z \Omega = 0\}$. *This is automatically satisfied if* G *is semi-simple.*

Proof. – This is basically a direct consequence of (2.1). Note that H is the maximal subgroup of G satisfying (2.1) ii. (2.1) i is satisfied by hypothesis as is (2.1) iii by our restriction on $\mathfrak{h}$. On the other hand, Ω_M on $M = G/H$ – given that it is induced from Ω – will have maximum rank if and only if $i_Z\Omega = 0$ exactly if $Z \in \mathfrak{h}$. To see that the condition on $\mathfrak{h}$ is automatic if $\mathfrak{g}$ is semi-simple we note that by (2.5)

$$\mathfrak{h} = \{Z \in \mathfrak{g} \mid di_Z\Omega = 0\} = \{Z \in \mathfrak{g} \mid \Omega(Z, [\mathfrak{g}, \mathfrak{g}] = 0\}.$$

Since $\mathfrak{g}$ semi-simple implies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, the statement follows.

Combining this with the fact that when Ω decomposes to $\Omega = \Omega_1 + \ldots + \Omega_m$ the form Ω is closed if and only if each Ω_i and Ω'_i is closed, we easily obtain the following consequence.

(5.6) COROLLARY. – *Let* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ *be a semi-simple Lie algebra* G *a simply connected group with Lie algebra* $\mathfrak{g}$, *and* G_i *the analytic subgroups corresponding to* $\mathfrak{g}_i$. *If* Ω *is a closed 2-form on* G, *then* G/H, $H = H_\Omega$, *and* G_i/H_i, $H_i = H \cap G_i$, *are symplectic with forms induced by* Ω, Ω'_i *respectively.*

The next theorem leads to one of our basic results in the analysis of compact, homogeneous symplectic spaces since it makes possible the application of powerful known results of Borel [3]. We suppose $M = G/K$ to be a CHS-space with G acting almost effectively. For convenience we take a realization in which G is simply connected.

(5.7) THEOREM. – *If* G *is semi-simple, then it must be compact.*

Proof. – If $G = G_1 \times \ldots \times G_m$ is the decomposition of G into simple groups and $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ the corresponding Lie algebra decomposition, we consider $\Omega = \pi^*\Omega_M$ and H. Since $G \supset H \supset K$, G/H is compact and also has a homogeneous symplectic structure induced by Ω. The same holds for each G_i/H_i which must be compact also and have a homogeneous symplectic structure derived from Ω'_i. Clearly it is enough to show that each G_i is compact.

For convenience of notation then, we drop the subscript i and suppose G/H to be a compact, symplectic homogeneous space with G *simple.* We shall assume G noncompact and show that this results in a contradiction. Let $n = \dim G/H$, $n > 0$, then Ω_M^n is a G-invariant volume element on G/H. This implies, according to arguments of Selberg (Borel [4]) that H has the Selberg property in G: if $x \in G$, U a neighborhood of e in G, then there exists an integer $k > 0$ such that $x^k \in UHU$. According to Borel [4], it follows that $\mathfrak{h}$ is stable under $Ad\,G$ so $\mathfrak{h}$ is an ideal. However $\mathfrak{g}$ is simple so $\mathfrak{h} = \mathfrak{g}$ or $\mathfrak{h} = (0)$. The former is impossible since $\dim G/H > 0$. If $\mathfrak{h} = (0)$, then G has a bi-invariant closed form Ω of rank equal to the dimension of G. Since the real 2-dimensional cohomology of a simple Lie algebra is zero (Chevalley-Eilenberg [5]), there must be a 1-form, $\theta \in \Lambda^1(\mathfrak{g})$, such that $d\theta = \Omega$. Let $Z \in \mathfrak{g}$ be dual to this form relative to the Killing form: $\langle Z, X\rangle = \theta(X)$. Then for all $X \in \mathfrak{g}$

$$\Omega(Z,X) = d\theta(Z,X) = \theta([Z,X]) = \langle Z, [Z,X]\rangle .$$

But $\langle Z, [Z,X]\rangle = \langle [Z,Z], X\rangle = 0$ by a standard property of the Killing form. Hence $Z = 0$ since Ω has maximum rank. This implies that θ and $d\theta = \Omega$ are zero, an obvious contradiction. The theorem then follows.

The theorem just proved makes it possible to apply the following result of Borel [3].

(5.8) THEOREM (Borel). – *If* G *is a compact semi-simple Lie group acting effectively on the homogeneous manifold* $M = G/K$ *and leaving invariant a symplectic form* Ω_M *on* M, *then* G/K *is simply connected, the center of* G *is* $\{e\}$, K *is connected and is the centralizer of a torus. Moreover* $G/K = G_1/K_1 \times \ldots \times G_m/K_m$ *when* $G = G_1 \times \ldots \times G_m$ *with* G_i *compact simple groups,* $K_i = K \cap G_i$ *and the restriction of* Ω_M *to each* G_i/K_i *is a* G_i*-invariant symplectic structure on* G_i/K_i.

If we combine this with Theorem (5.7), we have the following result.

(5.9) THEOREM. – *Suppose* $M = G/K$ *is a compact, symplectic homogeneous (CHS-) space, that* G *acts almost effectively on* M, *and that* G *is semi-simple. Then* G *is compact,* M *is simply connected and* K *is connected and contains the center of* G.

Further M *decomposes naturally into the product of* CHS-*spaces corresponding to the simple parts.*

Proof. – Let D be the maximal normal subgroup of G contained in K. It is exactly the set of elements of G which act as the identity on $M = G/K$, hence it is discrete. Being a discrete normal subgroup, it is in the center of G. Let $G' = G/D$ and $K' = K/D$, then $G/K \approx G'/K' \approx M$ where $\approx$ means naturally diffeomorphic – in fact, they have equivalent symplectic structures. By Theorem (5.7) G is compact. Thus G' is compact, semi-simple and effective on M. All of the conclusions of the theorem follow from Borel's Theorem (5.8) except that we must verify that the center of G lies in K. But this follows since the image of the center of G lies in the center of G', which is $\{e\}$ the identity and thus center of G lies in D. Thus, D = center of G.

