ON COMPLEX STRUCTURES IN 8-DIMENSIONAL VECTOR BUNDLES

Martin Čadek, Jiří Vanžura

Abstract. Necessary and sufficient conditions for the existence of a complex structure in 8-dimensional spin vector bundle over a closed connected spin manifold of dimension 8 are given in terms of characteristic classes. The result completes the papers by Heaps [H] and Thomas [T] on the same topic.

The problem of the existence of an almost complex structure on 8-dimensional closed smooth manifold was solved by T. Heaps in [H]. This result was generalized to arbitrary 8-dimensional vector bundles over 8-manifolds by E. Thomas in [T]. However, his necessary and sufficient conditions for the existence of a complex structure in 8-dimensional vector bundle contain a secondary cohomology operation which was computed only under special conditions. The aim of this note is to use the methods developed in [CV1] to improve the results from [T] for vector bundles with vanishing second Stiefel–Whitney class. The main result together with our description of the characteristic classes of such bundles in [CV3] enable us to decide which of them have a complex structure. We will show it on examples of S^8 and $G_{4,2}(\mathbb{C})$.

We will use the term "spin vector bundle" exclusively for oriented vector bundles which admit a spin structure, i. e. with the trivial first and second Stiefel–Whitney classes and we will not have in mind any fixed spin structure.

Let ξ be an oriented 8-dimensional spin vector bundle. We prove that the structure group $SO(8)$ of ξ can be reduced to $U(4)$ if and only if the vector bundle associated to ξ via a certain outer automorphism of the group $Spin(8)$ contains a 2-dimensional subbundle. This comparison enables to give necessary and sufficient conditions for the existence of a complex structure in ξ over a closed connected spin manifold of dimension 8 in terms of characteristic classes and the cohomology ring of the manifold without using any higher order cohomology operation. So our main theorem completes the results of Heaps [H] and Thomas [T].

In what follows we use the notation from [CV1]. We recall that we consider the Cayley numbers $\mathbb{O}$ as a right quaternionic vector space. In the Lie algebra $\mathfrak{so}(8)$ we consider the subalgebra $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$. Obviously, $c \in \mathfrak{so}(8)$ lies in $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$ if and only if the following conditions are satisfied

$$c([1,i]) \subset [1,i], \quad c([j,k,e,f,g,h]) \subset [j,k,e,f,g,h].$$

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Moreover $c|[1,i] \in \mathfrak{so}(2)$, and consequently there exists $l(c) \in \mathbb{R}$ such that

$$c(1) = l(c)i, \quad c(i) = -l(c)1.$$

Using the outer automorphisms λ and κ of $\mathfrak{so}(8)$ (see [CV1] for definition) we can formulate the following lemma.

Lemma 1. *Homomorphism $\kappa\lambda$ restricted to the Lie algebra $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$ is an isomorphism between $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$ and $\mathfrak{u}(4)$.*

Proof. The triality principle says that for every $x, y \in \mathbb{O}$ and every $a \in \mathfrak{so}(8)$ we have

$$a(xy) = b(x)y + xc(y)$$

where

$$a = (\kappa\lambda)(c) \quad , \quad b = \lambda^2(c).$$

So we get

$$a(x) = a(x1) = b(x)1 + xc(1) = b(x) + xl(c)i.$$

Taking into account this equality we obtain

$$a(xi) = b(x)i + xc(i) = a(x)i + xl(c) - xl(c) = a(x)i,$$

which shows that $a = (\kappa\lambda)(c) \in \mathfrak{u}(4)$. Since $\kappa\lambda$ is a monomorphism and $\dim \mathfrak{u}(4) = \dim(\mathfrak{so}(2) \oplus \mathfrak{so}(6)) = 16$, we get that $\kappa\lambda$ is an isomorphism between $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$ and $\mathfrak{u}(4)$.

We will denote the corresponding homomorphisms of corresponding Lie groups and Lie algebras by the same letters.

