ON COMPLETE HYPERSURFACES OF NONNEGATIVE SECTIONAL CURVATURES AND CONSTANT mTH MEAN CURVATURE[1]

BY

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ABSTRACT. The main result is that if $M = M^n$ is a complete Riemann manifold of nonnegative sectional curvature and $X: M \to R^{n+1}$ is an isometric immersion such that $X(M)$ has a positive constant mth mean curvature, then $X(M)$ is the product of a Euclidean space R^{n-d} and a d-dimensional sphere, $m \leqslant d \leqslant n$.

1. Introduction. Let $n \geqslant 2$, $M = M^n$ a Riemann manifold, and $X: M \to R^{n+1}$ an isometric immersion into Euclidean space. Let $\lambda_1(x) \leqslant \cdots \leqslant \lambda_n(x)$ be ordered principal curvatures of $X(M)$. In analogy with the case $n = 2$, we call $X(M)$ a Weingarten hypersurface (or a W-hypersurface) if $W(\lambda_1(x), \ldots, \lambda_n(x)) \equiv \text{const}$ for some nontrivial function $W(\lambda) = W(\lambda_1, \ldots, \lambda_n)$.

Let M be complete and have nonnegative sectional curvatures. Without any assumption that $X(M)$ is a Weingarten hypersurface, a theorem of Sacksteder [15] implies that M and X have factorizations,

$$M = R^{n-d} \times M_0^d \quad \text{and} \quad X = X_1 \times X_0 \tag{1.1}$$

such that M_0^d is a complete Riemann manifold, the first map X_1 in

$$X_1: R^{n-d} \to R^{n-d} \quad \text{and} \quad X_0: M_0^d \to R^{d+1} \tag{1.2}$$

is the identity and the second is an isometric immersion; $X_0(M_0^d)$ does not contain any (complete) lines and is the boundary of a convex body; cf. [8] for an analogue when $X: M^n \to R^{n+p}$. Of course, either factor R^{n-d} or M_0^d can be missing (i.e., reduce to a point). If, in addition, it is supposed that $X(M)$ is a W-Weingarten hypersurface, then it is of interest to find conditions on W which assure that, in the Sacksteder decomposition (1.1) of $X(M)$,

(a) $X_0(M_0^d)$ is compact, or even

(b) $X_0(M_0^d)$ is a sphere.

For the most part, this paper deals with the situation where $W(\lambda) = \sigma_m(\lambda)$ is the mth elementary symmetric function of $(\lambda_1, \ldots, \lambda_n)$, $1 \leqslant m \leqslant n$,

$$\sigma_m(\lambda) = \Sigma \cdots \Sigma \, \lambda_{i(1)}\lambda_{i(2)} \cdots \lambda_{i(m)}, \quad \text{where } i(1) < \cdots < i(m). \quad (1.3)$$

$H(x) = \sigma_1(\lambda(x))/n$ is the mean curvature, $R(x) = 2!(n-2)!\sigma_2(\lambda(x))/n!$ the scalar curvature, $K(x) = \sigma_n(\lambda(x))$ the Gauss-Kronecker curvature; and more generally, $H_m(x) = m!(n-m)!\sigma_m(\lambda(x))/n!$ the mth mean curvature $(1 \leqslant m \leqslant n)$. Let $X(M)$ be a σ_m-Weingarten hypersurface, with $\sigma_m(\lambda(x)) \equiv C_0 > 0$. It is known that (b) holds if $m = 1$ ([5]; cf. [12] for $n = 2$ and [7] for a related result) or if $m = 2$ ([6]). The object of this note is to prove:

THEOREM (∗). *Let $M = M^n$ be a complete connected Riemann manifold of class C^2 with nonnegative sectional curvature. Let $X: M \to R^{n+1}$ be an isometric immersion of class C^2 such that $X(M)$ has a positive constant mth mean curvature $H_m(x) \equiv C_0 > 0$, $x \in M$, for some m, $1 < m < n$. Then, in the Sacksteder decomposition (1.1)–(1.2), $X_0(M_0^d)$ is a sphere (of dimension d, $m \leqslant d \leqslant n$).*

We might remark that since M and X are of class C^2, they are real analytic, for $\sigma_m(\lambda(x)) \equiv C_0 > 0$ is a nonlinear elliptic analytic partial differential equation.