(5.10) COROLLARY. – *With the hypotheses of the theorem satisfied, it follows that* $H = K$.

Proof. – Since G is semi-simple, Lemma (5.5) asserts that $\mathfrak{h} = \{Z \in \mathfrak{g} \mid i_Z \Omega = 0\}$. But by (4.2) the condition of maximum rank (and (2.1)) require that $\mathfrak{k} = \{Z \in \mathfrak{g} \mid i_Z \Omega = 0\}$. Hence H and K have the same Lie algebra. However both G/K and G/H are CHS-spaces with form induced by $\Omega = \pi^* \Omega_M$ (see (5.5)). Since the theorem implies H and K are connected we see that $H = K$.

We turn next to the general case, except that we shall assume that the connected Lie group G decomposes into a product $G = S \times R$ corresponding to the Levi decomposition. This involves no real loss of generality, since as has been noted, we may replace G by its universal covering group and K by its preimage under the covering map. With this reservation we have:

(5.11) THEOREM. – *Let* $M = G/K$ *be a* CHS-*space,* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ *the Levi decomposition of the Lie algebra and* $G = S \times R$ *the corresponding decomposition of* G. *Then* Ω *decomposes into* $\Omega = \Omega_s + \Omega_r$, $K = K_s \times K_r$, $K_s = K \cap S$ *and* $K_r = K \cap R$ *a compatible direct product to that of* G *and* $G/K \approx S/K_s \times R/K_r$ *where* S/K_s *and* R/K_r *are* CHS-*spaces with forms induced by* Ω_s *and* Ω_r.

Proof. – Since $\Omega(\mathfrak{s}, \mathfrak{r}) = 0$ by (4.5) and both $\mathfrak{s}$ and $\mathfrak{r}$ are ideals by (4.11), Ω is decomposable – this was Example (5.1). By Lemma (2.9) H decomposes into $H = H_s \times H_r$, with $H_s = H \cap S$ and $H_r = H \cap R$; moreover these are exactly the subgroups of S and R whose action leaves Ω_s and Ω_r, respectively, invariant. According to (5.6) S/H_s is a CHS-space and by (5.10) we see that $H_r = K \cap S$. Since $H \supset K$ any $x \in K$ decomposes uniquely into $x = x_s . x_r$ with $x_s \in H_s$ and $x_r \in H_r$. It follows that x_s and x_r are also in K. Thus K splits into a direct product $K = K_s \times K_r$ with $K_s = H_s$. This means that $G/K = S/K_s \times R/K_r$. $Ad\, K_r$ leaves Ω_r on R invariant and it is easy to verify that $i_Z \Omega = 0$ if and only if $i_{Z_r}\Omega_r = 0$ and $i_{Z_s}\Omega_s = 0$, $Z = Z_r + Z_s$ being the direct sum decomposition of $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{k}_r$. This guarantees that the form induced on R/K_r by Ω_r will have maximum rank. Hence it induces a symplectic structure on R/K_r as does Ω_s on $S/K_s = S/H_s$. This completes the proof.

As a result of this theorem and (5.8) and (5.9) it remains only to study CHS-spaces of the form G/K with G solvable. This is done in the next section.

6. The solvable case.

We now consider exclusively the case of a compact homogeneous symplectic (CHS) manifold $M = G/K$ with G a simply connected *solvable* Lie group acting almost effectively on M, which carries the G-invariant symplectic form Ω_M. Other notations and assumptions are as before: $\Omega = \pi^*\Omega_M$ on $\mathfrak{g}$ determines the subgroup $H = H_\Omega$ which contains K, $\mathfrak{n}$ is the nil-radical and N the corresponding analytic (i.e. connected) subgroup of G, etc. The simplest example of such a manifold M is the torus $T^{2n} = \mathbf{R}^{2n}/\mathbf{Z}^{2n}$ with $\Omega = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}$ and $H = \mathbf{R}^{2n}$. Here, of course, G is the simply connected abelian group $\mathbf{R}^{2n}$ and K is necessarily a lattice of G – since the action is almost effective. As we shall see, G need not be abelian. However, the compactness of G/K, together with the invariant symplectic structure, impose strong restrictions on the structure of G.

We begin by restating some of the facts already proved, and add a few further observations.

(6.1) *Since* $H \supset K$, G/H *is compact.*

(6.2) *The nilradical* $\mathfrak{n}$ *and the corresponding connected subgroup* N *are abelian.*

The following facts concerning analytic subgroups of simply connected solvable groups are useful.

(6.3) *Any analytic subgroup of* G, *for example* N, *is closed and simply connected. The coset space of any such group is also simply connected and* G *is diffeomorphic to the cartesian product of the subgroup and the coset space.*

Proofs of these statements may be found in Hochschild [9], Chapter XII.

We have seen earlier that $\Omega(\mathfrak{n}, [\mathfrak{g}, \mathfrak{n}]) = 0$. This can be strengthened in this case to

(6.4) $\Omega(\mathfrak{n}, [\mathfrak{g}, \mathfrak{g}]) = 0$ *and further,* $H \supset N$.

