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# On completeness of left-invariant Lorentz metrics on solvable Lie groups.

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**Abstract**

We study geodesic completeness for left-invariant Lorentz metrics on solvable Lie groups.

## 1 Introduction

In [4], we have shown (among other things) that a generic left-invariant Lorentz metric on $Sl(2,\mathbf{R})$ is non-complete.

The nilpotent case has, as well, been studied in [5]. It was shown that every left-invariant pseudo-riemannian metric on a 2-step nilpotent Lie group is complete. However, an example of a 3-step nilpotent Lie group with a non-complete left-invariant Lorentz metric is given.

In this paper we study completeness for the left-invariant Lorentz metricx on some solvable Lie groups. First, after J. Milnor [6] and K. Nomizu [7] we consider a special class $\mathcal{F}$ of solvable Lie groups. A non commutative Lie group $G$ belong to $\mathcal{F}$ if its Lie algebra $\mathcal{G}$ has the property that for any elements $x, y$ in $\mathcal{G}$ the bracket product $[x, y]$ is a linear combination of $x$ and $y$.

For such a group we show that every left-invariant Lorentz metric is non-complete. This case is a generalization of the well-known example of the Lorentz half-plane (i.e the affine group $A(1,\mathbf{R})$ with its left-invariant Lorentz metric).

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Also, we investigate the completeness of left-invariant Lorentz metrics on the unimodular 3-dimensional Lie group $E(2)$ (resp. $E(1,1)$) of rigid motions of Euclidean (resp. Minkowski) 2-space. We prove that all left-invariant Lorentz metrics on $E(2)$ are complete, while such a metric on $E(1,1)$ is complete if and only if it realizes a Lorentzian submersion on Minkowski 2-space.

# 2 Preliminaries

## 2.1 Geodesics of left-invariant pseudo-metrics.

Let $G$ be a Lie group, and $\mathcal{G}$ its Lie algebra. It is well known that the data of a left-invariant pseudo-riemannian metric on $G$ is equivalent to that of a non-degenerate quadratic form on $\mathcal{G}$. Furthermore, every $C^1$-curve $t \mapsto c(t)$ of $G$ gives rise (up to a left translation) to the curve $L^{-1}_{c(t)*}\dot{c}(t)$ on $\mathcal{G}$.

**Lemma 2.1** *The curves of $\mathcal{G}$ associated to geodesic are solutions of the equation*

$$\dot{x} = ad^*_x x \qquad (*)$$

*where $ad^*_x$ stands for the adjoint of $ad_x$ relative to the inner product on $\mathcal{G}$.*

**Proof.** It is an immediate consequence of the formula (see [2])

$$\forall X, Y \in \mathcal{G} \quad \nabla_X Y = \frac{1}{2}\{[X,Y] - ad^*_X Y - ad^*_Y X\}$$

where $\nabla$ is the Levi-Civita connesion associated to the metric.

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The general study of $(*)$ may be very complicated. If $G$ is semi-simple, it takes the more remarkable form

$$\phi(\dot{x}) = [\phi(x), x],$$

where $\phi$ stands for the endomorphism on $\mathcal{G}$ which is associated to the metric via the Killing form (see [4] for some consequences).

### 2.2 General fact

Now, the groups we study here satisfy the following property: *There exists a codimension one commutative ideal (so that the Lie algebra is 2-step solvable).*

Denote by $E$ this ideal. Consider a left-invariant Lorentz metric on $G$, and let its associate inner product on $\mathcal{G}$ be $\langle\cdot,\cdot\rangle$.

If $\langle\cdot,\cdot\rangle_{|E}$ is nondegenerate, let $e_0 \notin E$ such that

$$\langle e_0, E\rangle = 0 \quad \text{and} \quad \mathcal{G} = \mathbf{R}e_0 \oplus E.$$

Now, it is easy to check that

$$ad^*_{e_0}e_0 = 0 \quad \text{and} \quad \forall y \in E \quad ad^*_{e_0}y \in E.$$

Thus equation $(*)$ takes the form

$$\begin{cases} \dot{x}_0 = -\frac{\langle [e_0,x],x\rangle}{\langle e_0,e_0\rangle} = -\frac{\langle Sx,x\rangle}{\langle e_0,e_0\rangle} \\ \dot{x} = x_0(ad^*_{e_0}x) \end{cases}$$

where $S = \frac{1}{2}(ad_{e_0} + ad^*_{e_0})$ and $L^{-1}_{c(t)*}\dot{c}(t) = x_0e_0 + x$, $x \in E$.

