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# ON COMPLETE FLAT SURFACES IN HYPERBOLIC 3-SPACE

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## § 1. Introduction.

"Any 2-dimensional, connected complete and flat Riemannian manifold $M$ isometrically immersed in the Euclidean space $E^3$ is a plane or a cylinder." This theorem was first proved by Pogorelov [3, 4] in 1956 and an elementary proof was given by Massey [2] in 1962. Correspinding to it, the problems to characterize 2-dimensional connected, complete and flat Riemannian manifolds isometrically immersed in 3-sphere $S^3$ and in hyperbolic 3-space $H^3$ arise. The author studied the $S^3$ case in [7]. In this paper we shall study the $H^3$ case. Main theorems are Theorem 3 in §3 and Theorem 6 in § 5 which tell us that "any complete flat surface in $H^3$ is either a horosphere or an equidistant surface of a geodesic line." For the sake of simplicity, all functions are assumed to be smooth, i.e. of class $C^\infty$.

## § 2. Basic considerations.

As the model of the hyperbolic 3-space $H^3$ we take the upper half space $x^3>0$ in the sense of Poincaré's representation. Without any loss of generality, we may assume that the sectional curvature of $H^3$ is $-1$. In this case the metric tensor of $H^3$ is given by

$$G_{\alpha\beta}=(x^3)^{-2}\delta_{\alpha\beta}. \tag{2.1}$$

Now, let us consider a connected complete surface $M$ (i.e. 2-dimensional Riemannian manifold $M$ immersed) in $H^3$ and take a coordinate neighborhood $U$ on $M$. Then, $U$ can be expressed parametrically in the form $x^\alpha=x^\alpha(u^1, u^2)$ $(\alpha, \beta, \gamma, \delta=1, 2, 3)$. If we put $X_i^\alpha\equiv\partial x^\alpha/\partial u^i$ $(i, j, k, l=1, 2)$ and choose the unit normal vector field $N^\alpha$ so that $|X_1, X_2, N|>0$. Then we have

$$g_{ij}=G_{\alpha\beta}X_i^\alpha X_j^\beta, \tag{2.2}$$

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A few months after I completed this paper, I found that the main theorems were also found independently by Yu. A. Volkov and S.M. Vladimirova a little earlier than me (Cf. Isometric immersions of a Euclidean plane in Lobachevskii space, Math. Notices, Acad. Sci. USSR **10** (1972), 619-622, Russian Original (1971)).

(2.3) $$D_{X_k}X_j^\alpha=\begin{Bmatrix} i \\ j \quad k \end{Bmatrix}X_i^\alpha+h_{jk}N^\alpha,$$

(2.4) $$D_{X_k}N^\alpha=-h_k^iX_i^\alpha,$$

where $g_{ij}$, $h_{ij}$, $\{{}_j{}^i{}_k\}$ and $N^\alpha$ are the first and the second fundamental tensors of $M$, the Christoffel's symbol with respect to $g_{ij}$ and the unit normal vector of $M$, and $D_{X_k}$ means the covariant derivative in $H^3$ in the direction of $X_k^\alpha$. (2.3), (2.4) are Gauss' and Weingarten's derived equations.

The integrability conditions of (2.3) and (2.4) are

(2.5) $$R_{ijkl}-h_{jk}h_{il}+h_{ik}h_{jl}=-g_{jk}g_{il}+g_{ik}g_{jl}$$

and

(2.6) $$\nabla_l h_{jk}-\nabla_k h_{jl}=0$$

known as Gauss' and Codazzi's equations, where $R_{ijkl}$ is the curvature tensor with respect to $g_{ij}$ and $\nabla_k$ means the covariant differentiation. When $M$ is flat, (2.5) is equivalent with

(2.5)′ $$h_{11}h_{22}-h_{12}^2=g_{11}g_{22}-g_{12}^2.$$

Now assume that $M$ is complete. Then, $M$ can be regarded as an isometric immersion of the Euclidean plane $E^2$ with rectangular coordinates $(u^1, u^2)$ and we have