See [13] for the situation when $C_m = 0$.

In §§2 and 3, we adapt the arguments used by Cheng and Yau [6] in the case $m = 2$ to show that $X_0(M_0^d)$ is compact for arbitrary m, $1 < m < n$. Our proof depends on a generalization of the elliptic operator used in [21] and [6]. It then follows from results of Alexandrov [2] that $X_0(M_0^d)$ is a sphere; cf. Proposition 1.1 and remarks below. A different proof of the result in the compact case has been given by Nakagawa and Yokote [13, p. 479]. It should be noted that it has also been shown by Süss [17], Hsiung [11], U. Simon [16] and Yano [19] that when M_0^d is compact with nonnegative sectional curvature and $X_0(M_0^d)$ is a σ_m-Weingarten hypersurface with $\sigma_m(\lambda(x)) \equiv C_0 > 0$ on M, and certain additional convexity properties are satisfied, then $X_0(M_0^d)$ is a sphere. By virtue of Sacksteder [15], Proposition 3.5 and its proof below, the additional convexity properties in Hsiung [11] and in U. Simon [16] are redundant and, in fact, Nakagawa and Yokote deduce their result from Hsiung's.

Actually, the fact that $X_0(M_0^d)$ is a sphere can be deduced from the following consequence (applied to $W = \sigma_m(\lambda)$ and $c > 0$) of results of A. D. Alexandrov [2]:

PROPOSITION 1.1. *Let $W(\lambda) = W(\lambda_1, \ldots, \lambda_n)$ be of class C^1 for $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n$ such that $\partial W/\partial \lambda_k > 0$ for $1 \leqslant k \leqslant n$. Let $M = M^n$ be a compact connected Riemann manifold of class C^2 with nonnegative sectional curvature. Let $X: M \to R^{n+1}$ be an isometric immersion of class C^2 such that $X(M)$ is a W-Weingarten hypersurface, say, with $W(\lambda(x)) \equiv c$ on M and that*

$$\partial W(\lambda)/\partial \lambda_k > 0 \quad for\ 1 < k < n \tag{1.4}$$

whenever $\lambda \equiv \lambda(x)$, $x \in M$ *(for example, let* (1.4) *hold on the set* $\{\lambda: W(\lambda) = c\}$*). Then* $X(M)$ *is a sphere.*

Self-intersections are not excluded a priori in this assertion, but it follows from Sacksteder's results [15] that, in fact, $X(M)$ is the boundary of a convex body, so that X is an embedding when M and X are sufficiently smooth.

Because A. D. Alexandrov strived for extreme generality in the statements of his theorems, some do not seem to be clearly worded; cf. the comments in [13, pp. 479–480] and in a footnote of the translator of [2, III, p. 391]. Nevertheless, it is not difficult to see that the proof of Proposition 1.1 is contained in Alexandrov's arguments in the proof of the result of [2, V], with due reference to [2, III, pp. 390–391], and of course to [2, II, pp. 361–375]. (For an argument similar to [2, II, p. 371], in the special case $W = \sigma_m(\lambda)$, see [1, p. 827].)

Proposition 1.1 applies not only to $W(\lambda) = \sigma_m(\lambda)$ and $c > 0$ (cf. Proposition 3.5 below) but, for example, also to $W(\lambda) = S_1^m - S_m$ and $c > 0$, where $S_m = \lambda_1^m + \cdots + \lambda_n^m$ and $m = 1, 2, \ldots$ (and W reduces to $S_1^2 - S_2 = 2\sigma_2$ for $m = 2$). Also, Proposition 1.1 and the proof of Theorem (∗) have the following consequence:

COROLLARY (∗). *Let* $a_1, \ldots, a_n$ *be nonnegative constants and*

$$W(\lambda) = \sum_{\mu=1}^{n} a_\mu \sigma_\mu(\lambda).$$

Let $M = M^n$ *be a complete Riemann manifold of class* C^2 *with nonnegative sectional curvatures. Let* $X: M \to R^{n+1}$ *be an isometric immersion of class* C^2 *such that* $X(M)$ *is a* W-*Weingarten hypersurface with* $W(\lambda(x)) \equiv c > 0$. *Then, in the Sacksteder decomposition* (1.1), $X_0(M_0^d)$ *is a sphere (of dimension d,* $0 < d < n$*).*

ADDED IN PROOF (6/20/78). Using different methods in a forthcoming paper, we shall show that Theorem (∗) remains valid if "$H_m(x) \equiv C_0$" is replaced by "$W(\lambda(x)) \equiv c$" where W satisfies the conditions of Proposition 1.1.