## 3 A remarkable class of solvable Lie groups

In this section, $\mathcal{F}$ denotes a special class of solvable Lie groups. A non commutative Lie group $G$ belongs to $\mathcal{F}$ if its Lie algebra $\mathcal{G}$ has the property that for any elements $x, y$ in $\mathcal{G}$ the bracket product $[x, y]$ is a linear combination of $x$ and $y$.

It is shown in [6] that this is equivalent to the existence of a codimension one commutative ideal $E$ and an element $e_0 \notin E$ such that

$$\forall x \in E \quad [e_0, x] = x.$$

The simplest example of such groups is given by

$$\begin{pmatrix} a & 0 & \cdots & b_1 \\ 0 & \ddots & & \vdots \\ \vdots & & a & b_{n-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{where } a > 0, \quad b_1, \cdots, b_{n-1} \in \mathbf{R}$$

The main result of this section is the following.

**Theorem 3.1** *If $G$ belongs $\mathcal{F}$, then every left-invariant Lorentz metric on $G$ is geodesically incomplete.*

**Proof.** We shall continue to denote by $\langle\cdot,\cdot\rangle$ the Lorentzian inner product given by the metric, and to further simplify notations $\langle L^{-1}_{c(t)*}\dot{c}(t), L^{-1}_{c(t)*}\dot{c}(t)\rangle$ will be denoted by $\langle\dot{c},\dot{c}\rangle$.

• First, we assume that $\langle\cdot,\cdot\rangle_{|E}$ is nondegenerate. Then, with the same notations as in 2.2, we have $S = I_E$. Hence, equation $(*)$ is now

$$\begin{cases} \dot{x}_0 = -\frac{\langle x,x\rangle}{\langle e_0,e_0\rangle} \\ \dot{x} = x_0 x \end{cases}$$

Next, $\langle x, x, \rangle = \langle\dot{c},\dot{c}\rangle - x_0^2\langle e_0, e_0\rangle$, thus

$$\dot{x}_0 = x_0^2 - \frac{\langle\dot{c},\dot{c}\rangle}{\langle e_0,e_0\rangle}.$$

Therefore, for a null geodesic (that is $\langle\dot{c},\dot{c}\rangle = 0$) we have $x_0 \to \infty$ as $t \to b$ with $b < \infty$, and the metric is non-complete.

• We assume now that $\langle\cdot,\cdot\rangle_{|E}$ is degenerate, which means, in geometric terms that the subspace $E$ is tangent to the null cone. Thus, $E$ contains a null vector $b$ and a $(n-2)$-dimensional subspace $E_1$ such that

$$E = \mathbf{R}b \oplus E_1 \quad \text{orthogonal sum}, \quad \langle b, b\rangle = \langle b, E_1\rangle = 0$$

and $\langle\cdot,\cdot\rangle_{|E}$ is positive-definite.

On the other hand, since the orthogonal complement $E_1^{\perp}$ of $E_1$ is Lorentzian, we can find a vector $c$ such that

$$\langle c, c\rangle = 0 \quad \text{and} \quad \langle b, c\rangle = -1.$$

Therefore, as in 2.2, we may replace $c$ by the vector $e_0$, and so we obtain the following orthogonal decomposition

$$\mathcal{G} = Span\,\{b, e_0\} \oplus E_1.$$

An easy computation shows that, for all $x_1 \in E_1$, we have

$$\begin{array}{ll} ad^*_{e_0}x_1 = x_1, & ad^*_{e_0}e_0 = e_0 \\ ad^*_{x_1}x_1 = \langle x_1, x_1\rangle b, & ad^*_b e_0 = -b \end{array}$$

the other terms being zero. So that, equations $(*)$ are now

$$\begin{cases} \dot{x}_0 = x_0^2 \\ \dot{y} = \langle \dot{c}, \dot{c} \rangle \\ \dot{x}_1 = x_0 x_1 \end{cases}$$

where $L_{c(t)*}^{-1}\dot{c}(t) = x_0 e_0 + yb + x_1$, and $x_1$ belongs to $E_1$.

