On Complemented Lattices,

by

R. P. DILWORTH, Pasadena, Calif. U.S.A..

1. *Introduction.* In a paper on the foundation of quantum mechanics, Kôdi Husimi(¹) conjectured that a lattice with a negation is modular if the chain law holds for every sublattice closed with respect to relative negation. Although the theorem in this form does not hold, as we show by an example, we prove a theorem of a similar nature for relatively complemented lattices.

We also show that any complemented, non-modular lattice of finite dimensions has a complemented non-modular sublattice of order five. This theorem is the analogue for complemented lattices of the theorem of Dedekind that any non-modular lattice contains a non-modular sublattice of order five. As an application, we give a new proof of the theorem due to G. Birkhoff and M. Ward(²) that a lattice of finite dimensions is a Boolean algebra if and only if every element has a unique complement.

2. *Notation and terminology.* We denote the fixed lattice of elements $a, b, c, \ldots$ by $\mathfrak{S}$. $(,), [,], \supset$ denote union, cross-cut, and lattice division respectively. German capitals will denote sublattices of $\mathfrak{S}$ and subsets of $\mathfrak{S}$ which are not necessarily sublattices will be denoted by latin capitals. If $a \supset x \supset b$, $a \neq b$ implies $x = a$ or $x = b$ we say that a "covers" b and write $a > b$. Elements which cover the null element z of a lattice are called *points* and elements which are covered by the unit element i are said to be *simple*.

A lattice $\mathfrak{S}$ is said to satisfy the *ascending chain condition* if every chain $a_1 \subset a_2 \subset a_3 \subset \ldots$ has only a finite number of distinct members. Similarly $\mathfrak{S}$ is said to satisfy the descending chain condition if every chain $a_1 \supset a_2 \supset a_3 \supset \ldots$ has only a finite number of distinct members. If both the ascending and descending chain conditions hold, $\mathfrak{S}$ is said to have finite dimensions. A chain $a = a_0 \supset a_1 \supset a_2 \supset \ldots \supset a_n = b$ joining two elements a and b is said to be *complete*

(¹) K. Husimi, *Studies on the foundations of quantum mechanics.* Proc. of the Physico-Math. Soc. of Japan, **19** (1937), pp. 766–789.

(²) G. Birkhoff and M. Ward, Bull. of the Amer. Math. Soc., abstract (45-1-78).

if $a_i > a_{i+1}$. A lattice of finite dimensions is said to satisfy the chain law if every complete chain joining any two elements a and b with $a \supset b$ has the same length.

An element a' is said to be a complement of a if $(a, a') = i$ and $[a, a'] = z$. If every element of $\mathfrak{S}$ has a complement, then $\mathfrak{S}$ is said to be complemented. $\mathfrak{S}$ is said to be relatively complemented if $a \supset b$ implies there exists an element b_1 such that $(b, b_1) = a$ and $[b, b_1] = z$.

An involutory automorphism $a \longleftrightarrow a'$ of $\mathfrak{S}$ is called a *negation* if $(a, a') = i$ and $[a, a'] = z$. A lattice with a negation is clearly complemented. If $\mathfrak{S}$ has a negation, then a sublattice $\mathfrak{A}$ of $\mathfrak{S}$ is said to be closed with respect to relative negation if with a and b, $a \supset b$ it contains $[a, b']$ and $a = (b, [a, b'])$. A lattice closed with respect to relative negation is clearly relatively complemented.

Definition 2. 1. A lattice $\mathfrak{S}$ is said to be a *Birkhoff* lattice if:

(1) $$a > [a, b] \text{ implies } (a, b) > b.$$

$\mathfrak{S}$ is said to be a dual Birkhoff lattice if

(2) $$(a, b) > a \text{ implies } b > [a, b].$$

Condition (1) and (2) are closely connected with modularity[1] as is shown by the following lemma proved by Garrett Birkhoff[2]:

Lemma 2.1. *A finite dimensional lattice $\mathfrak{S}$ is modular if and only if it is both a Birkhoff and dual Birkhoff lattice.*

3. *Relatively complemented lattices.* We are now ready to prove the first theorem mentioned in the introduction.