This statement is demonstrated as follows. By Mostow [15], since K contains no proper connected normal analytic subgroup of G, the group NK is closed in G. Therefore NK/K is compact and so is $N/N \cap K$ which is, in our case a torus. There exists then, a basis $N_1, \ldots, N_t$ of $\mathfrak{n}$ such that $\exp N_i \in K$, $i = 1, \ldots, t$. Since Ω is Ad K-invariant, and $Ad(\exp N_i) = e^{ad N_i}$,

$$\Omega(X,Y) = Ad(\exp N_i)^*\Omega(X,Y) = \Omega(e^{ad N_i}X, e^{ad N_i}Y)$$
$$= \Omega(X,Y) + \Omega([N_i, X], Y + \Omega(X, [N_i, Y])$$

for all $X, Y \in \mathfrak{g}$ and $i = 1, \ldots, t$. Higher brackets vanish since $\mathfrak{n}$ is abelian and contains $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. This implies

$$\Omega([N_i, X], Y) + \Omega(X, [N_i, Y]) = 0,$$

which, by linearity implies that for any $Z \in \mathfrak{n}$ we have

$$(ad\, Z)^*\Omega(X,Y) = \Omega([Z,X], Y) + \Omega(X, [Z,Y]) = 0.$$

Thus $\mathfrak{h} \supset \mathfrak{n}$ so $H_0 \supset N$, H_0 being the identity component of H. At the same time, since $d\Omega = 0$ application of (2.2), or better (4.4), gives $\Omega(Z, [X,Y]) = 0$. This proves $\Omega(\mathfrak{n}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

Having seen that $H \supset N$, we shall next prove a lemma.

(6.5) LEMMA. – H *is a normal subgroup of* G *and lies in the centralizer,* $C_G(N)$, *of* N *in* G.

Proof. – If $Z \in \mathfrak{g}$ such that $\exp Z \in H$, then just as above we see that for any $X, Y \in \mathfrak{g}$

$$\Omega(X, Y) = Ad(\exp Z)^* \Omega(X, Y) = \Omega(e^{ad\, Z}X, e^{ad\, Z}Y).$$

If now we write $e^{ad\, Z}X = X + [Z, X] + \frac{1}{2!}[Z[Z, X]] + \dots$ and further assume $Y \in \mathfrak{n}$ (so that $e^{ad\, Z}Y \in \mathfrak{n}$ also), then according to (6.4) the right side reduces to $\Omega(X, e^{ad\, Z}Y)$. Combining left and right sides of the equation gives for all $X \in \mathfrak{g}$, $Y \in \mathfrak{n}$, and Z such that $\exp Z \in H$,

$$\Omega(X, (e^{ad\, Z} - I)Y) = 0.$$

It follows that $(e^{ad\, Z} - I)(\mathfrak{n}) \subset \mathfrak{k}$. It is easy to verify directly that $(e^{ad\, Z} - I)(\mathfrak{n})$ is an ideal of $\mathfrak{g}$. In fact let $X, Y \in \mathfrak{g}$ and $V \in \mathfrak{n}$, then the Jacobi identity reads

$$[Y, [X, V]] + [X, [V, Y]] + [V, [X, Y]] = 0.$$

The last term is zero since $[X, Y] \in \mathfrak{g}' \subset \mathfrak{n}$ and $V \in \mathfrak{n}$ which is abelian. Thus $ad\, X\, ad\, Y(V) = ad\, Y\, ad\, X(V)$ and hence

$$ad\, Y(e^{ad\, X} - I)(\mathfrak{n}) = (e^{ad\, X} - I)\, ad\, Y(\mathfrak{n}) \subset (e^{ad\, X} - I)(\mathfrak{n}).$$

But $\mathfrak{k}$ contains no ideals except (0), thus $(e^{ad\, Z} - I)(\mathfrak{n}) = (0)$. Therefore $e^{ad\, Z} = Ad(\exp Z)$ is the identity on $\mathfrak{n}$ or equivalently, $\exp Z$ commutes with every element of N whenever Z is such that $\exp Z \in H$. In particular this holds for all $Z \in \mathfrak{h}$, so $\mathfrak{h} \subset c_{\mathfrak{g}}(\mathfrak{n})$, the centralizer in $\mathfrak{g}$ of $\mathfrak{k}$. We still must show H is normal and every element of H is of the form $\exp X$ for some $X \in \mathfrak{g}$. H is normal because of the following observation.

(6.6) *If* $L \supset N$, *then* L *is a normal subgroup of* G *and* L/N *is abelian.*

Indeed $\mathfrak{n} \supset \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, hence G/G' is abelian – G' being the analytic subgroup of G corresponding to $\mathfrak{g}'$. Further, if $L \supset G'$, then L is the complete inverse image in G of the (normal) subgroup L/G' in G/G'. Hence L is normal. Clearly if $L \supset N \supset G'$, then $L/N \cong L/G'/N/G'$ is abelian.

Returning to the proof of (6.5), G/H_0 must then be an abelian group and simply connected since H_0 is a connected subgroup of a simply connected group. Thus $G/H_0 \cong \mathbf{R}^d$, a vector space and the compact group G/H is a factor group of G/H_0 by the discrete (lattice) group H/H_0. Let $\rho : G \longrightarrow G/H_0$ be the natural homomorphism and $\rho^* : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ the corresponding Lie algebra homomorphism. Clearly given $x \in H$ there is an $X' \in \mathfrak{g}/\mathfrak{h}$ such that $\exp X' = xH_0$, and an $X \in \mathfrak{g}$ such that $\rho^*(X) = X'$. It follows that $\rho(\exp X) = xH_0 \in H/H_0$. This can only be so if $\exp X = xx_0, x_0 \in H_0$. Then $x = (\exp X)x_0^{-1}$ and since both x_0 and $\exp X$ commute with every element of N, so must x. This completes the proof of (6.5).