Consequently, all geodesic (unless $x_0 \equiv 0$) are incomplete.

∎

**Remark 3.1** According to [7], if a Lie group $G$ is of type $\mathcal{F}$, then it admits left-invariant Lorentz metrics with positive constant sectional curvatures.

Clearly, such a group is not unimodular, and therefore has no compact quotients. On the other hand, Calabi and Marcus have shown (cf. [1]) that any complete Lorentz manifold of positive constant curvature is not compact. So it is reasonable to conjecture that there is no compact, complete or not, Lorentz manifold of positive constant curvature.

# 4 Unimodular 3-dimensinal Lie groups

It is well known (see for instance [3]) that simply-connected unimodular Lie groups of dimension 3 are classified as follows.

1) $\widetilde{So(3)} = S^3$.
2) $\widetilde{Sl(2,\mathbf{R})}$.
3) $\widetilde{E(2)},\mathbf{R})$ (the universal covering of the group $E(2)$ of rigid motion of Euclidian 2-space).
4) $E\widetilde{(1,1)}$ (the universal covering of the group $E(1,1)$ of rigid motions of Minkowski 2-space).
5) $H_3$ (the Heisenberg group).
6) $\mathbf{R}^3$.

In order to finish with dimension 3, we study here the cases 3) and 4). In these cases the Lie algebra has a codimension one commutative ideal. Our study still relies on the properties of some $ad_{e_0|_E}(e_0 \notin E)$. Of course now $ad_{e_0} \neq Id$.

### 4.1 The case of $E(2)$

We look here $E(2)$ as the semi-direct product $O(2) \propto \mathbf{R}^2$ is the group of orthogonal transformations of Euclidean 2-space.

Our result in this subsection is the following.

**Theorem 4.1** *All left-invariant Lorentz metrics on $E(2)$ are geodesically complete.*

**Proof.** Let $E = [\mathcal{E}(2), \mathcal{E}(2)]$ where $\mathcal{E}(2)$ is the Lie algebra of $E(2)$. We denote by $\langle \cdot, \cdot \rangle$ the inner product over $\mathcal{E}(2)$ associated to the metric on $E(2)$.

**Case 1.** *The subspace $E$ is non-degenerate, that is $\langle \cdot, \cdot \rangle_{|E}$ is non-degenerate.*

As in 2.2, we may choose $e_0 \notin E$ such that

$$\langle e_0, E \rangle = 0 \text{ and } \mathcal{E}(2) = \mathbf{R}e_0 \oplus E.$$

In terms of the infinitesimal representation of $O(2)$ in the vector space $\mathbf{R}^2 \simeq E$ we can find a basis $\{e_1, e_2\}$ of $E$ for which both $\langle \cdot, \cdot \rangle_{|E}$ and the usual positive-definite inner product on $\mathbf{R}^2$ are diagonal. Thus, $ad_{e_0}$ is antisymmetric with respect to the basis. That is $\{e_0, e_1, e_2\}$ is an orthogonal basis of $\mathcal{E}(2)$ satisfying

$$[e_0, e_1] = -e_2, \quad [e_0, e_2] = e_1 \text{ and } [e_1, e_2] = 0. \tag{1}$$

We put the inner product $\langle \cdot, \cdot \rangle$ under the form

$$\langle e_0, e_0 \rangle \omega_0^2 + \lambda_1 \omega_1^2 + \lambda_2 \omega_2^2$$

where $\omega_i$ is the dual form of $e_i$, and $\lambda_1 \lambda_2 \neq 0$. Then, we get easily

$$ad^*_{e_0} = \begin{pmatrix} 0 & -\frac{\lambda_2}{\lambda_1} \\ \frac{\lambda_1}{\lambda_2} & 0 \end{pmatrix}.$$