(2.7) $$g_{11}=g_{22}=1, \qquad g_{12}=0.$$

So (2.5)′ and (2.6) reduce to

(2.8) $$h_{11}h_{22}-h_{12}^2=1,$$

(2.9) $$\partial h_{11}/\partial u^2=\partial h_{12}/\partial u^1,\ \partial h_{22}/\partial u^1=\partial h_{12}/\partial u^2.$$

(2.9) tells us that there exists a smooth function $\phi(u^1, u^2)$ defined on the whole plane $E^2$ such that

(2.10) $$h_{11}=\phi_{11},\ h_{12}=\phi_{12},\ h_{22}=\phi_{22},$$

where we have put $\phi_{ij}=\partial^2\phi/\partial u^i\partial u^j$. Thus (2.8) reduces to

(2.11) $$\phi_{11}\phi_{22}-\phi_{12}^2=1.$$

Now, by a theorem of Jörgens [1], the differential equation of elliptic type (2.11) admits as solutions only polynomials of the second degree of the variables $u^1$ and $u^2$. So $h_{ij}$'s are constants. If $h_{12}=0$, then

(2.12) $$h_{11}=\lambda_1,\ h_{22}=\lambda_2 \qquad (h_{12}=0),$$

where $\lambda_1$, $\lambda_2$ are principal curvatures, i.e. eigenvalues of the second fundamental tensor and satisfy

$$\lambda_1\lambda_2=1. \tag{2.13}$$

In the case $h_{12}\neq 0$, we can reduce it to the first case by a suitable orthogonal transformation of rectangular coordinates $u^1, u^2$ in $E^2$.

Thus we get the following theorem:

THEOREM 1. *For each complete flat surface $M$ in $H^3$ regarded as an isometric immersion of the Euclidean plane $E^2$ with rectangular coordinates $(u^1, u^2)$, the principal curvatures $\lambda_1$ and $\lambda_2$ are constant and their product is equal to* 1. *Conversely, if we take two constants $\lambda_1$ and $\lambda_2$ so that their product is equal to* 1, *then there exists a complete flat surface in $E^3$ such that its principal curvatures coincide with the given $\lambda_1$ and $\lambda_2$.*

The proof of the latter part follows easily if we define $g_{ij}$ and $h_{ij}$ by (2.7) and (2.12) and apply the first fundamental theorem of surfaces in space forms (Cf. [6]). On $M$ parameter curves are lines of curvature and isothermal.

COROLLARY. *Every complete flat surface in $H^3$ can not be a minimal surface.*

As both of the first and second fundamental tensors have constant components with respect to a parameter system which covers $M$, we see, by the fundamental theorem of surfaces in $H^3$ again, that the following theorem is true.

THEOREM 2. *Every complete flat surface $M$ in $H^3$ is an orbit space of a* 2-*parametric subgroup of the isometry group $I(H^3)$ of $H^3$.*

The constants $\lambda_1$ and $\lambda_2$ have the same sign. If $\lambda_1$ and $\lambda_2$ are negative, we may change parameters and the unit normal vector so that

$$\bar{u}^1=-u^1,\ \bar{u}^2=u^2,\ \bar{N}^\alpha=-N^\alpha.$$

And the determinant $|\bar{X}_1\bar{X}_2\bar{N}|$ of the new Gaussian frame is positive and $\bar{\lambda}_1=-\lambda_1$, $\bar{\lambda}_2=-\lambda_2$. Hence, we may hereafter assume without any loss of generality that $\lambda_1$ and $\lambda_2$ are positive.

From the above arguments, we may, without any loss of generality, classify complete flat surfaces into following two types by their principal curvatures:

Umbilical type: $\lambda_1=\lambda_2=1$,

Non-umbilical type: $\lambda_2>1>\lambda_1>0 \quad (\lambda_1\lambda_2=1)$.

## § 3. Complete flat totally umbilical surfaces.

(2.1) shows that the Riemannian metrics of $H^3$ and $E^3$ in the upper half space $x^3>0$ are conformal with each other.