We now consider the adjoint representation $Ad_{\mathfrak{n}}(G)$ of G on the ideal $\mathfrak{n}$ and the corresponding Lie algebra representation $ad_{\mathfrak{n}}(\mathfrak{g})$. The restriction of $Ad_{\mathfrak{n}}(G)$ to the subgroup H is trivial by the preceding lemma, hence $Ad_{\mathfrak{n}}(G)$ determines a representation ρ of G/H on $\mathfrak{n}$ by $\rho(xH) = Ad_{\mathfrak{n}}(x)$, i.e. $\rho \circ \pi(x) = Ad_{\mathfrak{n}}(x)$ where π is the natural homomorphism of G to G/H. Since $G/H \cong T^d$ a torus, the representation space $\mathfrak{n}$ of ρ (and hence of $Ad_{\mathfrak{n}}$) has a (non-unique) invariant inner product. It decomposes into a direct sum of (orthogonal) invariant subspaces $\mathfrak{n}_i$ of dimension 2 and $\mathfrak{c}$, $\mathfrak{c}$ being the center of $\mathfrak{g}$ and kernel of $Ad_{\mathfrak{n}}$ and ρ. To each $\mathfrak{n}_i$ corresponds a weight $\theta_i \neq 0$ and to $\mathfrak{c}$ corresponds the weight 0. Choosing once and for all an orthonormal basis X_i, Y_i of $\mathfrak{n}_i$, the representations (restricted to $\mathfrak{n}_i$) $Ad_{\mathfrak{n}_i}(\exp tX)$ and $ad_{\mathfrak{n}_i}(tX)$ are given by the matrices

$$R_i = \begin{pmatrix} \cos 2\pi\, \theta_i(tX) & \sin 2\pi\, \theta_i(tX) \\ -\sin 2\pi\, \theta_i(tX) & \cos 2\pi\, \theta_i(tX) \end{pmatrix}, \quad Q_i = \begin{pmatrix} 0 & 2\pi\, t\theta_i(X) \\ -2\pi\, t\theta_i(X) & 0 \end{pmatrix}$$

respectively. Choosing $Z_1, \ldots, Z_s$ a basis of $\mathfrak{c}$, then

$$X_1, Y_1, \ldots, X_r, Y_r, Z_1, \ldots, Z_s$$

is a basis of $\mathfrak{n}$ which we will fix. Relative to these bases $Ad_{\mathfrak{n}}(\exp tX)$ and $ad_{\mathfrak{n}}(tX)$ are given by matrices:

$$\begin{pmatrix} R_1 & & 0 & \dots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \dots & R_r & & \vdots \\ \vdots & & & 1 & \vdots \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_1 & & 0 & \dots & 0 \\ & \ddots & & & \\ 0 & \dots & Q_r & \dots & 0 \\ & & & 0 & \\ 0 & & & & \ddots 0 \end{pmatrix}$$

The weights $\theta_1, \dots, \theta_r$ take integer values on any $X \in \mathfrak{g}$ such that $\exp X \in H$, reflecting the fact that $Ad_{\mathfrak{n}}(\exp X) =$ identity.

(6.7) DEFINITION. – *We shall call a set of m elements $P_1, \dots, P_m$ of $\mathfrak{g}$ a complementary set* (to $\mathfrak{n}$) *if* (i)

$$P_1, \dots, P_m, X_1, Y_1, \dots, X_r, Y_r, Z_1, \dots, Z_s$$

is a basis of $\mathfrak{g}$, called the associated basis, and (ii) $\exp P_i \in H$, $i = 1, 2, \dots, m$.

Condition (ii) implies that $\theta_j(P_i)$ is an integer for $j = 1, \dots, r$ and $i = 1, \dots, m$. Note that if $X \in \mathfrak{h}$, then $Ad_{\mathfrak{n}}(\exp tX) = I_{\mathfrak{n}}$ and all $\theta_j(X) = 0$.

(6.8) *There exist complementary sets. If $P_1, \dots, P_m$ is such a set, then so is any set $P_1, \dots, P_m$ of $\mathfrak{g}$ which is linearly independent mod $\mathfrak{n}$ and whose expressions in the associated basis to $P_1, \dots, P_m$ have integral coefficients with respect to the $P_i' s$.*

Proof. – We know $G/H_0 \cong \mathbf{R}^d$ and $G/H \cong T^d$, with the integral lattice H/H_0 as kernel of the natural homomorphism $G/H_0 \longrightarrow G/H$. Thus we may choose a basis $Y_1, \dots, Y_d$ of $\mathfrak{g}/\mathfrak{h}$ such that $\exp Y_i \in H/H_0$. Let $\rho : G \longrightarrow G/H_0$ be the natural projection and $\rho^* : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ be the corresponding Lie algebra homomorphism. There exists a linearly independent set of m vectors $P_1, \dots, P_m$ in $\mathfrak{g}$, spanning a space complementary to $\mathfrak{n}$ and such that $\rho^*(P_i) = Y_i$ for $i = 1, \dots, d$ and $\rho^*(P_i) = 0$ for $i = d + 1, \dots, m$, i.e. $P_{d+1}, \dots, P_m \in \mathfrak{h}$. Since

$$\rho(\exp P_i) = \exp \rho^*(P_i) = \exp Y_i \ \text{ if } \ 1 \leqslant i \leqslant d$$

and the identity if $i > d$, we see that $\exp P_i \in H$ for $i = 1, \dots, m$.

Now let $P = \sum_{i=1}^{m} n_i P_i + X$, $n_i \in \mathbf{Z}$ and $X \in \mathfrak{n}$. Then

$$\rho(\exp P) = \exp \rho^*\left(\sum n_i P_i + X\right) = \exp \sum n_i Y_i = \pi(\exp Y_i)^{n_i}.$$

Hence $\exp P \in H$ and the proposition is proved. With this established we state a theorem.

(6.9) THEOREM. -- *The complementary set* $P_1, \ldots, P_m$ *may be chosen so that* $[P_i, P_j] = 0$, *i.e. so that they span an abelian Lie algebra* $\mathfrak{a}$ *complementary to* $\mathfrak{n}$.