Let $\upsilon_2 : Spin(2) \to Spin(8)$ and $\upsilon_6 : Spin(6) \to Spin(8)$ be the canonical inclusions. It is easy to see that $\mathrm{im}\ \upsilon_2 \cap \mathrm{im}\ \upsilon_6 = \{1, -1\}$. Because every element from $\mathrm{im}\ \upsilon_2$ commutes with every element from $\mathrm{im}\ \upsilon_6$, we get a homomorphism $\upsilon_2 \times \upsilon_6 : Spin(2) \times Spin(6) \to Spin(8)$ with kernel $K = \{(1,1), (-1,-1)\}$. We denote $Spin(2) \cdot Spin(6) = (Spin(2) \times Spin(6))/K$. Then we obtain an induced monomorphism $\upsilon : Spin(2) \cdot Spin(6) \to Spin(8)$. Let $\pi : Spin(8) \to SO(8)$ be the standard epimorphism. Obviously, we have $\mathrm{im}\ (\pi\upsilon) = SO(2) \times SO(6)$. We denote by $\widetilde{U(4)} = \pi^{-1}U(4)$ the inverse image of the subgroup $U(4) \subset SO(8)$. Because the Lie group $U(4)$ is connected we can see that the Lie group $\widetilde{U(4)}$ has either one or two components. Taking an orthonormal basis $e_1 = 1$, $e_2 = i$, $e_3 = j$, $e_4 = k$, $e_5 = e$, $e_6 = f$, $e_7 = g$, $e_8 = h$ in $\mathbb{O}$, we can consider a curve $\varphi(t) = \cos t + \sin t \cdot e_1 e_2$, $t \in < 0, \pi >$ in $Spin(8)$. It is easy to verify that this curve lies in $\widetilde{U(4)}$ and joins the elements 1 and -1. This shows that the group $\widetilde{U(4)}$ is connected. By virtue of Lemma 1 we get a commutative diagram

$$\begin{array}{ccc}
Spin(2) \cdot Spin(6) & \xrightarrow{\ \upsilon\ } & Spin(8) \\
\downarrow{\scriptstyle \kappa\lambda} & & \downarrow{\scriptstyle \kappa\lambda} \\
\widetilde{U(4)} & \xrightarrow{\ \gamma\ } & Spin(8) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
U(4) & \xrightarrow{\ \beta\ } & SO(8)
\end{array}$$

where β and γ denote the inclusions, and $\kappa\lambda$ are isomorphisms. From this diagram we obtain easily the following lemma.

Lemma 2. *The homogeneous space $Spin(8)/\widetilde{U(4)}$ determined by the inclusion γ is diffeomorpfic to the Grassmann manifold $\widetilde{G}_{8,2}$ of oriented 2-planes.*

Let X be a CW-complex. Applying first the classifying space functor B and then the functor $[X,-]$ to the above diagram, we get the following commutative diagram

$$\begin{array}{ccc}
[X, B(Spin(2)\cdot Spin(6))] & \xrightarrow{\upsilon_*} & [X, BSpin(8)] \\
{\scriptstyle (\kappa\lambda)_*}\Big\uparrow & & \Big\uparrow{\scriptstyle (\kappa\lambda)_*} \\
[X, B\widetilde{U(4)}] & \xrightarrow{\gamma_*} & [X, BSpin(8)] \\
\Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \pi_*} \\
[X, BU(4)] & \xrightarrow{\beta_*} & [X, BSO(8)]
\end{array}$$

where $(\kappa\lambda)_*$ are bijections.

Now let us consider an oriented 8-dimensional vector bundle (V, p_V, X) over X. Choosing a riemannian metric on it we can consider the corresponding principal $SO(8)$-bundle $\xi = (Q, p_Q, X)$. We can consider this bundle as an element $\xi \in [X, BSO(8)]$. We have $V = Q \times_{SO(8)} \mathbb{R}^8$. Let us assume that ξ is a spin bundle. This means that there exists a principal $Spin(8)$-bundle $\bar{\xi} = (P, p_P, X)$ and an π-equivariant principal bundle epimorphism $\chi : P \to Q$. Equivalently, we can say that there exists an element $\bar{\xi} \in [X, BSpin(8)]$ such that $\pi_*\bar{\xi} = \xi$. If $\xi = (Q, p_Q, X)$ has a reduction $\widetilde{Q} \subset Q$ to the subgroup $U(4)$, then $\bar{\xi} = (P, p_P, X)$ has a reduction $\widetilde{P} = \chi^{-1}(\widetilde{Q})$ to the subgroup $\widetilde{U(4)}$. Conversely, if $\bar{\xi} = (P, p_P, X)$ has a reduction $\widetilde{P} \subset P$ to the subgroup $\widetilde{U(4)}$, then $\xi = (Q, p_Q, X)$ has a reduction $\widetilde{Q} = \chi(\widetilde{P})$ to the subgroup $U(4)$.