In general, for a conformal change of Riemannian metrics $G_{\alpha\beta}=\sigma^2 G^0_{\alpha\beta}$ on a differentiable manifold $V^3$ we have

$$\left\{\begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix}\right\} = \left\{\begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix}\right\}_0 + \delta^\alpha_\beta \sigma_\gamma + \delta^\alpha_\gamma \sigma_\beta - G_0^{\alpha\delta} \sigma_\delta G^0_{\beta\gamma}, \tag{3.1}$$

where we have put $\sigma_\alpha = \partial \log \sigma / \partial x^\alpha$. We consider a surface $F$ immersed in $V^3$ and denote its unit normal vector, its first and second fundamental tensors with respect to the Riemannian metric $G^0_{\alpha\beta}$ by $N_0^\alpha$, $g^0_{ij}$ and $h^0_{ij}$ respectively, and those with respect to the Riemannian metric $G_{\alpha\beta}$ by $N^\alpha$, $g_{ij}$ and $h_{ij}$ respectively. Then there exist following relations as we can easily verify them:

$$N^\alpha = (1/\sigma) N_0^\alpha, \tag{3.2}$$

$$g_{ij} = \sigma^2 g^0_{ij}, \tag{3.3}$$

$$h_{ij} = \sigma \{h^0_{ij} - (N_0^\alpha \sigma_\alpha) g^0_{ij}\}. \tag{3.4}$$

From (3.4) we see that the following lemma is true. (Cf. [5])

LEEMA. *The totally umbilical property of a surface in a Riemannian manifold $V^3$ is invariant under any conformal change of metrics.*

When $F$ is totally umbilical, then we see easily that

$$\Omega = (1/\sigma)(\Omega^0 - N_0^\alpha \sigma_\alpha). \qquad (\Omega, \Omega^0\text{: mean curvatures}) \tag{3.5}$$

By virtue of the Lemma, a complete flat totally umbilical surface $M$ in $H^3$ is also a totally umbilical surface in $E^3$. So, it is a piece or the whole of an ordinary sphere or plane in $E^3$. This tells us that $M$ in consideration is one of proper spheres, horo-spheres, equidistant surfaces or $H$-planes in $H^3$ where $H$-plane means a plane in the sense of hyperbolic geometry. Thus, we have reduced our problem to calculate the function $\lambda$ for each of these surfaces and to pick up the one for which $\lambda = \pm 1$.

Now, without any loss of generality, we may express any one of surfaces in $H^3$ described above by an equation of the type

$$(x^1)^2 + (x^2)^2 + (x^3 - c)^2 = R^2 (R > 0), \tag{3.6}$$

the cases $c > R$; $c = R$; $R > c > -R (c \neq 0)$ and $c = 0$ corresponding to a proper sphere, a horo-sphere, an equidistant surface and an $H$-plane respectively. If we express (3.6) parametrically by

$$x^1 = u^1,\ x^1 = u^2,\ x^3 = x^3(u^1, u^2), \tag{3.6$'$}$$

then we see first that

$$X^\alpha_j = \begin{cases} \delta^i_j & \text{for } \alpha = i\ (=1 \text{ or } 2), \\ -x^j/(x^3 - c) & \text{for } \alpha = 3. \end{cases} \tag{3.7}$$

As $\sigma = 1/x^3$ and $G^0_{\alpha\beta} = \delta_{\alpha\beta}$ in our case

$$g^0_{ij} = G^0_{\alpha\beta} X^\alpha_i X^\beta_j = \delta_{ij} + x^i x^j / (x^3 - c)^2, \tag{3.8}$$

(3.9)
$$N_0^\alpha=\begin{cases} x^i/R & \text{for } \alpha=i \ (=1 \text{ or } 2), \\ (x^3-c)/R & \text{for } \alpha=3. \end{cases}$$

(We took the normal direction toward outside as the positive direction of the normal.) Then, as

(3.10)
$$h_{ij}^0=-(1/R)g_{ij}^0,\ \Omega^0=-1/R,$$

we see by (3.5) that

(3.11)
$$\Omega=-c/R,$$

i.e. the mean curvature of the surface (3.6) is $-c/R$. Hence $\lambda=\pm 1$ if and only if the surface in consideration is a horo-sphere. Thus we get the following.

THEOREM 3. *Any complete flat totally umbilical surface in the hyperbolic* 3-*space is a horo-sphere. It is isometric with the Euclidean plane.*

N. B. A similar theorem holds good for any complete flat totally umbilical hypersurface $M^n$ in $H^{n+1}$ too.

## § 4. Geometrical construction of complete flat surfaces in $H^3$.

In order to study complete flat non-umbilical surface, we shall study here some geometric properties of complete flat surfaces.