Consequently, equation $(*)$ are

$$\begin{cases} \dot{x}_0 = \frac{\lambda_2 - \lambda_1}{\langle e_0, e_0 \rangle} x_1 x_2 \\ \dot{x}_1 = -\frac{\lambda_2}{\lambda_1} x_0 x_2 \\ \dot{x}_2 = \frac{\lambda_1}{\lambda_2} x_0 x_1 \end{cases}$$

where $L^{-1}_{c(t)*}\dot{c}(t) = x_0e_0 + x_1e_1 + x_2e_2$.

When $\lambda_1 = \lambda_2$, we show by easy trigonometric computation that the metric is complete. Otherwise (i.e when $\lambda_1 \neq \lambda_2$) we have the two first-integrals

$$\begin{aligned} \langle e_0, e_0\rangle x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 &= e \\ \lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 &= m \end{aligned}$$

So, $x_0, x_1$ and $x_2$ are bounded, and hence the metric is complete.

**Case 2.** *Suppose now that $\langle\cdot,\cdot\rangle_{|E}$ is degenerate. Then we can find a vector $b$ such that*

$$\forall x \in E \quad \langle b, b\rangle = \langle b, x\rangle = 0.$$

Let $\{e_0, e_1, e_2\}$ be a basis of type (1). Then, by an appropriate rotation of axis $e_0$, which is in fact and automorphism of $\mathcal{E}(2)$, we can take $b = e_1$. This implies that $e_2$ is space-like (i.e $\langle e_2, e_2\rangle > 0$) since a null vector is never orthogonal to a time-like one. Then, by considering the automorphis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\langle e_2,e_2\rangle}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\langle e_2,e_2\rangle}} \end{pmatrix}.$$

we can suppose that $\langle e_2, e_2\rangle = 1$.

On the other hand, we have necessarilly $\langle e_0, e_1\rangle \neq 0$ since the metric is non-degenerate. Hence, up to an automorphism of type

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \mu & 0 & 1 \end{pmatrix},$$

we can assume that $\langle e_0, e_0\rangle = \langle e_0, e_2\rangle = 0$.

Now, the metric only depends on the value $\langle e_0, e_1\rangle$.In fact, by replacing $e_0$ by $-e_0/\langle e_0, e_1\rangle$, we may assume that $\{e_0, e_1 e_2\}$ satisfies

$$\langle e_0, e_0\rangle = \langle e_0, e_2\rangle = \langle e_1, e_1\rangle = \langle e_1, e_2\rangle = 0, \text{ and } \langle e_2, e_2\rangle = -\langle e_0, e_1\rangle = 1,$$

(it does not change completeness properties).

An easy calculation shows that

$$\begin{array}{lll} ad^*_{e_0}e_0 = -e_2, & ad^*_{e_0}e_1 = 0 & ad^*_{e_0}e_2 = e_0 \\ ad^*_{e_1}e_0 = ad^*_{e_1}e_1 = 0, & ad^*_{e_1}e_2 = -e_1 & \\ ad^*_{e_2}e_0 = -e_1, & ad^*_{e_2}e_1 = ad^*_{e_2}e_2 = 0 & \end{array}$$

Therefore, equations $(*)$ are given by

$$\begin{cases} \dot{x}_0 = x_0 x_2 \\ \dot{x}_1 = -(x_0 + x_1)x_2 \\ \dot{x}_2 = -x_0^2 \end{cases}$$

where $x_0, x_1, x_2$ are the componets of $L^{-1}_{c(t)*}\dot{c}(t)$ with respect to $e_0, e_1 e_2$.
Obviously, we get

$$x_2^2 - 2x_0x_1 = e, \quad x_0^2 + x_2^2 = m.$$

Hence, $x_0, x_1$, and $x_2$ are bounded along every bounded interval, and the metric is complete.