2. Preliminaries for (a). As mentioned above, Cheng and Yau [6] show that a complete hypersurface in R^{n+1} with nonnegative sectional curvatures and positive constant scalar curvature satisfies (b). From their arguments, we shall extract a general result (Lemma 2.1 below) dealing with the question of the validity of (a).

Unless otherwise indicated, all sums are over the range $1, \ldots, n$.

If M is a Riemann manifold of class C^3, let $\omega_1, \ldots, \omega_n$ be a local C^2 orthonormal field of 1-forms on M satisfying the usual structure equations

$$d\omega_i = -\sum_m \omega_{im}\omega_m, \qquad \omega_{im} + \omega_{mi} = 0,$$

$$d\omega_{ij} = -\sum_m \omega_{im}\omega_{mj} + \Omega_{ij},$$

$$\Omega_{ij} = \frac{1}{2}\sum_k \sum_m R_{ijkm}\omega_k\omega_m, \qquad R_{ijkm} + R_{ijmk} = 0.$$

Here and below, products of exterior forms are understood to be exterior products. If a function f is of class $C^2(M)$, its gradient or covariant derivative is defined by

$$df = \sum_m f_{,m}\omega_m,$$

and its second covariant derivative by

$$\sum_m f_{,im}\omega_m = df_{,i} - \sum_m f_{,m}\omega_{mi}.$$

Similarly, if $\sum\sum h_{ij}\omega_i \otimes \omega_j$ is a C^2 symmetric tensor on M, its first and second covariant derivatives are defined by

$$\sum_m h_{ij,m}\omega_m = dh_{ij} - \sum_m h_{mj}\omega_{mi} - \sum_m h_{im}\omega_{mj},$$

$$\sum_m h_{ij,mk}\omega_m = dh_{ij,k} - \sum_m h_{ij,m}\omega_{mk} - \sum_m h_{im,k}\omega_{mj} - \sum_m h_{mj,k}\omega_{mi}.$$

Correspondingly, the Laplacian of the function f and of the tensor $\sum\sum h_{ij}\omega_i \otimes \omega_j$ are defined by

$$\Delta f = \sum_k f_{,kk} \quad \text{and} \quad \Delta h_{ij} = \sum_k h_{ij,kk}. \tag{2.1}$$

For any continuous symmetric tensor $\sum\sum\phi_{ij}\omega_i \otimes \omega_j$, define the differential operator L_ϕ of second order by

$$L_\phi f = \sum_1 \sum_j \phi_{ij} f_{,ij}. \tag{2.2}$$

This operator is called elliptic if (ϕ_{ij}) is positive definite, and degenerate elliptic if (ϕ_{ij}) is nonnegative definite.

PROPOSITION 2.1 [6]. *Let $M \in C^3$ and $\phi \in C^1$. Then L_ϕ is formally selfadjoint if and only if*

$$\sum_j \phi_{ij,j} \equiv 0 \quad \text{for } 1 < i < n \text{ on } M. \tag{2.3}$$

A form of the maximum principle gives

PROPOSITION 2.2 [5], [6]. *Let L_ϕ be formally selfadjoint and elliptic (possibly degenerate). Let $E \subset M$ be open with compact closure $\bar{E}$, $0 < f \in C^2(\bar{E})$,*

$0 < g \in C^2(\overline{E})$, $g \not\equiv 0$, *and* $g = 0$ *on* ∂E. *Then*

$$-\int_E g(L_\phi g)\, dx / \int_E g^2\, dx > \inf_E \left[-(L_\phi f)/f \right]. \tag{2.4}$$

It will be convenient to list here, for ready reference, some assumptions which we will use from time to time in this and the following section.