For any curve $x^\alpha=x^\alpha(u^1(s), u^2(s))$ on $M$ defined in some interval of $s$, we get easily

$$D_T T^\alpha=X_i^\alpha(\nabla_T T^i)+(h_{ij}T^iT^j)N^\alpha,$$

where $T^\alpha=X_i^\alpha T^i$ is the unit tangent vector. Any $u^1$-curve on $M$ is a geodesic of $M$ as it is the image of a straight line by an isometric immersion of $E^2$ into $H^3$. So for a $u$-curve, we have

(4.1)
$$D_T T^\alpha=h_{11}N^\alpha.$$

Now, the Frenet formulas of the $u^1$-curve are of the form

(4.2)
$$\begin{aligned} D_T T^\alpha&=\kappa_1 H, \\ D_T H^\alpha&=-\kappa_1 T^\alpha+\tau_1 B^\alpha, \\ D_T B^\alpha&=-\tau_1 H^\alpha, \end{aligned}$$

Comparing (4.1) with (4.2), we see first $H^\alpha=N^\alpha$ as $\kappa_1>0$ by assumption and $h_{11}=\lambda_1>0$. So we get

(4.3)
$$T^\alpha=X_1^\alpha,\ H^\alpha=N^\alpha,\ B^\alpha=-X_2^\alpha,$$

$(4.4)_1$
$$\kappa_1=h_{11}=\lambda_1.$$

On the other hand, we have

$$D_T H^\alpha = D_{X_1} N^\alpha = -h_1^i X_i^\alpha$$
$$= -h_{11} T^\alpha + h_{12} B^\alpha.$$

Comparing this with $(4.2)_2$, we get

$$(4.2)_2 \qquad \tau_1 = h_{12} = 0.$$

In the same way, we see that the Frenet's frame of any $u^2$-curve on $M$ is given by

$$(4.5) \qquad \bar{T} = X_2,\ \bar{H} = N,\ \bar{B} = X_1$$

and the curvature and torsion are given by

$$(4.6) \qquad \kappa_2 = h_{22} = \lambda_2,\ \tau_2 = -h_{12} = 0$$

respectively. Thus, we get the following

THEOREM 4. *For each complete flat surface $M$ in $H^3$ with the principal curvatures $\lambda_1$ and $\lambda_2$, the curvature and torsion of a family of lines of curvature are given by* (4.4) *and those of another family of lines of curvature are given by* (4.6).

Now the above argument suggests us a method how to construct complete flat surfaces in $H^3$.

THEOREM 5. *Let $\lambda_1$ and $\lambda_2$ be two positive constants such that their product is equal to* 1. *We first draw a curve $\Gamma_1$ with curvature $\kappa_1(u_1, 0) = \lambda_1$ and torsion $\tau_1(u^1, 0) = 0$ in $H^3$, the parameter being the arc length. Using the moving Frenet's frame $(T, H, B)$ of $\Gamma_1$, we draw, for each fixed value $u^1$, a curve $\Gamma_2(u^1)$ with curvature $\kappa_2(u^1, u^2) = \lambda_2$ and torsion $\tau_2(u^1, u^2) = 0$ with initial Frenet's frame*

$$(4.7) \qquad \bar{T}(u^1, 0) = -B(u^1),\ \bar{H}(u^1, 0) = H(u^1),\ \bar{B}(u^1, 0) = T(u^1),$$

*the parameter $u^2$ being arc length. Then, the locus of all $\Gamma_2(u^1)$ $(u^1 \in R)$ is a complete flat surface in $H^3$.*

*Proof*. By the latter half of Theorem 1, there exists complete flat surfaces in $H^3$ such that (2.7) and (2.12) hold good and any two of them are congruent under a motion of $H^3$. We take any one of them and denote it by $M$. $M$ can be regarded as an isometric immersion of $E^2$ into $H^3$ by a map $f$.