Theorem 3.1. *Let $\mathfrak{S}$ be a relatively complemented lattice of finite dimensions. Then if every relatively complemented sublattice satisfies the chain law, $\mathfrak{S}$ is a dual Birkhoff lattice.*

Proof. If $\mathfrak{S}$ is not a dual Birkhoff lattice there is an element x such that there exist two elements x_1 and x_2 for which $x > x_1$, $x \supset x_2$, $x_1 \not\supset x_2$, $x_2 \not> [x_1, x_2]$. For clearly $(x_1, x_2) = x > x_1$. Now let S be the set of all such elements x. Then since the descending chain condition holds in $\mathfrak{S}$, S must have at least one minimal element a. Since $a \varepsilon S$ there exist elements a_1 and y such that $a > a_1$, $a \supset y$, $a_1 \not\supset y$, $y \not> [a_1, y]$. For a and a_1 fixed let T be the set of all such elements y. Then T must have at least one minimal element b.

---

(1) A lattice $\mathfrak{S}$ is modular if $a \supset b$ implies $[a, (b, c)] = (b, [a, c])$.

(2) Garrett Birkhoff. *On combination of subalgebra*, Proc. of the Cambridge Phil. Soc., 29 (1933), pp. 441-464.

Hence $a>a_1$, $a\supset b$, $a_1\not\supset b$, $b\not>[a_1, b]$. Since $b\not>[a_1, b]$ there exists an x such that $b>x\supset[a_1, b]$, $x\neq[a_1, b]$. But if $x\not>[a_1, b]$, then x belongs to T which contradicts the minimal property of b. Thus $b>x>[a_1, b]$. We will now show that $[a_1, b]=z$. Suppose that $[a_1, b]\neq z$. Since $\mathfrak{S}$ is relatively complemented, there exists an element y such that $(y, [a_1, b])=b$, $[y, [a_1, b]]=z$. Since $[a_1, b]\neq z$, we have $b\neq y$ and there exists an element m such that $b>m\supset y$. Then $m\neq x$ since otherwise $x\supset y$ and $x\supset[a_1, b]$. Whence $x\supset(y, [a_1, b])=b$ contradicting $b>\chi$. Also $[m, \chi]\neq[a_1, b]$ since otherwise $m\supset(y, [a_1, b])$ $=b$ which contradicts $b>m$. Now $b=(m, x)>x$ and hence $m>[m, x]$ by the minimal property of a. Similarly $x>[m, x]$. Now $x=([m, x]$, $[a_1, b])>[a_1, b]$. Hence $[m, x]>[[m, x], [a_1, b]]=[m, a_1, b]=[m, a_1]$ by the minimal property of a. But then $m>[m, x]>[m, a_1]$. Hence $a\supset m$, $a_1\not\supset m$, $m\not>[m, a_1]$ and m then belongs to T. This however contradicts the minimal property of b. Hence we have $[a_1, b]=z$.

Since $b\supset x$ there exists an element x_1 such that $(x, x_1)=b$, $[x, x_1]=z$. Now $a\supset(a_1, x)\supset a_1$ and since $a>a_1$, either $a=(a_1, x)$ or $a_1\supset x$. But if $a_1\supset x$, then since $b\supset x$ we have $z=[a_1, b]\supset x$ and hence $x=z$ which contradicts $x>[a_1, b]$. Hence $(a_1, x)=a$ and similarly $(a_1, x_1)=a$. But $[a_1, (x, x_1)]=[a_1, b]=z$. Hence $\{a, a_1, b, x, x_1, z\}$ is a sublattice closed with respect to relative complement in which the chain law does *not hold*. This contradicts the hypothesis of the theorem and $\mathfrak{S}$ is thus a dual Birkhoff lattice.

Corollary: *Let $\mathfrak{S}$ satisfy the hypotheses of theorem* 3.1. *Then if $\mathfrak{S}$ has a negation, $\mathfrak{S}$ is modular.*

For $\mathfrak{S}$ is a dual Birkhoff lattice by theorem 3.1 and since $\mathfrak{S}$ has a dual automorphism $\mathfrak{S}$ is also a Birkhoff lattice. Hence by lemma 2.1 $\mathfrak{S}$ is modular.

It will be noted that the converse of theorem 3.1 does not hold in general; that is, in a relatively complemented dual Birkhoff lattice every relatively complemented sublattice need *not* satisfy the chain law. Consider for example the lattice diagramed in Fig. 1.