Proof. -- We have decomposed $\mathfrak{n}$ into the direct sum of the center $\mathfrak{c}$ of $\mathfrak{g}$ and ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$. If $U \in \mathfrak{n}$ we let U $U_{(0)} + U_{(1)} + \ldots + U_{(r)}$ be the unique decomposition of U corresponding to $\mathfrak{n} = \mathfrak{c} \oplus \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_r$. For any $U \in \mathfrak{n}$ there is an integer k, $0 \leqslant k \leqslant r$, such that $U \in \mathfrak{c} + \mathfrak{n}_1 + \ldots + \mathfrak{n}_k$, i.e. such that $U_{(k+1)} = \ldots = U_{(r)} = 0$. $U \in \mathfrak{c}$ is equivalent to $k = 0$.

Let $P_1, \ldots, P_m$ be a complementary set such that for $1 \leqslant i$, $j \leqslant m$ we have $[P_i, P_j] \in \mathfrak{c} + \mathfrak{n}_1 + \ldots + \mathfrak{n}_k$, $k \leqslant r$. We shall demonstrate that a new complementary set $P'_1, \ldots, P'_m$ may be chosen such that $[P'_i, P'_j] \in \mathfrak{c} + \mathfrak{n}_1 + \ldots + \mathfrak{n}_{k-1}$ for all i, j. This shows by recursion that there exists such a set with all brackets lying in the center $\mathfrak{c}$.

As a first step we note that by renumbering the given complementary set if necessary we may assume that $\theta_k(P_1) \neq 0$. (Remark that no θ_ℓ vanishes on all the $P_i's$). With this assumption satisfied, we define

$$\overline{P}_1 = P_1 \; ; \; \overline{P}_i = \theta_k(P_1) P_i - \theta_k(P_i) P_1 \, , \quad i = 2, \ldots, m \, .$$

Since θ_k takes integer values on $P_1, \ldots, P_m$ this is a new complementary set. $[\overline{P}_i, \overline{P}_j]$ are linear combinations of $[P_i, P_j]$, hence also lie in $\mathfrak{c} + \mathfrak{n}_1 + \ldots + \mathfrak{n}_k$. Moreover for $i \geqslant 2$ and X_k, Y_k the basis of $\mathfrak{n}_k$

$$ad\, \overline{P}_i(X_k) = 2\pi(\theta_k(P_1)\,\theta_k(P_i) - \theta_k(P_i)\,\theta_k(P_1))\, X_k = 0 \, ,$$

similarly $ad\, \overline{P}_i(Y_n) = 0$. Thus $ad\, \overline{P}_i|_{\mathfrak{n}_k} = 0$ for $i = 2, \ldots, k$. It follows that $[\overline{P}_i, [\mathfrak{g}, \mathfrak{g}]]_k = [\overline{P}_i, [\mathfrak{g}, \mathfrak{g}]_k] = 0$ for $i > 2$, since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. More particularly for any $i, j \geqslant 2$ we have

$$0 = [\overline{P}_1, [\overline{P}_i, \overline{P}_j]]_{(k)} + [\overline{P}_i, [\overline{P}_j, \overline{P}_1]]_{(k)} + [\overline{P}_j, [\overline{P}_1, \overline{P}_i]]_{(k)}$$
$$= [\overline{P}_1, [\overline{P}_i, \overline{P}_j]_{(k)}] \, .$$

However, $\theta_k(\overline{P}_1) \neq 0$ so $ad\,\overline{P}_1|_{\mathfrak{n}_1}$ is non-singular, thus $[\overline{P}_i, \overline{P}_j]_{(k)} = 0$, i.e. $[\overline{P}_i, \overline{P}_j] \in \mathfrak{c} + \mathfrak{n}_1 + \ldots + \mathfrak{n}_{k-1}$ if $i, j \geqslant 2$.

However, we are unable to make the same assertion about $[\overline{P}_1, \overline{P}_j]$, so we choose another complementary set:

$$P_1' = \overline{P}_1 \quad \text{and} \quad P_i' = \overline{P}_i - (ad\,\overline{P}_1)_{\mathfrak{n}_k}^{-1}[\overline{P}_1, \overline{P}_i]_{(k)}.$$

For $i, j \geqslant 2$, $[P_i', P_j'] = [\overline{P}_i, \overline{P}_j]$, if we make use of the facts that $\mathfrak{n}$ is abelian and $ad\,\overline{P}_i|_{\mathfrak{n}_k} = 0$ for $i \geqslant 2$. Thus

$$[P_i, P_j] \in \mathfrak{c} + \mathfrak{n}_1 + \ldots + \mathfrak{n}_{k-1}$$

in this case. Finally consider $[P_1', P_i']$

$$[P_1', P_i'] = [\overline{P}_1, \overline{P}_i] - ad\,\overline{P}_1((ad\,\overline{P}_1)_{\mathfrak{n}_k}^{-1}[\overline{P}_1, \overline{P}_i]_{(k)})$$
$$= [\overline{P}_1, \overline{P}_i] - [\overline{P}_1, \overline{P}_i]_{(k)}.$$

But we have already seen that $[P_i, P_j] \in \mathfrak{c} + \mathfrak{n}_1 + \ldots + \mathfrak{n}_k$ for all i, j, including $[\overline{P}_1, \overline{P}_i]$, $i = 1, \ldots, m$. Thus

$$[P_1', P_i'] = [\overline{P}_1, \overline{P}_i] - [\overline{P}_1, \overline{P}_i]_{(k)} \in \mathfrak{c} + \mathfrak{n}_1 + \ldots + \mathfrak{n}_{k-1}.$$

It follows then by recursion that we may choose a complementary set $P_1', \ldots, P_m'$ such that all the brackets $[P_i', P_j']$ are in $\mathfrak{c}$. Assume such a set has been chosen.