At each point $f(u^1, u^2)$ of $M$, we define an orthonormal frame $(T, H, B)$ by

$$(4.8) \qquad T(u^1, u^2) = X_1(u^1, u^2), \quad H(u^1, u^2) = N(u^1, u^2), \quad B(u^1, u^2) = -X_2(u^1, u^2).$$

We fix the value $u^2$, then they constitute the moving Frenet's frame for the $u^1$-curve and the curvature and torsion are given by

$$(4.9) \qquad \kappa_1(u^1, u^2) = \lambda_1,\ \tau_1(u^1, u^2) = 0.$$

Especially, the moving Frenet frame $(T(u^1), H(u^1), B(u^1))$ of the $u^1$-curve $u^2=0$ relates to the Gauss' frame of $M$ on the curve by

$$T(u^1)=X_1(u^1, 0),\ H(u^1)=N(u^1, 0), \qquad B(u^1)=-X_2(u^1, 0). \tag{4.10}$$

In the same way, the moving frame

$$\bar{T}(u^1, u^2)=X_2(u^1, u^2),\ \bar{H}(u^1, u^2)=N(u^1, u^2), \qquad \bar{B}(u^1, u^2)=\bar{X}(u^1, u^2) \tag{4.11}$$

gives, for each fixed value of $u^1$, the moving Frenet's frame of the $u^2$-curve. The curvature and torsion of the latter curve are given by

$$\kappa_2(u^1, u^2)=\lambda_2, \qquad \tau_2(u^1, u^2)=0. \tag{4.12}$$

By (4.8), (4.10) and (4.11) we get (4.7). This completes the proof.

## § 5. Complete flat non-totally umbilical surfaces.

As a preparation we remark, by (2.1) and (3.1), that

$$\begin{cases} \left\{ {i \atop j\ k} \right\}=0, & \left\{ {3 \atop j\ k} \right\}=-\sigma\delta_{jk}, \\ \left\{ {i \atop 3\ k} \right\}=-\delta^i_k\sigma, & \left\{ {3 \atop 3\ k} \right\}=0, \\ \left\{ {i \atop 3\ 3} \right\}=0, & \left\{ {3 \atop 3\ 3} \right\}=-\sigma \end{cases} \tag{5.1}$$

hold good, where $\sigma=1/x^3$.

First, let us consider a half line $\Gamma_1$

$$x^1=t,\ x^2=0,\ x^3=\tan\omega\cdot t \qquad (t>0) \tag{5.2}$$

in the plane $x^2=0$, where $\omega$ is the angle such that $\tan\omega=\sqrt{1-\lambda_1^2}/\lambda_1$ and $0<\omega<\pi/2$. Then, the line element $du^1$ and the unit tangent vector $T$ are given by

$$du^1=\frac{dt}{x^3\cos\omega}, \tag{5.3}$$

$$T=(x^3\cos\omega,\ 0,\ x^3\sin\omega). \tag{5.4}$$

As $x^2=0$ is an $H$-plane and each $H$-plane is a totally geodesic surface in $H^3$, the unit principal normal vector $H$ lies in the $H$-plane $x^2=0$ and so we see that $H=(-x^3\sin\omega, 0, x^3\cos\omega)$ and the unit binormal vector is given by $B=(0, x^3, 0)$. Putting (5.1) and (5.3) into

$$\frac{\delta T^\alpha}{du^1}=\frac{dt}{du^1}\frac{\delta T^\alpha}{du^1}=\frac{dt}{du^1}\frac{dT^\alpha}{dt}+\left\{ {\alpha \atop \beta\ \gamma} \right\}T^\beta T^\gamma$$

we can easily verify that

$$\frac{\delta T^\alpha}{du^1}=\kappa_1 H^\alpha, \qquad \kappa_1=\cos\omega=\lambda_1 \tag{5.5}$$

holds good, where $\kappa_1$ is the curvature of $\Gamma_1$. In the same way, we can easily get

$$\frac{\delta H^\alpha}{du^1}=-\kappa_1 T^\alpha,\ \tau_1=0 \tag{5.6}$$

where $\tau_1$ is the torsion of $\Gamma_1$.