Let us consider a reduction $\widetilde{P} \subset P$ of the principal $Spin(8)$-bundle $\bar{\xi}$ to the subgroup $\widetilde{U(4)}$. We can consider the Lie group $Spin(2)\cdot Spin(6)$ as a left $\widetilde{U(4)}$ -space when we define for $h \in \widetilde{U(4)}$ and $g \in Spin(2)\cdot Spin(6)$ the action $h \cdot g = (\kappa\lambda(h))g$. Then the associated bundle $\widetilde{P} \times_{\kappa\lambda} Spin(2)\cdot Spin(6)$ is a principal $Spin(2)\cdot Spin(6)$-bundle. It is easy to see that this is the bundle $(\kappa\lambda)_*(\bar{\xi})$.

The same argument as above shows that a principal $SO(8)$-bundle $\xi = (Q, p_Q, X)$ admitting a spin structure $\chi : P \to Q$ has a reduction to the subgroup $SO(2) \oplus SO(6)$ if and only if the principal $Spin(8)$-bundle $\bar{\xi} = (P, p_P, X)$ has a reduction to the subgroup $Spin(2)\cdot Spin(6)$.

In the sequel we shall not distinguish between an oriented 8-dimensional vector bundle, the corresponding principal $SO(8)$-bundle, and the corresponding element from $[X, BSO(8)]$. We have proved the following lemma.

Lemma 3. *Let X be a CW-complex and let $\xi \in [X, BSO(8)]$ be a spin vector bundle. Then ξ has a complex structure (i. e. an $U(4)$-structure) if and only if there exists an element $\bar{\xi} \in [X, BSpin(8)]$ such that*

(1) $\pi_*\bar{\xi} = \xi$;
(2) *The vector bundle $\zeta = \pi_*(\kappa\lambda)_*(\bar{\xi})$ has an oriented 2-dimensional subbundle.*

We will use $w_m(\xi)$ for the m-th Stiefel–Whitney class of the vector bundle ξ, $p_m(\xi)$ for the m-th Pontrjagin class, and $e(\xi)$ for the Euler class. The letters w_m, p_m and e will stand for the characteristic classes of the universal bundles over the classifying spaces $BSO(8)$ and $BSpin(8)$. The mapping $\rho_m : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z}_m)$ is induced from the reduction mod m.

Now, we shall recall some facts about the cohomology of $BSpin(8)$. These results can be found in [Q] and [CV1].

Lemma 4. *The cohomology rings of $BSpin(8)$ are*

$$H^*(BSpin(8);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, \varepsilon]$$

and

$$H^*(BSpin(8);\mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, e, \delta w_6]/\langle 2\delta w_6\rangle$$

where q_1, q_2 and ε are defined by the relations

$$p_1 = 2q_1 \quad , \quad p_2 = q_1^2 + 2e + 4q_2 \quad , \quad \rho_2 q_2 = \varepsilon.$$

Moreover,

$$\rho_2 q_1 = w_4 \quad , \quad \rho_2 e = w_8.$$

Let ξ be an oriented 8-dimensional vector bundle over a CW-complex X given by the homotopy class of some mapping $\xi : X \to BSO(8)$. ξ has a spinor structure iff $w_2(\xi) = 0$. If some lifting $\bar{\xi} : X \to BSpin(8)$ is fixed we can define spin characteristic classes

$$q_1(\xi) = \bar{\xi}^* q_1 \quad , \quad q_2(\xi) = \bar{\xi}^* q_2.$$

The first spin characteristic class is always independent of the choice of $\bar{\xi}$. Moreover, if $H^4(X;\mathbb{Z})$ has no element of order 4, then it is uniquely determined by the relations

$$2q_1(\xi) = p_1(\xi) \quad , \quad \rho_2 q_1(\xi) = w_4(\xi).$$

The second spin characteristic class is independent of the spinor structure $\bar{\xi}$ if X is simply connected or $H^8(X;\mathbb{Z}) \cong \mathbb{Z}$. In the case of 8-dimensional manifold $q_2(\xi)$ is uniquely determined by the relation

$$16q_2(\xi) = 4p_2(\xi) - p_1^2(\xi) - 8e(\xi).$$

We shall also need information about the action of the homeomorphisms κ and λ on the above mentioned cohomology rings. See [CV1] for the next lemma.

