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ON COMPOSITION OF FOUR-SYMBOL δ-CODES AND HADAMARD MATRICES

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ABSTRACT. It is shown that key instruments for composition of four-symbol δ-codes are the Lagrange identity for polynomials, a certain type of quasi-symmetric sequences (i.e., a set of normal or near normal sequences) and base sequences. The following is proved: If a set of base sequences for length t and a set of normal (or near normal) sequences for length n exist then four-symbol δ-codes of length $(2n+1)t$ (or nt) can be composed by application of the Lagrange identity. Consequently a new infinite family of Hadamard matrices of order $4uw$ can be constructed, where w is the order of Williamson matrices and $u = (2n+1)t$ (or nt). Other related topics are also discussed.

1. INTRODUCTION

Turyn [T1] constructed Hadamard matrices of order $4tw$ from a 4-symbol δ-code of length t and Williamson matrices of order w using Baumert-Hall units. (See formal definitions for an Hadamard matrix and others below.) He [T1, T2] used certain binary sequences for construction of 4-symbol δ-codes which are defined in terms of nonperiodic auto-correlation functions (whose concept originated in optics and signal transmission problems).

A method to compose 2-symbol δ-codes of length $2^k mn$ from 2-symbol δ-codes of lengths $2m$ and $2n$ was found by Golay [G] for $k \geq 3$ and improved by Turyn [T1] for $k \geq 2$. It is easy to construct 4-symbol δ-codes of length $2^k t$ from a 4-symbol δ-code of length t, which Turyn found for $t \leq 59$ (except 49, 57) and $t = g+1$, where $g = 2^a 10^b 26^c$ (Golay numbers), a, b and c are nonnegative integers [T1, T2]. However using only the definition of auto-correlation functions to prove composition of a 4-symbol δ-code of length st from a 4-symbol δ-code of length t for odd s is more difficult. This difficulty was solved by introduction of an algebraic approach, polynomials defined on the unit circle [Y4] and the Lagrange identity for polynomials [Y1,Y2]. For $s = 3, 7, 13$ and $2g+1$, i.e., $s = 5, 9, 17, 21, 27, 33, 41, 53, 65, 81, \ldots,$ it was solved using this approach in [Y1,Y2,Y3].

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In this paper, constructions for $s = 4n+1$, and for $s \leq 31$, $s = 51, 59$ and $s = 2g+1$ are made. The former depends on the existence of near normal sequences (a 4-symbol code) for length $4n+1$, which exist for $n \leq 11$ (and likely to exist for all n) and the latter on the existence of normal sequences (a 3-symbol code) for length $m = (s-1)/2$, which exist for odd $m \leq 15$, $m = 25$, 29 and $m = g$ (a Golay number). Therefore new cases are solved for $s = 11, 15, 19, 23, 25, 29, 31, 37, 45, 51$ and 59; and new constructions are made for the known $4n+1$ and $2g+1$ cases.

These new results also lead to construction of four complementary $(1, -1)$-sequences of length $u = rst$ ($r = m$ above, or $2g+1$), or equivalently 4-symbol δ-codes of length $2u$; consequently, to construction of Goethals-Seidel (Hadamard) matrices of orders $4u$ and $8uw$, where w is the order of Williamson matrices which exist for all $w < 100$ (except 35, 39, 47, 53, 59, 65, 67, 70, 71, 73, 77, 83, 89 and 94) [W], $w = (q+1)/2$, where q (a prime power) $\equiv 1 \pmod 4$ [T3], and others (see [A], [M], [S] and [W]). Each new case actually leads to construction of infinitely many new matrices.

2. Preliminaries, notations and definitions

A matrix whose every entry is either b or c is called a *(b, c)-matrix*. Similarly a *(b, c)-sequence* has each element b or c. An *Hadamard matrix* $H_n = [h_{ij}]$ is a square $(1, -1)$-matrix of order n such that $H_n^t H_n = nI_n$, where I_n is the identity square matrix of order n and t indicates the transposed matrix. In H_n, distinct *column vectors* $v_i = [h_{1i}, h_{2i}, \dots, h_{ni}]$ are orthogonal, i.e., $v_i^t \cdot v_j = \sum_k h_{ki} h_{kj} = 0$, for $i \neq j$; similarly distinct row vectors are also orthogonal since the matrix and its transpose commute. H_n exists only if $n = 1, 2$, or $4k$, and the converse is conjectured to be valid. Hadamard matrices H_{4n} have been constructed for all $n \leq 100$ and for infinitely many n. For various properties and applications of Hadamard matrices, see [A], [GS], [H], [HS], [HW] and [K] in the references.