Since $[P_j', P_\ell'] \in \mathfrak{c}$ for $j, \ell = 1, 2, \ldots, m$, they span an ideal $\mathfrak{l}$ of $\mathfrak{g}$. Because $\exp P_j' \in H$, $Ad(\exp P_j')^*\Omega = \Omega$, from which it follows that

$$\Omega(P_j, P_\ell') = \Omega(e^{ad\,P_j'}P_k'\, e^{ad\,P_j'}P_\ell')$$
$$= \Omega(P_k', P_\ell') + \Omega([P_j', P_k'], P_\ell') + \Omega(P_k', [P_j', P_\ell'])$$

with other terms vanishing since $[P_i', P_j'] \in \mathfrak{c}$ for all i, j. Thus

$$\Omega([P_i', P_k'], P_\ell') + \Omega(P_k', [P_i', P_\ell']) = 0 \quad \text{for all } i, k, \ell.$$

Since Ω is closed this gives via (4.4) that $\Omega(P_i', [P_k', P_\ell']) = 0$. Using this and $\Omega(\mathfrak{n}, [\mathfrak{g}, \mathfrak{g}] = 0$ we see that $\Omega(\mathfrak{g}, \mathfrak{l}) = 0$. Thus $\mathfrak{l} \subset \mathfrak{k}$ which contains only the ideal (0). It follows that $[P_j', P_\ell'] = 0$ for $j, \ell = 1, 2, \ldots, m$ and therefore $P_1', \ldots, P_m'$ span an abelian subalgebra $\mathfrak{a}$, which is complementary to $\mathfrak{n}$.

This proves the theorem. We have several corollaries.

(6.10) COROLLARY. – *If* A *is the analytic group corresponding to* $\mathfrak{a}$ *then* A *is closed and simply connected and* G = AN, *a semi-direct product of* A *and the normal subgroup* N.

Proof. – According to (6.3) A is closed and simply connected. From the Lie algebra $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ we can construct a semi-direct product AN of A and N which will be simply connected and locally isomorphic to G. Since G is also simply connected both the local isomorphism and its inverse can be extended so that G is isomorphic to AN.

(6.11) COROLLARY. – $\mathfrak{h} = \mathfrak{n}$ *and* $H_0 = N$. $C_G(N)$ *is abelian, is the direct sum of* $C_G(N) \cap A$ *and* N, *and* $C_G(N) = H$.

That $H_0 = N$ follows once we establish $\mathfrak{h} = \mathfrak{n}$. We know that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{n}) \supset \mathfrak{h} \supset \mathfrak{n}$, $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{n})$ being the centralizer in $\mathfrak{g}$ of $\mathfrak{n}$. If $Z \in \mathfrak{a}$, then $[Z, \mathfrak{n}] \neq (0)$ for otherwise $Z \in \mathfrak{c}$ and $\mathfrak{a} \cap \mathfrak{c} = (0)$. Since any element of $\mathfrak{g}$ may be written uniquely as a sum of its component in $\mathfrak{a}$ and its component in $\mathfrak{n}$, we see that if it centralises $\mathfrak{n}$ it must lie in $\mathfrak{n}$. Thus $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{n}) = \mathfrak{h} = \mathfrak{n}$.

To prove the last statement, note that $C_G(N)$ consists of elements of the form an, where $a \in A$ and $n \in N$ are uniquely determined. Moreover $an = na$ since $n \in C_G(N)$, which implies that $a \in C_G(N)$. Now $Ad(an) = Ad(a)Ad(n)$ leaves Ω invariant as does $Ad(n)$, hence $a \in H$, i.e. $an \in H$. This shows that $H = C_G(N)$.

(6.12) COROLLARY. – $[\mathfrak{a}, \mathfrak{n}] = [\mathfrak{g}, \mathfrak{g}]$ *and the adjoint representation of* $\mathfrak{a}$ *on* $\mathfrak{n}_1 + \ldots + \mathfrak{n}_r = [\mathfrak{g}, \mathfrak{g}]$ *is faithful, i.e. if* $X \in \mathfrak{a}$ *then* $X = 0$ *if and only if* $\theta_1(X) = \theta_2(X) = \ldots = \theta_r(X) = 0$.

This is an immediate consequence of the fact that $\mathfrak{a}$ is spanned by a complementary set $P_1, \ldots, P_m$ and for each p_j we have $\theta_i(P_j) \neq 0$ for some θ_i. Otherwise $ad(P_j)|_{\mathfrak{n}} = 0$ and since $\mathfrak{a}$ is abelian $ad(P_j)|_{\mathfrak{a}} = 0$, i.e. $[P_j, \mathfrak{g}] = 0$ and $P_j \in \mathfrak{c}$, a contradiction. This implies the following fact.

(6.13) COROLLARY. – $\dim \mathfrak{a} \leqslant \dim \{\theta_1, \ldots, \theta_r\} \leqslant r$, $\{\theta_1, \ldots, \theta_r\}$ *being the subspace of* $\mathfrak{a}^*$ *spanned by* $\theta_1, \ldots, \theta_r$.

Otherwise there would be non-zero elements of $\mathfrak{a}$ vanishing on $\theta_1, \dots, \theta_r$, which we have seen to be impossible.

7. A further result in the solvable case. An example.

We continue our discussion of the solvable case making use of the characterization of G obtained in the previous section: $G = AN$, a semi-direct product with $A \cong \mathbf{R}^m$, with $N \cong \mathbf{R}^{2r+s}$, and with the homomorphism of A into the automorphisms of N given by the forms (or "weights") $\theta_1, \dots, \theta_r$ on $\mathfrak{a}$ defined in Section 6. Recall that H may be described as the centralizer of N in G, $H = C_G(N)$, and in fact, $H = DN$ a direct product with

$$D = \{a \in A \mid a = \exp\Big(\sum_{i=1}^{m} n_j P_j\Big),\ n_j \in \mathbf{Z}\}.$$

The vectors $P_1, \dots, P_m$ are the vectors of the basis of $\mathfrak{a}$ defined in Theorem (6.9). Although above we began with G, K and Ω_M and arrived at this description of G, here we will not assume any prescribed K or Ω_M for the present, but just work with the structure on G outlined above.