(H1) $M = M^n$ *is a complete connected Riemann manifold of class* C^4 *of nonnegative sectional curvature.*

(H2) $X: M \to R^{n+1}$ *is an isometric immersion of class* C^4 *such that* $X(M)$ *contains no complete lines, and so is the boundary of a convex body (for otherwise, we can replace* M, X *by* M_0^d, X_0 *in the Sacksteder decomposition).*

(H3) $X(M)$ *is not compact.*

(H4) $W = W(\lambda)$ *is a* C^2 *function* Φ *of* $S_1, \ldots, S_m$ *for some* $m \geqslant 1$, *where* $S_\mu = \lambda_1^\mu + \cdots + \lambda_n^\mu$, *i.e.,* $W(\lambda) = \Phi(S_1, \ldots, S_m)$.

Under the assumptions (H1)–(H2), there is a unit normal vector field N: $M \to S^{n+1}$ such that the second fundamental form $\Sigma\Sigma\, h_{ij}\omega_i \otimes \omega_j$ is nonnegative definite. We, of course, have the Codazzi relations

$$h_{ij,k} = h_{ji,k} = h_{ik,j} = h_{jk,i}. \tag{2.5}$$

When (H3) holds, the normal image $N(M)$ is contained in a hemisphere, so that there exists a constant vector $U \in R^{n+1}$ such that the Euclidean scalar product

$$N(x) \cdot U \geqslant 0 \quad \text{on } M; \tag{2.6}$$

Sacksteder (cf., e.g., [18]). If $\{e_i, \ldots, e_n\}$ is a local orthonormal frame field on $X(M)$, then the Gauss and Weingarten formulas are

$$X_{,ij} = h_{ij}N \quad \text{and} \quad N_{,i} = -\sum_j h_{ij}e_j,$$

so that

$$N_{,ij} = -\sum_k h_{ik}h_{kj}N - \sum_k h_{ki,j}e_k;$$

and consequently

$$L_\phi X = \sum_i \sum_j \phi_{ij} h_{ji} N \quad \text{or} \quad L_\phi(X \cdot U) = \sum_i \sum_j \phi_{ij} h_{ji} (N \cdot U), \tag{2.7}$$

$$L_\phi(N \cdot U) = -\sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki} (N \cdot U) - \sum_i \sum_j \sum_k \phi_{ij} h_{ji,k} (e_k \cdot U); \tag{2.8}$$

cf. [7, p. 83].

PROPOSITION 2.3. *Assume* (H1)–(H3) *and* (2.6). *Let* L_ϕ *be elliptic and satisfy*

$$\sum_i \sum_j \phi_{ij} h_{ji,k} \equiv 0 \quad \text{for } 1 < k < n \text{ on } M. \tag{2.9}$$

Then $N(x) \cdot U > 0$ on M

PROOF. Even if L_ϕ is degenerate elliptic, we have that

$$\sum_i \sum_j \phi_{ij} h_{jk} h_{ik} > 0 \quad \text{for } 1 < k < n.$$

Hence, by (2.6) and (2.9), $L_\phi(N \cdot U) < 0$ on M. The strong minimum principle of E. Hopf [10] for elliptic operators implies that if $N \cdot U = 0$ at some point of M, then $N \cdot U \equiv 0$ on M. But this would show that, for every $x^0 \in M$, the line $X(x^0) + tU$, $-\infty < t < \infty$, is on $X(M)$, which contradicts (H2).

PROPOSITION 2.4. *Assume* (H1)–(H3). *Let L_ϕ be formally selfadjoint, elliptic, and let* (2.9) *hold. Let $E = E(r) = \{x \in M: X(x) \cdot U < r\}$ for $r > 0$. Then*

$$4 \int_{E(r)} \sum_i \sum_j \phi_{ij} h_{ji} \, dx / r \text{ vol } E(r/2) > \inf_{E(r)} \sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki}. \quad (2.10)$$

PROOF. If we let $g = r - X(x) \cdot U$ and $f = N(x) \cdot U > 0$, then Proposition 2.2 gives

$$\int_{E(r)} (r - X(x) \cdot U) \sum_i \sum_j \phi_{ij} h_{ji} (N \cdot U) \, dx / \int_{E(r)} (r - X(x) \cdot U)^2 \, dx$$

$$> \inf_{E(r)} \sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki}.$$

Since the left side of this relation is at most the left side of (2.10), the assertion follows.