■

## 4.2 The case of $E(1,1)$

As before, $E(1,1)$ will be considered as the semi-direct product $O(1,1) \propto \mathbf{R}^2$, where $O(1,1)$ is now the group of orthogonal transformations of Minkowski 2-space. However, the present case is more delicate because we are going to compare two indefinite inner products on $\mathbf{R}^2$: the first is the inner product $\langle \cdot, \cdot \rangle_{|E}$ associated to the metric, and the second is the usual Lorentz inner product on $\mathbf{R}^2$ given by

$$(x, y) = x_1y_1 - x_2y_2 \quad \text{where} \quad x = (x_1, x_2), y = (y_1, y_2).$$

Also, we will consider the submersion $\pi : E(1,1) \to \mathbf{R}^2$ given by the projection upon the second factor.
We shall now prove the following:

**Theorem 4.2** *A left-invariant Lorentz metric on $E(1,1)$ is complete if and only if it realizes a Lorentz submersion from $E(1,1)$ into $(\mathbf{R}^2, (\cdot,\cdot))$.*

**Proof.** Let $E = [\mathcal{E}(1,1), \mathcal{E}(1,1)]$, where $\mathcal{E}(1,1)$ is the Lie algebra of $E(1,1)$.

• *Suppose that* $\langle\cdot,\cdot\rangle_{|E}$ *is non-degenerate.*

Then, according to 2.2, we may choose $e_0 \notin E$ such that

$$\mathcal{E}(1,1) = \mathbf{R}e_0 \oplus E \quad \text{and} \quad \langle e_0, E\rangle = 0.$$

Therefore, $\langle\cdot,\cdot\rangle_{|E}$ is determined by a $(\cdot,\cdot)$-self-adjoint isomorphism $\phi$ such that

$$\forall x, y \in E, \quad \langle x, y\rangle = (\phi(x), y)$$

**Case 1.** *$\phi$ is diagonizable over* **R**.

Since it is $(\cdot,\cdot)$-self-adjoint, $\phi$ is diagonizable in an $(\cdot,\cdot)$-orthonormal basis $\{e_1, e_2\}$, let $\lambda_1, \lambda_2$ its eigenvalues. In this basis, $ad_{e_0}$ is now symmetric, that is

$$[e_0, e_1] = e_2, \quad [e_0, e_2] = e_1 \quad \text{and} \quad [e_1, e_2] = 0. \tag{2}$$

An easy computation shows that

$$ad^*_{e_0} = \begin{pmatrix} 0 & -\frac{\lambda_2}{\lambda_1} \\ -\frac{\lambda_1}{\lambda_2} & 0 \end{pmatrix}.$$

The equation $(*)$ are now

$$\begin{cases} \dot{x} = \frac{(\lambda_2-\lambda_1)}{\langle e_0,e_0\rangle} x_1 x_2 \\ \dot{x}_1 = -\frac{\lambda_2}{\lambda_1} x_0 x_2 \\ \dot{x}_2 = -\frac{\lambda_1}{\lambda_2} x_0 x_1 \end{cases} \tag{3}$$

where $x_0, x_1, x_2$ are respectively components of $L^{-1}_{c(t)*}\dot{c}(t)$ with respect to $e_0, e_1, e_2$.

On the other hand, we have the two first-integrals

$$\begin{aligned} \langle e_0, e_0\rangle x_0^2 + \lambda_1 x_1^2 - \lambda_2 x_2^2 &= e \\ \lambda_1^2 x_1^2 - \lambda_2^2 x_2^2 &= m. \end{aligned}$$

If $\lambda_1 = \lambda_2$, it is straightforward to verify that the metric is complete. *Suppose now* $\lambda_1 \neq \lambda_2$. Then, according to the above expressions, the first equation of (3) is given by

$$\dot{x}_0 = \pm\sqrt{\left(\frac{\lambda_2 - \lambda_1}{\langle e_0, e_0\rangle}\right)^2 (ax_0^2 + b)(cx_0^2 + d)}$$

where

$$a = \frac{\langle e_0, e_0\rangle\lambda_1}{\lambda_2(\lambda_1 - \lambda_2)}, \quad c = \frac{\langle e_0, e_0\rangle\lambda_2}{\lambda_1(\lambda_1 - \lambda_2)}$$

the values of $b$ and $d$ have little importance. Now, obviously

$$ac = \left(\frac{\langle e_0, e_0\rangle}{\lambda_1 - \lambda_2}\right)^2 > 0.$$

Thus, there exist solutions for which $x_0 \to \infty$ when $t \to b$, where $b < \infty$ and $t$ is an affine parameter. Consequently, the metric is non-complete.