Lemma 5. *For $\kappa : BSpin(8) \to BSpin(8)$ and $\lambda : BSpin(8) \to BSpin(8)$ we have*

$$\begin{aligned} \kappa^*(q_1) &= q_1 & \qquad \lambda^*(q_1) &= q_1 \\ \kappa^*(q_2) &= q_2 + e & \qquad \lambda^*(q_2) &= -e - q_2 \\ \kappa^*(e) &= -e & \qquad \lambda^*(e) &= q_2. \end{aligned}$$

Theorem 6. *Let ξ be a real 8-dimensional oriented vector bundle over a closed connected oriented smooth manifold M of dimension 8. Let $w_2(\xi) = w_2(M) = 0$. Then in ξ there exists the structure of a complex vector bundle if and only if there are $u \in H^2(M;\mathbb{Z})$ and $v \in H^6(M;\mathbb{Z})$ such that*

(i) $\rho_2 v = w_6(\xi) + w_4(\xi)\rho_2 u + \rho_2 u^3$ *and* $16uv = -4p_2(\xi) + p_1^2(\xi) + 8e(\xi)$,

(ii) $\{p_1^2(\xi) - p_1(\xi)p_1(M) + 8e(\xi)\}[M] \equiv 0$ *mod 16.*

Proof. Let us choose $\bar{\xi} \in [M, BSpin(8)]$ such that $\pi_*\bar{\xi} = \xi \in [M, BSO(8)]$, and consider the bundle $\zeta = \pi_*(\kappa\lambda)_*\bar{\xi} \in [M, BSO(8)]$. By virtue of Lemma 5 we have $\lambda^*\kappa^*(q_1) = q_1$, which implies $\lambda^*\kappa^*(w_4) = \lambda^*\kappa^*\rho_2(q_1) = \rho_2(q_1) = w_4$. Further, $\lambda^*\kappa^*(w_6) = \lambda^*\kappa^* Sq^2(w_4) = Sq^2(w_4) = w_6$. Consequently, we get

$$q_1(\zeta) = q_1(\xi), \quad w_4(\zeta) = w_4(\xi), \quad w_6(\zeta) = w_6(\xi).$$

Similarly, we have $\lambda^*\kappa^*(q_2) = -e$ and $\lambda^*\kappa^*(e) = -q_2$, which implies

$$q_2(\zeta) = -e(\xi), \quad e(\zeta) = -q_2(\xi).$$

Now, using Theorem 3.1 and Theorem 3.5 from [CV2], we can see that ζ has an oriented 2-dimensional subbundle if and only if there are $u \in H^2(M;\mathbb{Z})$ and $v \in H^6(M;\mathbb{Z})$ such that

(i) $\rho_2 v = w_6(\zeta) + w_4(\zeta)\rho_2 u + \rho_2 u^3$ and $uv = e(\zeta)$,

(ii) $\rho_2 q_2(\zeta) = \rho_2 \frac{1}{2}\{q_1(\zeta)q_1(M) - q_1^2(\zeta)\}$.

Expressing the characteristic classes of ζ in terms of the characteristic classes of ξ, we obtain the two conditions of the theorem. We can also see that the proof does not depend on the choice of the spin structure $\bar{\xi}$.

Remark 7. In [T] explicit necessary and sufficient conditions for the existence of a complex structure are given only for 8-dimensional vector bundles ξ satisfying the conditions

$$\delta w_2(\xi) = 0 \quad , \quad w_4(\xi) = w_4(M).$$

Example 8. We shall consider the complex Grassmann manifold $G_{4,2}(\mathbb{C})$. (See also [CV2], Example 3.6.) Let us recall that $H^*(G_{4,2}(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/(x_1^3 - 2x_1x_2, x_2^2 - x_1^2x_2)$. The isomorphism is given by $x_1 \mapsto c_1$, $x_2 \mapsto c_2$, where c_1 and c_2 are the Chern classes of the canonical complex vector bundle γ_2 over $G_{4,2}(\mathbb{C})$. A standard computation shows that

$$c_1(G) = 4c_1, \quad c_2(G) = 7c_1^2, \quad c_3(G) = 12c_1c_2, \quad c_4(G) = 6c_1^2c_2,$$

$$p_1(G) = 2c_1^2, \quad p_2(G) = 14c_1^2c_2, \quad e(G) = 6c_1^2c_2,$$

where $G = G_{4,2}(\mathbb{C})$. We can immediately see that $G_{4,2}(\mathbb{C})$ is a spin manifold.