COROLLARY 2.1. *If, in addition to the assumptions of Proposition 2.4, we assume that*

$$\int_{E(r)} \sum_i \sum_j \phi_{ij} h_{ji} \, dx / r \text{ vol } E(r/2) \to 0 \quad \text{as } r \to \infty, \quad (2.11)$$

e.g., that

$$\sum_i \sum_j \phi_{ij} h_{ji} \quad \text{is bounded on } M, \quad (2.12)$$

then

$$\inf_M \sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki} = 0. \quad (2.13)$$

Thus the arguments above lead to the following general result.

LEMMA 2.1. *Assume* (H1)–(H2), *and that* (2.11) *(say,* (2.12)*) holds but that* (2.13) *does not. Also, assume that L_ϕ is elliptic, formally selfadjoint, and that*

(2.9) *holds. Then $X(M)$ is compact, i.e., (H3) cannot hold.*

It will be seen, in the next section, that the assumption that "L_ϕ is formally selfadjoint" is quite restrictive.

3. **Compactness.** The object of this section is to prove the following:

LEMMA 3.1. *Let m be fixed, $1 < m < n$. Assume conditions* (H1)–(H2) *of* §2 *and that $X(M)$ is a σ_m-Weingarten hypersurface with $\sigma_m(\lambda(x)) = C_0 > 0$ on M. Then $X(M)$ is compact.*

This is a consequence of Lemma 2.1 and Propositions 3.1 and 3.4–3.7 below. In this section, we assume (H1)–(H2) and (H4). We associate with W, the symmetric tensor ϕ or $\Sigma\Sigma\phi_{ij}\omega_i \otimes \omega_j$ defined by

$$\phi = \sum_{\mu=1}^{m} \mu(\partial W/\partial S_\mu)h^{\mu-1}, \tag{3.1}$$

where $h^\mu = (h_{ij})^\mu$ is the μth power of $h = (h_{ij})$, the second fundamental matrix. Thus $h^0 = I$ is the identity matrix and the ijth element of h^μ_{ij} is

$$h^\mu_{ij} = \sum_{i(1)} \cdots \sum_{i(\mu-1)} h_{ii(1)}h_{i(1)i(2)} \cdots h_{i(\mu-1)j}. \tag{3.2}$$

Hence (3.1) means that

$$\phi_{ij} = \sum_{\mu=1}^{m} \mu(\partial W/\partial S_\mu)h^{\mu-1}_{ij}, \quad \text{where } h^0_{ij} = \delta_{ij}. \tag{3.3}$$

Correspondingly, the associated operator is

$$L_\phi f = \sum_i \sum_j \phi_{ij}f_{,ij} = \sum_i \sum_j \sum_{\mu=1}^{m} \mu(\partial W/\partial S_\mu)h^{\mu-1}_{ij}f_{,ij}. \tag{3.4}$$

PROPOSITION 3.1. *Under the assumptions* (H1)–(H2), (H4) *and $W(\lambda(x)) \equiv$ const on M, the relations* (2.9) *hold.*

PROOF. Let $s_\mu(x) = S_\mu(\lambda(x))$. Then

$$s_\mu(x) = \operatorname{tr} h^\mu = \sum_i \sum_{i(1)} \cdots \sum_{i(\mu-1)} h_{ii(1)}h_{i(1)i(2)} \cdots h_{i(\mu-1)i}.$$

Hence, we have, with $i = i(\mu)$,

$$s_{\mu,k} = \sum_{\kappa=1}^{\mu} \sum_i \sum_{i(1)} \cdots \sum_{i(\mu-1)} h_{ii(1)} \cdots h_{i(\kappa-1)i(\kappa),k}$$

$$\cdot h_{i(\kappa)i(\kappa+1)} \cdots h_{i(\mu-1)i}, \tag{3.5}$$

so that

$$s_{\mu,k} = \mu \operatorname{tr}(h^{\mu-1}h_{,k}) = \mu\sum_i \sum_j h^{\mu-1}_{ij}h_{ji,k}. \tag{3.6}$$

Note that if $w(x) = W(\lambda(x))$, then $w(x) \equiv c_0$ on M implies that

$$0 = w_{,k} = \sum_{\mu=1}^{m} (\partial W/\partial S_\mu)s_{\mu,k} = \sum_i \sum_j \phi_{ij}h_{ji,k} \quad \text{for } 1 < k < n.$$