Let $G^+ = A \oplus N$ denote the (abelian) direct sum of A, N. We shall write its elements as pairs (a, n) and continue to use multiplicative notation: $(a, n)(a', n') = (aa', nn')$ for the group operation. There is an obvious 1:1 correspondence $\psi : G^+ \longrightarrow G$. In fact $\psi(a, n) = an$ (juxtaposition denotes the product in $G = AN$). This correspondence is a diffeomorphism and, restricted to A or to N (and even to H) is a group isomorphism as well. Let $A^+ = \psi^{-1}(A)$, $N^+ = \psi^{-1}(N)$ and $H^+ = \psi^{-1}(H)$, then ψ^{-1} and ψ are group isomorphisms. To any closed subgroup $K \subset H$ will correspond a closed subgroup $K^+ \subset H^+$, and K is uniform in G if and only if K^+ is uniform in G^+. But more surprising is the fact that given such K^+ and K, cosets of K^+ correspond to (left) cosets of K under the mapping ψ. For suppose $x = (a_1, n_1)$, $y = (a_2, n_2)$ are in the same coset of K^+, i.e.

$$y^{-1}x = (a_2, n_2)^{-1}(a_1, n_1) = (a_2^{-1}a_1, n_2^{-1}n_1) \in K^+.$$

Applying ψ we obtain, first, that

$$\psi(y^{-1}x) = \psi(a_2^{-1}a_1, n_2^{-1}n_1) = a_2^{-1}a_1 n_2^{-1} \in \mathrm{K}$$

and, second, that $\psi(x) = a_1 n_1$, $\psi(y) = a_2 n_2$, thus

$$\psi(y)^{-1}\psi(x) = n_2^{-1}a_2^{-1}a_1 n_1 .$$

However, since $\mathrm{K} \subset \mathrm{H} = \mathrm{DN}$, a direct product, it follows that $a_2^{-1}a_1 \in \mathrm{D} \subset \mathrm{h}$ and hence commutes with elements of N. Therefore $n_2^{-1}a_2^{-1}a_1 n_1 = a_2^{-1}a_1 n_2^{-1} n_1$, which is in K; so $\psi(x)$ and $\psi(y)$ are in the same coset of K. This proves that ψ induces a fibre space isomorphism of the fibre bundle $\mathrm{G}^+/\mathrm{K}^+$ onto G/K, in fact it is a bundle isomorphism since if $(1, n) \in \mathrm{K}^+$ then $\psi \circ \mathbf{R}_{(1,n)} = \mathbf{R}_n \circ \psi$, i.e. right translation by elements of K is preserved by ψ.

Now suppose that Ω is any differential form on G which is left invariant, i.e. $\Omega \in \Lambda(\mathfrak{g})$, and is also $Ad(\mathrm{H})$-invariant. Let $\Omega^+ = \psi^*\Omega$, then we claim Ω^+ is an invariant form on the abelian group G^+. Let (a_0, n_0) be an element of G^+ and $a_0 n_0$ the corresponding element of G, and denote by $\mathrm{I}_{n_0^{-1}}$ the inner automorphism of G determined by n_0^{-1}. Then for any $(a, n) \in \mathrm{G}^+$

$$\begin{aligned}\psi^{-1} \circ \mathrm{L}_{a_0 n_0} \circ \mathrm{I}_{n_0^{-1}} \circ \psi(a, n) &= \psi^{-1}(a_0 n_0 n_0^{-1}(an) n_0) \\ &= (a_0, n_0)\,(a, n) = \mathrm{L}_{(a_0, n_0)}\,(a, n).\end{aligned}$$

Thus $\mathrm{L}^*_{(a_0,n_0)}\Omega^+ = \psi^* \circ Ad(n_0^{-1})^* \circ \mathrm{L}^*_{a_0 n_0} \circ \psi^{-1*}\Omega^+ = \psi^*\Omega = \Omega^+$ as claimed. Since any invariant form on a abelian group is closed, Ω^+ and hence Ω are closed. In summary

(7.1) *If* Ω *is any left invariant and* $Ad(\mathrm{H})$ *invariant exterior form on* G, *then* $\Omega^+ = \psi^*\Omega$ *is invariant under the translations of the abelian group* G^+ *and* $d\Omega^+ = 0 = d\Omega$.

If we apply this to the case of interest to us, namely $\mathrm{M} = \mathrm{G}/\mathrm{K}$ a CHS-space with symplectic form Ω_{M} we see that we have a commutative diagram

$$\begin{tikzcd} \mathrm{G}^+ \arrow[r, "\psi"] \arrow[d, "\pi_+"] & \mathrm{G} \arrow[d, "\pi"] \\ \mathrm{G}^+ \arrow[r, "\psi'"] & \mathrm{G}/\mathrm{K} \end{tikzcd}$$

corresponding to the bundle map and $\psi'^*\Omega_{\mathrm{M}}$ is invariant under the action of G^+ on G^+/K^+ just as Ω_M is invariant under the action of G on $G/K = M$. Since ψ, ψ' are diffeomorphisms, G/K is diffeomorphic to G^+/K^+ which is a factor group of $G^+ \cong \mathbf{R}^{m+2r+s}$ by a uniform subgroup K^+ and hence is a toral group, T^{2q}, $2q = m + 2r + s - \dim K$. Thus M is diffeomorphic to a torus T^{2q} by ψ', and $\psi'^*\Omega_M$ is an invariant symplectic form on T^{2q}. However, it is important to realize that G^+ acting on G^+/K^+ and G on G/K are *not equivariantly related by* ψ'.