**Case 2.** *$\phi$ is non-diagonizable.*

We choose a basis $\{X_1, X_2\}$ for which

$$\phi = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix} \quad \text{where } a \neq 0.$$

**Lemma 4.3** *We have* $(X_1, X_1) = 0$ *and* $(X_1, X_2) \neq 0$.

**Proof.** The first equality is the consequence of $(\phi(X_1), X_2) = (X_1, \phi(X_2))$, as for as to the inequality, it follows from the fact that $(\cdot, \cdot)$ is nondegenerate or, equivalently, from the fact that in a Lorentzian 2-space if $x$ is a null vector then $x^\perp = \mathbf{R}x$.

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Replacing $X_2$ by $X_2 + tX_1$, where $t = -\frac{(X_2,X_2)}{(X_1,X_2)}$ we may assume that $(X_2, X_2) = 0$. Then, by putting

$$e_1 = \frac{X_1 + X_2}{\sqrt{2\,|\,(X_1, X_2)\,|}} \text{ and } \quad e_2 = \frac{X_1 - X_2}{\sqrt{2\,|\,(X_1, X_2)\,|}}$$

we may assume that $\{e_1, e_2\}$ is an $(\cdot, \cdot)$-orthonormal basis. Thus, $ad_{e_0}$ is symmetric with respect to this basis, and

$$\phi = \begin{pmatrix} (\lambda + \frac{a}{2}) & -\frac{a}{2} \\ \frac{a}{2} & (\lambda - \frac{a}{2}) \end{pmatrix}.$$

Hence

$$\langle e_1, e_1\rangle = \frac{a}{2} + \lambda,\ \langle e_2, e_2\rangle = \frac{a}{2} - \lambda,\ \langle e_1, e_2\rangle = -\frac{a}{2}$$

We easily obtain

$$ad^*_{e_0}e_1 = -\frac{a}{\lambda}e_1 - \left(\frac{a}{\lambda}+1\right)e_2 \text{ and } ad^*_{e_0}e_2 = \left(\frac{a}{\lambda}-1\right)e_1 + \frac{a}{\lambda}e_2$$

Thus, equations $(*)$ are given by

$$\begin{cases} \dot{x}_0 = \frac{a(x_1-x_2)^2}{2\langle e_0,e_0\rangle} \\ \dot{x}_1 = -\frac{a}{\lambda}x_0x_1 + \left(\frac{a}{\lambda}-1\right)x_0x_2 \\ \dot{x}_2 = -\left(\frac{a}{\lambda}+1\right)x_0x_1 + \frac{a}{\lambda}x_0x_2 \end{cases} \tag{4}$$

Furthermore, we have the first-integrals

$$\begin{aligned} \langle e_0,e_0\rangle x_0^2 + \tfrac{\lambda}{2}\left(x_1^2 - x_2^2\right) &= m \\ \langle e_0,e_0\rangle x_0^2 + \left(\tfrac{a}{2}+\lambda\right)x_1^2 + \left(\tfrac{a}{2}-\lambda\right)x_2^2 - ax_1x_2 &= e. \end{aligned}$$

Substituting these two formulas into the first equation of (4), we see that

$$\dot{x}_0 = x_0^2 + \frac{e-2m}{\langle e_0,e_0\rangle}.$$

So that, at the level $e = m = 0$, such a geodesic is never complete.