PROPOSITION 3.2. *Assume* (H1)–(H2), (H4) *and* (3.1). *Then* L_ϕ *is elliptic if and only if*

$$\partial W(\lambda)/\partial \lambda_i > 0 \quad \text{for } \lambda = \lambda(x), x \in M, 1 < i < n. \tag{3.7}$$

PROOF. In fact, the eigenvalues of ϕ are

$$\sum_{\mu=1}^{m} \mu(\partial W/\partial S_\mu)\lambda_i^{\mu-1} = \partial W/\partial \lambda_i \quad \text{with } \lambda = \lambda(x), 1 < i < n. \tag{3.8}$$

PROPOSITION 3.3. *Under the assumptions* (H1)–(H2), (H4) *and* (3.1), *a sufficient condition for* L_ϕ *to be formally selfadjoint is that*

$$\sum_{\kappa=1}^{m} \sum_{\mu=1}^{m} \mu\kappa(\partial^2 W/\partial S_\kappa \partial S_\mu)\lambda_i^{\mu-1}\lambda_k^{\kappa-1}$$

$$+ \sum_{\kappa=1}^{m-1} \sum_{\mu=\kappa+1}^{m} \mu(\partial W/\partial S_\mu)\lambda_i^{\mu-\kappa-1}\lambda_k^{\kappa-1} = 0 \tag{3.9}$$

for $1 < i, k < n$. *This is the case, e.g., if* W *satisfies*

$$\kappa \sum_{\mu=1}^{m} \mu(\partial^2 W/\partial S_\kappa \partial S_\mu)\lambda_i^{\mu-1} + \sum_{\mu=\kappa+1}^{m} \mu(\partial W/\partial S_\mu)\lambda_i^{\mu-\kappa-1} = 0 \tag{3.10}$$

for $1 < i < n, 1 < \kappa < m$.

PROOF. From (3.3),

$$\sum_j \phi_{ij,j} = \sum_j \left\{ \sum_{\mu=1}^{m} \sum_{k=1}^{m} \mu(\partial^2 W/\partial S_\kappa \partial S_\mu)s_{\kappa,j}h_{ij}^{\mu-1} + \sum_{\mu=2}^{m} \mu(\partial W/\partial S_\mu)h_{ij,j}^{\mu-1} \right\}.$$

Note that (3.2) implies that

$$h_{ij,j}^{\mu-1} = \sum_{\kappa=1}^{\mu-1} \sum_k \sum_p h_{ik}^{\kappa-1}h_{kp,j}h_{pj}^{\mu-\kappa-1}. \tag{3.11}$$

By the Codazzi relations (2.5), we have $h_{kp,j} = h_{jp,k}$, so that, by (3.6),

$$\sum_j h_{ij,j}^{\mu-1} = \sum_{\kappa=1}^{\mu-1} (\mu - \kappa)^{-1} \sum_k h_{ik}^{\kappa-1}s_{\mu-\kappa,k}$$

$$= \sum_{\kappa=1}^{\mu-1} \kappa^{-1} \sum_j h_{ij}^{\mu-\kappa-1}s_{\kappa,j}$$

for $\mu > 2$. If, at a fixed arbitrary point x of M, $h = \text{diag}(\lambda_1(x), \ldots, \lambda_n(x))$,

then, by (3.6) and the last relation,

$$s_{\kappa,j} = \kappa \sum_k \lambda_k^{\kappa-1} h_{kk,j},$$

$$\sum_j h_{ij,j}^\mu = \sum_{\kappa=1}^{\mu-1} \kappa^{-1} \lambda_i^{\mu-\kappa-1} s_{\kappa,i} = \sum_{\kappa=1}^{\mu-1} \sum_k \lambda_i^{\mu-\kappa-1} \lambda_k^{\kappa-1} h_{kk,i}.$$

Hence, $\sum_j \phi_{ij,j}$ becomes

$$\sum_k \left\{ \sum_{\kappa=1}^m \sum_{\mu=1}^m \mu\kappa \left(\partial^2 W/\partial S_\kappa \partial S_\mu\right) \lambda_i^{\mu-1} \lambda_k^{\kappa-1} \right.$$

$$\left. + \sum_{\mu=2}^m \sum_{\kappa=1}^{\mu-1} \mu \left(\partial W/\partial S_\mu\right) \lambda_i^{\mu-\kappa-1} \lambda_k^{\kappa-1} \right\} h_{kk,i}.$$

Thus, the first part of the assertion follows, and the second is obvious (since $\partial W/\partial S_{m+1} = 0$).