Let ξ be a spin vector bundle over $G_{4,2}(\mathbb{C})$ (i. e. $w_2(\xi) = 0$). As in [CV2] we can write

$$p_1(\xi) = 2ac_1^2 + 2bc_2, \quad p_2(\xi) = Cc_1^2c_2, \quad e(\xi) = Dc_1^2c_2.$$

Further, let us write

$$u = kc_1 \in H^2(G_{4,2}(\mathbb{C});\mathbb{Z}), \quad v = lc_1c_2 \in H^6(G_{4,2}(\mathbb{C});\mathbb{Z}).$$

In [CV2] we have shown that the condition (i) of Theorem 6 has the form

(i) $l \equiv (k+1)b$ mod 2, $\quad 4kl = -C + 2a^2 + 2ab + b^2 + 2D.$

Easy computation shows that the condition (ii) of the same theorem reads as

(ii) $2a^2 + 2ab + b^2 - 2a - b + 2D \equiv 0 \mod 4$.

It remains to investigate the existence of integers k and l satisfying the above equations (i). For this purpose we shall investigate two cases, namely $b \equiv 0 \bmod 2$ and $b \equiv 1 \bmod 2$. We start with the case $b \equiv 0 \bmod 2$. If b is even then l is also even and the integer $-C + 2a^2 + 2ab + b^2 + 2D$ is divisible by 8. We can now easily see that if $b \equiv 0 \bmod 2$ then the integers k and l satisfying (i) exist if and only if

$$8|(-C + 2a^2 + 2ab + b^2 + 2D).$$

We can start with the case $b \equiv 1 \bmod 2$. Here $l = k + 1 + 2r$, where r is an integer. Substituting into the second equation of (i), we obtain easily

$$(2k + (2r + 1))^2 - (2r + 1)^2 = X,$$

where we have denoted $X = -C + 2a^2 + 2ab + b^2 + 2D$. Now, it is obvious that the existence of integers k and l satisfying (i) is equivalent to the existence of integers k and r satisfying the above equation. Denoting $\alpha = 2k + (2r + 1)$ and $\beta = 2r + 1$, we can easily see that the existence of integers k and r is equivalent to the existence of odd integers α and β such that

$$\alpha^2 - \beta^2 = X.$$

Simple considerations show that odd integers α and β with the above property exist if and only if $8|X$. This means that in both cases we get the same result. Summarizing we can say that on a vector bundle ξ over $G_{4,2}(\mathbb{C})$ there exists a complex vector bundle structure if and only if

(i) $8|(-C + 2a^2 + 2ab + b^2 + 2D)$,

(ii) $2a^2 + 2ab + b^2 - 2a - b + 2D \equiv 0 \mod 4$.

This example also enables to test our theorem. Taking $\xi = T(G_{4,2}(\mathbb{C}))$, the tangent bundle of $G_{4,2}(\mathbb{C})$, we can easily find out that the conditions of Theorem 6 are satisfied. This corresponds with the fact that $G_{4,2}(\mathbb{C})$ is a complex manifold.

Example 9. Let us consider the unit sphere S^8. Using Theorem 2 from [CV3] we can see that isomorphism classes of 8-dimensional real vector bundles over S^8 are in bijective correspondence with elements of $\mathbb{Z} \times \mathbb{Z}$. A vector bundle $\xi_{k,l}$ corresponding to $(k, l) \in \mathbb{Z} \times \mathbb{Z}$ satisfies

$$p_2(\xi_{k,l})[S^8] = 6k \quad \text{and} \quad e(\xi_{k,l})[S^8] = 3k + 2l.$$

Using the above Theorem 6 we can easily find that there is a complex vector bundle structure on $\xi_{k,l}$ if and only if

$$k \equiv 0 \bmod 2, \quad l = 0.$$

This result can be reformulated in the following form. An 8-dimensional real vector bundle ξ over S^8 admits a complex vector bundle structure if and only if

$$12|p_2(\xi)[S^4] \quad \text{and} \quad p_2(\xi) = 2e(\xi).$$

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Faculty of Sciences, Masaryk University, Janáčkovo nam. 2a, 662 95 Brno, Czech Republic

E-mail address: cadek@math.muni.cz

Academy of Sciences of the Czech Republic, Institute of Mathematics, Žižkova 22, 616 62 Brno, Czech Republic

E-mail address: vanzura@ipm.cz