**Case 3.** *$\phi$ admits complex eigenvalues.*

Let $\lambda, \bar{\lambda}$ be the eigenvalues of $\phi$, and set $\lambda = \alpha + i\beta$. If $v, \bar{v}$ are the eigenvectors associates to $\lambda, \bar{\lambda}$, we get

$$(v,v) = (\bar{v},\bar{v}) \quad \text{and} \quad (v,\bar{v}) = 0.$$

Taking $v = e_1 + ie_2$, we get

$$(e_1,e_1) + (e_2,e_2) = 0 \quad \text{and} \quad (e_1,e_2) = 0.$$

In other words, we may assume (up to an automorphism of $\mathcal{E}(1,1)$) that $\{e_1,e_2\}$ is an $(\cdot,\cdot)$-orthonormal basis with $(e_2,e_2) = -1$.

With respect to this basis, we have

$$\phi = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

On the other hand, $ad_{e_0}$ is symmetric and we get easily

$$\langle e_1,e_1\rangle = \alpha,\ \langle e_2,e_2\rangle = -\alpha \text{ and } \langle e_1,e_2\rangle = \beta.$$

Taking

$$a = \frac{2\alpha\beta}{\alpha^2+\beta^2} \quad \text{and} \quad b = \frac{\beta^2-\alpha^2}{\alpha^2+\beta^2}.$$

we obtain

$$ad^*_{e_0}e_1 = ae_1 + be_2 \text{ and } ad^*_{e_0}e_2 = be_1 - ae_2$$

Now, equations $(*)$ are given by

$$\begin{cases} \dot{x}_0 = \frac{-\beta(x_1^2+x_2^2)}{\langle e_0,e_0\rangle} \\ \dot{x}_1 = x_0(ax_1+bx_2) \\ \dot{x}_2 = x_0(bx_1-ax_2) \end{cases} \tag{5}$$

We have the two first-integrals

$$\begin{aligned} \langle e_0,e_0\rangle x_0^2 + \alpha\left(x_1^2-x_2^2\right) + 2\beta x_1x_2 &= e \\ b\langle e_0,e_0\rangle x_0^2 + 2\beta x_1x_2 &= m \end{aligned}$$

Suppose that $\alpha \neq 0$, and choose the level $e = m = 0$, we obtain

$$x_1^2 + x_2^2 = \pm\frac{\langle e_0,e_0\rangle}{\beta}x_0^2.$$

Substituting this formula into the first equation of (5), we get

$$\dot{x}_0 = \pm x_0^2.$$

Thus, $x_0$ tends to $\infty$ when $t \to b$, where $b < \infty$. Hence, the metric is non-complete. The case where $\alpha = 0$ is elementary.

• *Assume now that* $\langle\cdot,\cdot\rangle_{|E}$ *is degenerate.* Then, we can find a vector $b$ such that

$$\forall x \in E \quad \langle b,b\rangle = \langle b,x\rangle = 0.$$

Let $\{e_1', e_1'\}$ be a basis of $E$ such that

$$(e_1',e_1') = (e_2',e_2') = 0 \text{ and } (e_1',e_2') = -1.$$

Then

$$[e_0,e_1'] = e_1', [e_0,e_2'] = -e_2' \quad \text{and} \quad [e_1',e_2'] = 0.$$

There are two cases which we may consider:

**Case 1.** *$b$ is colinear to $e_1'$ or $e_2'$.*

Assume for example $b = e_1'$, then $\langle e_2', e_2' \rangle > 0$ (since $e_1'^{\perp}$ could not contain time-like vectors).

By an appropriate automorphism of $\mathcal{E}(1,1)$) (which is an isometry for the metric) we may assume that $\langle e_2', e_2' \rangle = 1$.

The first equation of $(*)$ is then given by

$$\dot{x}_0 = x_0^2.$$

Thus, the metric is incomplete.

**Case 2.** *$b$ is not colinear to $e_1'$, neither to $e_2'$.*

By an appropriate hyperbolic rotation (which is an automorphism of $\mathcal{E}(1,1)$), we may assume that $b = e_1'$. Next, with similar approach as for the case of $E(2)$ the first equation of $(*)$ gives

$$\dot{x}_0 = \pm x_0^2$$

so that, the metric is incomplete, and the conclusion follows.

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