PROPOSITION 3.4. *If $W = \sigma_m(\lambda)$, then (3.10) holds. Hence if (H1)–(H2) holds and $W = \sigma_m(\lambda)$ in (3.1), then L_ϕ is selfadjoint.*

It is not difficult to see that if $W(\lambda)$ is a symmetric homogeneous polynomial of degree m in $(\lambda_1, \dots, \lambda_n)$, then (3.10) holds only if W is a constant multiple of σ_m.

PROOF. Standard formulas show that

$$\mu\partial\sigma_m/\partial S_\mu = (-1)^{\mu-1}\sigma_{m-\mu} \quad \text{for } 1 \le \mu \le m; \tag{3.12}$$

cf. [4, §80 or 161], where $p_\mu = (-1)^\mu \sigma_\mu$. Hence

$$\mu\kappa\partial^2\sigma_m/\partial S_\kappa \partial S_\mu = (-1)^{\mu+\kappa} \sigma_{m-\mu-\kappa} \quad \text{for } 1 \le \kappa, \mu \le m-1, \text{ and } \mu+\kappa < m. \tag{3.13}$$

Thus, if $W = \sigma_m$, then the left side of (3.10) is

$$\sum_{\mu=1}^{m-\kappa} (-1)^{\mu+\kappa} \sigma_{m-\mu-\kappa} \lambda_i^{\mu-1} - \sum_{\mu=\kappa+1}^m (-1)^\mu \sigma_{m-\mu} \lambda_i^{\mu-\kappa-1} = 0.$$

PROPOSITION 3.5. *If $0 < \lambda_1^0 < \cdots < \lambda_n^0$ and $\sigma_m(\lambda^0) > 0$, then $\partial\sigma_m(\lambda^0)/\partial\lambda_k > 0$ for $1 \le k \le n$. Hence, if (H1)–(H2) holds, $W = \sigma_m(\lambda)$ in (3.1), and $\sigma_m(\lambda(x)) > 0$ on M, then L_ϕ is elliptic.*

PROOF. Note that

$$\partial\sigma_m(\lambda)/\partial\lambda_i = \sigma_{m-1}^{(i)}(\lambda), \tag{3.14}$$

where $\sigma_{m-1}^{(i)}$ denotes the $(m-1)$st elementary symmetric function in $\lambda_1, \dots, \lambda_{i+1}, \lambda_{i+1}, \dots, \lambda_n$. We shall first verify the following:

CLAIM. Let $0 \leqslant \lambda_1^0 \leqslant \cdots \leqslant \lambda_n^0$. Then $\sigma_m(\lambda^0) = 0$ if and only if there exist $m - 1$ indices $(1 \leqslant) \, i(1) < \cdots < i(m - 1) \, (\leqslant n)$ such that $\lambda_k^0 = 0$ if $k \neq i(\mu), 1 \leqslant \mu \leqslant m - 1$.

This is clear if $m = 1$. Assume that $m > 1$ and that the claim holds if m is replaced by $m - 1$. Suppose that $\lambda_j^0 \neq 0$ for some j. Since $m\sigma_m(\lambda) = \Sigma \lambda_i \sigma_{m-1}^{(i)}(\lambda)$, $\sigma_m(\lambda^0) = 0$ implies that $\sigma_{m-1}^{(j)}(\lambda^0) = 0$. Thus, by the induction hypothesis, there exist $m - 2$ indices $(1 \leqslant) \, i(1) < \cdots < i(m - 2) \, (\leqslant n)$, $i(\mu) \neq j$, such that $\lambda_k^0 = 0$ if $k \neq j$ or $k \neq i(\mu), 1 \leqslant \mu \leqslant m - 2$. This gives the claim.

COMPLETION OF THE PROOF. Suppose that $0 < \lambda_1^0 \leqslant \cdots \leqslant \lambda_n^0$ and that $\partial\sigma_m(\lambda^0)/\partial\lambda_i = 0$ for some i, $1 \leqslant i \leqslant n$. Then $\sigma_{m-1}^{(i)}(\lambda^0) = 0$, and so there exist $m - 2$ indices $i(1) < \cdots < i(m - 2)$, $i(\mu) \neq i$, such that $\lambda_j^0 = 0$ if $j \neq i$ or $j \neq i(\mu)$ for $1 \leqslant \mu \leqslant m - 2$. Thus $\sigma_m(\lambda^0) = 0$. This proves the first part of the proposition, and the other part follows from Proposition 3.2.