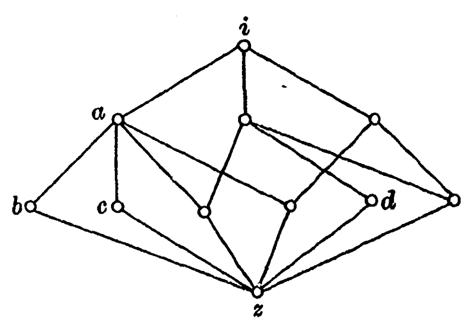
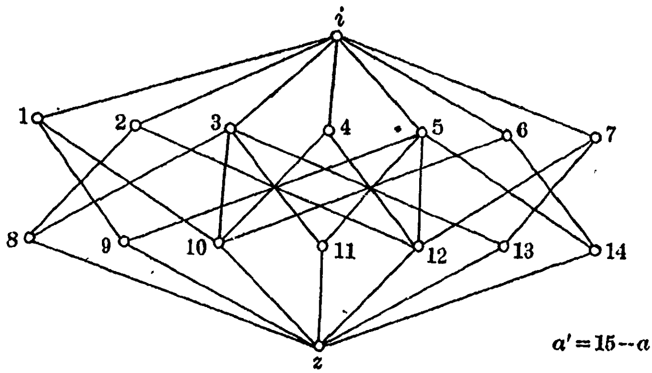


Fig. 1

The sublattice $\{i, a, b, c, d, z\}$ does not satisfy the chain law.

We conclude this section with an example of a lattice in which the theorem conjectured by Husimi does not hold.



$\mathfrak{S}$

Fig. 2

$\mathfrak{S}$ has a negation and is closed with respect to relative negation. Furthermore every sublattice closed with respect to relative negation satisfies the chain law. However $\mathfrak{S}$ is neither a Birkhoff lattice nor a dual Birkhoff lattice and hence is non-modular by lemma 2.1.

4. *Complemented non-modular lattices.* We prove now the second theorem mentioned in the introduction:

Theorem 4.1. *Every complemented, non-modular lattice of finite dimensions contains a complemented non-modular sublattice of order five.*

Proof. Let q be a cross-cut irreducible(1) of the lattice $\mathfrak{S}$. Then if q is not a simple element, there exists an element $q_1 \neq i$ such that $q_1 > q$. Let q_1' be the complement of q_1. Then $(q_1, q_1') = (q, q_1') = i$ and $[q_1, q_1'] = [q, q_1'] = z$. Hence $\{i, q_1, q_1', q, z\}$ is a sublattice of the desired type. We may thus assume that the only cross-cut irreducibles are simple elements and similarly that the only union irreducibles are points.

We show now that if $\mathfrak{S}$ contains no complemented, non-modular

(1) An element q is said to be cross-cut irreducible if $q = [a, b]$ implies either $q = a$ or $q = b$. If the lattice satisfies the descending chain condition q is cross-cut irreducible if and only if there is only one element covering q. Similarly p is union irreducible if $p = (a, b)$ implies either $p = a$ or $p = b$. p then covers only one element of $\mathfrak{S}$ if $\mathfrak{S}$ satisfies the ascending chain condition.

sublattice of order five, then $\mathfrak{S}$ is a **Birknoff** lattice. Dualizing the proof then shows that $\mathfrak{S}$ is a dual **Birkhoff** lattice and hence is modular by lemma 2.1 thus contradicting the hypothesis of the theorem.