We now can construct an example of a CHS manifold $M = G/K$ with G solvable (but not abelian). According to Section 6, $G = AN$ is a semi-direct product so we begin with connected, simply connected abelian groups A and N of dimensions m and $2m$ respectively. Using the fact that $\exp : \mathfrak{a} \longrightarrow A$ is an isomorphism of the vector group $\mathfrak{a}$ onto A we will identify A with its Lie algebra $\mathfrak{a} = \mathbf{R}^m$ and let $t = (t_1, \ldots, t_m)$ denote a typical element. N is isomorphic to the real vector space $\mathbf{R}^{2m}$ which it is often convenient to identify with $\mathbf{C}^m$, writing an element as $(x_1, y_1, \ldots, x_m, y_m)$ or $z = (z_1, \ldots, z_m)$, with $z_j = x_j + iy_j$, as suits our purpose. The structure of G is then determined by an arbitrary choice of m linear forms $\theta_1, \ldots, \theta_m$ on $\mathfrak{a}$ which (1) take values which are integral multiples of 2π on the lattice $\mathbf{Z}^m$ in $\mathbf{R}^m = \mathfrak{a}$ and (2) are linearly independent, i.e. a basis of a^*, the dual space to $\mathfrak{a}$. We may then denote an element of G by $(t, z) = (t_1, \ldots, t_m, z_1, \ldots, z_m)$. With this notation, the group product is expressed as follows:

$$(t, z)(t', z') = (t + t', e^{i\theta_1(t)} z_1' + z_1, e^{i\theta_2(t)} z_2' + z_2, \ldots, e^{i\theta_m(t)} z_m' + z_m) \tag{7.2}$$

where $t + t' = (t_1 + t_1', t_2 + t_2', \ldots, t_m + t_m')$. Note that the subgroup $C_G(N) = \mathbf{Z}^m N$ since $e^{i\theta_j(t)} = 1$ if and only if $t = (n_1, \ldots, n_m)$ with n_k integral. We define the form Ω at $(t, z) \in G$ by

$$\Omega_{(t,z)} = \frac{1}{2} \sum_{j=1}^{m} (e^{-i\theta_j(t)} dz_j + e^{i\theta_j(t)} d\bar{z}_j) \wedge d\theta_j. \tag{7.3}$$

Here, if $\theta_j(t_1, \ldots, t_m) = 2\pi \sum_{k=1}^{m} a_{jk} dt_k$, then $d\theta_j = 2\pi \sum a_{jk} dt_k$. The a_{jk} are integers and $d\theta_1, \ldots, d\theta_m$ are linearly independent. Of course, this can be written out in real form as

$$(7.4)\ \Omega_{(t,z)} = 2\pi \sum_j (\cos\theta_j(t)\,dx_j + \sin\theta_j(t)\,dy_j) \wedge \sum_k a_{jk}dt_k\,.$$

Using (7.2) and (7.3) it is easy to verify directly that $L^*_{(t^0,z^0)}\Omega = \Omega$ and that Ω is AdH invariant. For the latter we use the formula which follows for the inner automorphism $I_0 : G \longrightarrow G$ determined by the group element (t^0, z^0) :

$$(7.5)\ I_0(t, z) = (t, z_1 + z_1^0 - e^{-i\theta_1(t)}z_1^0, \ldots, z_m + z_m^0 - e^{-i\theta_m(t)}z_m^0).$$

It follows from (7.1) that $d\Omega = 0$ and since at the identity $(t, z) = (0,0)$ we have

$$(7.6)\qquad \Omega = \frac{1}{2}\sum_{j=1}^m (dz_j + d\overline{z_j}) \wedge d\theta_j = \sum_{j=1}^m dx_j \wedge d\theta_j\,,$$

we see that the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ must be spanned by $\dfrac{\partial}{\partial y_1}, \ldots, \dfrac{\partial}{\partial y_m}$. Thus if we take

$$K = \{(t, z) \in G \mid t \in \mathbf{Z}^m,\ z_j + \overline{z_j} = 0,\ j = 1, \ldots, m\}$$

or, in real notation,

$$K = \{(n_1, \ldots, n_m, 0, y_1, 0, y_2, \ldots, 0, y_m) \mid n_j \in \mathbf{Z},\ j = 1, \ldots, m\}$$

we will have $G/K = M$ compact of dimension $2m$, and Ω_M determined by Ω a G-invariant symplectic form (of rank $2m$).

Finally, it is of some interest to see explicitly the maps ψ, ψ' and the groups G^+, K^+ in this example.

Let $G^+ = A \oplus N = \mathbf{R}^m \times \mathbf{R}^{2m}$ be the vector group of all $3m$-tuples $(t_1, \ldots, t_m, w_1, \ldots, w_m)$ where $w_j = u_j + iv_j$ (we have again identified $\mathbf{R}^{2m}$ with $\mathbf{C}^m$). Then $\psi : G^+ \longrightarrow G$ is defined by

$$\psi(t, w) = (t, 0)(0, w) = (t_1, \ldots, t_m, e^{i\theta_1(t)}w_1, \ldots, e^{i\theta_m(t)}w_m).$$
(7.7)

The multiplication on the right is in $G = AN$. This gives $\Omega^+ = \psi^*\Omega$ on G^+ as follows

$$(7.8)\qquad \Omega^+_{(t,w)} = \frac{1}{2}\sum (dw_j + d\overline{w_j}) \wedge d\theta_j = \frac{1}{2}\sum du_j \wedge d\theta_j\,,$$

the last expression being in real terms. This gives a form with the same expression on the torus $T^{2m} = G^+/K^+$,

$$\begin{aligned} K^+ &= \mathbf{Z}^m \times \mathbf{Z}^m \times \mathbf{R}^m \\ &= \{(n_1, \ldots, n_m, k_1, \ldots, k_m, v_1, \ldots, v_m) \mid n_j, k_j \in \mathbf{Z}, v_j \in \mathbf{R}\} \end{aligned}$$

ψ induces the diffeomorphism ψ' of $T^{2m} = G^+/K^+$ onto $M = G/K$ and Ω_M corresponds to the form $\Omega_M^+ = \frac{1}{2} \sum du_j \wedge d\theta_j$ on T^{2m}.

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Manuscrit reçu le 27 juin 1979.

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