REMARK. It is clear that this proof shows that if $c_0 > 0$ and $c_1 > 0$, then there exists a $\delta = \delta(c_0, c_1) > 0$ such that

$$\partial\sigma_m(\lambda)/\partial\lambda_k \geqslant \delta > 0 \quad \text{if } \sigma_m(\lambda) \geqslant c_0 > 0, 0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant c_1. \quad (3.15)$$

PROPOSITION 3.6. *Assume* (H1)–(H2), (H4) *and that* $W(\lambda)$ *is a homogeneous function of degree* m *in* (3.1). *Then*

$$\operatorname{tr}(\phi h) = \sum_i \sum_j \phi_{ij} h_{ji} = mW(\lambda(x)), \quad (3.16)$$

so that (2.12) *holds if* $W(\lambda(x)) \equiv$ const *on* M.

PROOF. By (3.8), the eigenvalues of ϕh are

$$\sum_{\mu=1}^m \mu(\partial W/\partial S_\mu)\lambda_i^\mu = \lambda_i \partial W/\partial\lambda_i, \quad 1 \leqslant i \leqslant n,$$

so that (3.16) follows from Euler's theorem.

PROPOSITION 3.7. *If* (H1)–(H2) *holds,* $W = \sigma_m(\lambda)$ *in* (3.1), *and* $\sigma_m(\lambda(x)) \geqslant C_0 > 0$ *on* M, *then*

$$\inf_M \operatorname{tr}(\phi h^2) = \inf_M \sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki} > 0. \quad (3.17)$$

PROOF. The eigenvalues of ϕh^2 are

$$\sum_{\mu=1}^m \mu(\partial W/\partial S_\mu)\lambda_i^{\mu+1} \quad \text{for } 1 \leqslant i \leqslant n,$$

so that

$$\operatorname{tr}(\phi h^2) = \sum_{\mu=1}^m \mu(\partial W/\partial S_\mu)S_{\mu+1}.$$

When $W = \sigma_m$, (3.12) implies that

$$\operatorname{tr}\left(\phi h^2\right) = \sum_{\mu=1}^{m} (-1)^{\mu-1}\sigma_{m-\mu}S_{\mu+1} = \sum_{\mu=2}^{m+1} (-1)^{\mu}\sigma_{m+1-\mu}S_{\mu}.$$

Thus, by Newton's formula (cf. [3, p. 244]),

$$\operatorname{tr}\left(\phi h^2\right) = \sigma_m\sigma_1 - (m+1)\sigma_{m+1},$$

where $\sigma_{m+1} = 0$ if $m = n$. If $m < n$, then each term $\lambda_{i(1)} \cdots \lambda_{i(m+1)}$ of σ_{m+1} occurs $m + 1$ times in $\sigma_m\sigma_1$, for the factor $\lambda_{i(j)}$ can be considered to be in σ_1 for $j = 1, \ldots, m + 1$. Thus

$$\operatorname{tr}\left(\phi h^2\right) \geqslant \sum_{\mu=1}^{m} \left\{ \sum \cdots \sum \lambda_{i(1)} \cdots \lambda_{i(\mu-1)}\lambda_{i(\mu)}^2\lambda_{i(\mu+1)} \cdots \lambda_{i(m)} \right\},$$

where the inner sum is over $1 \leqslant i(1) < \cdots < i(m) \leqslant m$. Hence

$$\operatorname{tr}\left(\phi h^2\right) \geqslant \lambda_n(\lambda_n\lambda_{n-1} \cdots \lambda_{n-m+1}).$$

Since $\sigma_m \leqslant (\lambda_n \cdots \lambda_{n-m+1})n!/(n-m)!m! \leqslant \lambda_n^m n!/(n-m)!m!$, we see that

$$\operatorname{tr}\left(\phi h^2\right) \geqslant \left[m!\,(n-m)!\sigma_m/n! \right]^{1+1/m} \geqslant \left[m!\,(n-m)!\,C_0/n! \right]^{1+1/m} > 0.$$

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