

On Compact Quantum Semigroup QS_{red}

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Abstract—We study some properties of a reduced semigroup C^* -algebra of a semigroup S . For the semigroup C^* -algebra generated by the deformation of the algebra of continuous functions on a compact abelian group we obtain a structure of a compact quantum semigroup. We also consider morphisms of constructed compact quantum semigroups.

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1. Introduction. There exist two different approaches to quantization, namely, the algebraic and topological ones. The first of them (proposed by V. G. Drinfeld [1]) implies the deformation of universal enveloping algebras. The second approach (proposed later by S. L. Woronowicz [2]) is related to the theory of compact quantum groups and semigroups. In [3] one has proved that these two approaches are equivalent. Let us illustrate the quantization process in the framework of the theory of C^* -algebras.

Let P be a compact semigroup, i.e., a compact Hausdorff space with a continuous associative operation $(x, y) \rightarrow xy$. Denote by $C(P)$ the algebra of continuous functions on P . Then $C(P)$ is a commutative unital C^* -algebra which contains all topological information on the space P . Let us identify $C(P \times P)$ and $C(P) \otimes C(P)$, and define a map $\Delta : C(P) \rightarrow C(P) \otimes C(P)$ as follows:

$$\Delta(f)(x, y) = f(xy).$$

Evidently, Δ is a continuous unital $*$ -homomorphism. The associativity of multiplication in P consists in the so-called *co-associativity* condition for Δ , namely,

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta.$$

Thus, all information on the compact semigroup P is contained in the pair $(C(P), \Delta)$.

Now, conversely, let \mathcal{A} be some commutative unital C^* -algebra. Then by the Gelfand theorem, \mathcal{A} is isomorphic to the algebra of continuous functions $C(P)$ on some compact Hausdorff space P . If on \mathcal{A} an $*$ -homomorphism $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ satisfying the co-associativity condition is given, then the equality $f(xy) = \Delta(f)(x, y)$ defines a semigroup structure on P .

In essence, a quantization results in the transfer from the commutative algebra $C(P)$ to the noncommutative unital C^* -algebra \mathcal{A} . We can treat \mathcal{A} as the algebra of continuous functions on some imaginary compact geometric object, which is said to be a *quantum space*.

A unital $*$ -homomorphism $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ satisfying the co-associativity condition is called the *co-product*. Analogously to the classical case, Δ turns a quantum space into a quantum semigroup. Then the algebra \mathcal{A} with a (given on it) co-product is the *algebra of functions on a quantum semigroup*. The pair (\mathcal{A}, Δ) is usually called a *compact quantum semigroup* [4]. See [5] for an example of a compact quantum semigroup on a noncommutative C^* -algebra. Such objects were also studied by K. Kawamura [6].

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