Secondly, let us consider a circle $\Gamma_\gamma$ defined by

$$x^1=\gamma\cos\theta,\ x^2=\gamma\sin\theta,\ x^3=k\ \ (k=\gamma\tan\omega) \tag{5.7}$$

on a plane (a horo-sphere) $x^3=k$, where $\gamma>0$ is a constant. Then, we see that its arc length $u^2$ and the unit tangent vector $\bar{T}$ are given by

$$\frac{du^2}{d\theta}=\cot\omega \qquad (u^2=\theta\cot\omega), \tag{5.8}$$

$$\bar{T}=(-k\sin\theta,\ k\cos\theta,\ 0). \tag{5.9}$$

We denote the $H$-plane with center 0 (the origin) and radius $\gamma/\lambda_1$ by $\pi_\gamma$, then $\Gamma_\gamma$ lies on $\pi_\gamma$. So, the unit principal normal vector $\bar{H}$ is tangent to $\pi_\gamma$, normal to $\bar{T}$ and is given by

$$\bar{H}=\left(\frac{-k^2\cos\theta}{\sqrt{k^2+\gamma^2}},\ \frac{-k^2\sin\theta}{\sqrt{k^2+\gamma^2}},\ \frac{k\gamma}{\sqrt{k^2+\gamma^2}}\right).$$

In the similar way as the case of $\Gamma_1$, we can easily verify that

$$\frac{\delta\bar{T}^\alpha}{du^2}=\kappa_2\bar{H}^\alpha, \qquad \kappa_2=\frac{1}{\lambda_1}=\lambda_2,$$

(5.10)

$$\frac{\delta H^\alpha}{du^2}=-\kappa_2\bar{T}^\alpha, \quad \tau_2=0.$$

Thirdly, let us consider a half cone $S$ which is a surface of revolution of the half line $\Gamma_1$ around the $x^3$-axis. Then, it is easy to see that (i) all generating lines have same curvature $\lambda_1$ and torsion 0 and are equidistant curves to the $x^3$-axis and (ii) all circles $\Gamma_\gamma$ are same curvature $\lambda_2$ and torsion 0 and are congruent in $H^3$. Thus, $S$ has similar properties as a circular cylinder in $E^3$.

Now, we may regard the curve $\Gamma_1$ defined by (5.2) as the curve $\Gamma_1$ in Theorem 5. Its arc length $u^1$ is given by $u^1=\operatorname{cosec}\omega\cdot\log t$. The circle $\Gamma_\gamma$ corresponds to the curve $\Gamma_2(u^1)$ in Theorem 5 for $u^1=\operatorname{cosec}\omega\cdot\log\gamma$. As $u^2=\theta\cot\omega$, we may easily verify that the Frenet's frame $(\bar{T},\bar{H},\bar{B})$ of $\Gamma_\gamma$ at the point $\theta=0$, coincides with $(-B,H,T)$ of $\Gamma_1$ at the same point. Hence, the cone $S$ is nothing but the complete flat surface corresponding to the given constants $\lambda_1$ and $\lambda_2$ assured in Theorem 5. (We may easily verify directly that the Gaussian curvature of $S$ is everywhere equal to zero.) $S$ is an orbit space of a 2-parametric subgroup of isometries of the form

$$\bar{x}^1=\rho(x^1\cos\gamma+x^2\sin\gamma),$$
$$\bar{x}^2=\rho(-x^1\sin\gamma+x^2\cos\gamma),$$
$$\bar{x}^3=\rho x^3$$

$\rho>0$ and $\gamma$ being parameters.

Suppose $T$ be an isometry of $H^3$, i.e. a composite of some inversions with respect to some $H$-planes, then $T(S)$ is again a half cone whose axis is orthogonal to the plane $x^3=0$ with vertex on $x^3=0$ or $T(S)$ is one half of a cyclide with two vertices on the plane $x^3=0$. The latter carries a family of congruent equidistant curves corresponding to the family of generating lines of the half cone $S$ and a family of congruent proper circles corresponding to the family of circles $\Gamma_\gamma$ $(0<\gamma<\infty)$ on $S$. There is no distinction between the half cone $S$ and the half cyclide $T(S)$ in hyperbolic geometry. Each of them is an equidistant surface from a geodesic line in $H^3$ and can be regarded as an analogue of a circular cylinder in the sense of hyperbolic geometry. Thus, we get the following

THEOREM 6. *Any complete flat non-totally umbilical surface in $H^3$ is an equidistant surface from a geodesic line in $H^3$.*

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