If $\mathfrak{S}$ is not a **Birkhoff** lattice there is an element x with the property that $(x, p) \not\succ x$, $(x, p) \neq x$ for some point p. For by definition 2.1 there exists an element x_1 such that $x_1 > [x, x_1]$ but $(x, x_1) \not\succ x$. Now by the first paragraph we may assume that each element is a union of points. Hence there exists a point p such that $x_1 = ([x, x_1], p)$. But then $(x, p) = (x, [x, x_1], p) = (x, x_1) \not\succ x$. Let S be the set of all such elements x and let a be a maximal element of S. Since a is in S there is an element a_1 such that $(a, p) \supset a_1 > a$. If $(a, p) \not\succ a_1$, then a_1 is in S contradicting the maximal property of a. Hence $(a, p) > a_1 > a$. We show now that a_1 is simple and hence $(a, p) = i$. If a_1 is not simple, there exists an element y such that $y > a_1$, $y \neq (a, p)$. Now let $a_2 = [y, (a, a_1')]$. Then $y \supset a_2 \supset a$. Suppose that $y \supset a_2 \supset a_1$. Then either $y = a_2$ or $a_2 = a_1$ since $y > a_1$. If $y = a_2$, then $(a, a_1') \supset y$ and hence $(a, a_1') = (a, a_1', a_1') \supset (y, a_1') \supset (a_1, a_1') = i$. Thus $(a, a_1') = (a_1, a_1') = i$ and $[a, a_1'] = [a_1, a_1'] = z$. But then $\{i, a, a_1, a_1', z\}$ is a complemented non-modular sublattice of order five which contradicts our assumption. If $a_2 = a_1$, then $(a, a_1') \supset a_1$ and hence $(a, a_1') = (a, a_1', a_1') \supset (a_1, a_1') = i$. Thus $\{i, a, a_1, a_1', z\}$ is again a complemented non-modular sublattice of order five which contradicts our assumption. Hence $y \supset a_2 \supset a_1$ does not hold. Suppose now that $a_1 \supset a_2 \supset a$. Since $a_1 > a$ and $a_1 \neq a_2$ we must have $a_2 = a$ and $a \supset [y, (a, a_1')] \supset (a, [y, a_1']) \supset a$. Hence $(a, [y, a_1']) = a$ and $a \supset [a_1', y]$. But then $z = [a_1, a_1'] \supset [a, a_1'] \supset [y, a_1', a_1'] = [y, a_1']$. Thus $[y, a_1'] = z = [a_1, a_1']$. Also $(y, a_1') = (a_1, a_1') = i$. Thus $\{i, y, a_1, a_1', z\}$ is a complemented non-modular sublattices of order five which contradicts our assumption. Hence $a_1 \supset a_2 \supset a$ does not hold. Since $y > a_1 > a$ we have $(a_1, a_2) = y$, $[a_1, a_2] = a$. Now $(p, a_2) \neq (a, p)$. For if $(p, a_2) = (a, p)$, thene $a_1 = [y, (a, p)] = [y, (p, a_2)] = [y, ((a, p), a_2)] \supset (a_2, [y, (a, p)]) = (a_2, a) = y$ which contradicts $y > a_1$. Also $(y, p) > (a, p)$. For if $(y, p) \not\succ (a, p)$, let $y = (a_1, p_1)$. Then $(p_1, (a, p)) = (p_1, (a_1, p)) = ((p_1, a), p) = (y, p)$. Also $(a, p) \not\supset p_1$ since otherwise $(a, p) \supset (a_1, p_1, p) = (y, p) \supset y$ which is impossible. But then $(p_1, (a, p)) \not\succ (a, p)$, $(p_1, (a, p)) \neq (a, p)$ and (a, p) is a proper divisor of a. But then (a, p) is in S which contradicts the maximal property of a.

We have $(y, p) \supset (a_2, p) \supset (a, p)$. Hence by the result we have just obtained $(y, p)=(a_2, p)$. Thus $(a_2, p)\supset y\supset a_2$, $(a_2, p)\neq y$ since otherwise $y\supset p$ implies $y\supset(a, p)\supset a_1$ implies $y=(a, p)$ which contradicts the definition of y. Also $y\neq a_2$ as has already been shown. Hence $(a_2, p)\not\succ a_2$ and $(a_2, p)\neq a_2$. But $a_2\supset a$ and $a_2\neq a$. This contradicts the maximal property of a. Thus a_1 is simple and $(a, p)=i$. But then $\{i, a_1, a, p, z\}$ is a complemented non-modular sublattice of order five which contradicts our assumptions. Hence $\mathfrak{S}$ is a Birkhoff lattice and the theorem is proved.

Theorem 4.1 may be used to give a new proof of the following theorem due to G. Birkhoff and M. Ward.

Theorem 4.2. *A lattice of finite dimensions is a Boolean algebra if and only if every element has a unique complement.*

For if every element of a lattice $\mathfrak{S}$ has a unique complement, then $\mathfrak{S}$ must be modular by theorem 4.1. But it is well known([1]) that a modular lattice with unique complement is a Boolean algebra. This completes the proof.

California Institute of Technology,
Pasadena, Calif..

(Received September 12, 1939)

([1]) See for example, Huntington, Trans. Amer. Math. Soc., 5 (1904), p. 288; Skolem, Videnskapsselskepets Skrifter (1919); Bergman, Monatshefte f. Math. u. Phys., 36 (1929).