Williamson matrices $(\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ are four square $(1, -1)$-matrices of order w satisfying $\mathbf{M}^t\mathbf{N} = \mathbf{N}^t\mathbf{M}$ for all $\mathbf{M}$ and $\mathbf{N} \in \{\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$, i.e., all $\mathbf{M}^t\mathbf{N}$ are symmetric, and

$$(1) \qquad \mathbf{W}^t\mathbf{W} + \mathbf{X}^t\mathbf{X} + \mathbf{Y}^t\mathbf{Y} + \mathbf{Z}^t\mathbf{Z} = 4w\mathbf{I}_w.$$

Williamson [Wi] constructed Hadamard matrices H_n (i.e., $H_n^t H_n = nI_n$, $n = 4w$), in which the first, second, third and fourth (block) columns are respectively.

$$(2) \quad e_1 = [\mathbf{W}, -\mathbf{X}, -\mathbf{Y}, -\mathbf{Z}], \quad e_2 = [\mathbf{X}, \mathbf{W}, \mathbf{Z}, -\mathbf{Y}], \quad e_3 = [\mathbf{Y}, -\mathbf{Z}, \mathbf{W}, \mathbf{X}] \quad \text{and} \quad e_4 = [\mathbf{Z}, \mathbf{Y}, -\mathbf{X}, \mathbf{W}].$$

We note that the four column vectors e_k are orthogonal, i.e., $e_i^t \cdot e_j = \mathbf{0}$, for $i \neq j$, where $\mathbf{0}$ is the zero matrix and $e_i^t \cdot e_i = \mathbf{W}^t\mathbf{W} + \mathbf{X}^t\mathbf{X} + \mathbf{Y}^t\mathbf{Y} + \mathbf{Z}^t\mathbf{Z} = 4w\mathbf{I}_w$.

Let $S = (s_k)_n = (s_1, s_2, \dots, s_n)$ be a sequence of real numbers, then $S(z) = \sum_k s_k z^{k-1}$ $(1 \le k \le n)$ is called *the associated polynomial of* S and $s(j) = \sum_k s_k s_{k+j}$ $(1 \le k \le n-j)$, *the jth nonperiodic auto-correlation function of* S, where $0 \le j \le n-1$, and $s(j) = 0$ for $j \ge n$. We note here that $|S|^2 = S(z)S(z^{-1}) = s(0) + \sum_k s(k)(z^k + z^{-k})$ $(1 \le k \le n-1)$ is the generating function for $s(k)$, where $z \in \mathbf{K} = \{z \in \mathbf{C}: |z| = 1\}$, the unit circle, and $\mathbf{C}$ is the complex field. We shall use the same letter to represent a sequence and its associated polynomial.

An *m-symbol* δ*-code of length n* is a sequence of vectors, $V = (v_1, v_2, \dots, v_n)$, where v_k is one of m orthonormal (column) vectors, $i_1, i_2, \dots, i_m$ or their negatives, such that $v(j) = 0$ for $j \ne 0$, where $v(j) = \sum_k v_k^t \cdot v_{k+j}$ $(1 \le k \le n-j)$ is the nonperiodic auto-correlation function of V.

When $m = 4$, i.e., for a 4-symbol δ-code V of length n, it is convenient to set $i_1 = [1, 1, 0, 0]$, $i_2 = [1, -1, 0, 0]$, $i_3 = [0, 0, 1, 1]$ and $i_4 = [0, 0, 1, -1]$ as four orthogonal column vectors with normalized length $\sqrt{2}$. By letting $v_k = [q_k, r_k, s_k, t_k]$, we have

(3) $$v(j) = q(j) + r(j) + s(j) + t(j) = 0 \qquad \text{for } j \ne 0,$$

where $q(j), r(j), s(j)$ and $t(j)$ are respectively the nonperiodic auto-correlation functions of the component sequences of V, i.e.,

(4) $$Q = (q_1, \dots, q_n), \quad R = (r_1, \dots, r_n), \quad S = (s_1, \dots, s_n) \quad \text{and} \quad T = (t_1, \dots, t_n).$$

A δ-code of length n represented by $4 \times n$ matrix $[Q, R; S, T]$ is called a *regular* δ*-code of length n* (abbreviated as $RD(n)$), where Q, R, S and T are the four component sequences of a 4-symbol δ-code $V = (v_1, \dots, v_n)$ in (4). We note here that $[Q, R; 0, 0]$ and $[0, 0; S, T]$ are in symbols i_1, i_2 and i_3, i_4 respectively. Since $q(0) + r(0) + s(0) + t(0) = 2n$, condition (3) is equivalent to

$$|Q|^2 + |R|^2 + |S|^2 + |T|^2 = 2n \quad \text{for any } z \text{ on the unit circle } \mathbf{K}.$$

And we also note that in $V = [Q, R; S, T]$, either $|q_k| = |r_k| = 1$ and $s_k = t_k = 0$, or $q_k = r_k = 0$ and $|s_k| = |t_k| = 1$, for each k.

A pair of $(1, -1)$-sequences, $\mathbf{F} = (f_1, \dots, f_g)$ and $\mathbf{H} = (h_1, \dots, h_g)$ is called a pair of *Golay complementary sequences of length g* (abbreviated as $GCS(g)$), if their auto-correlation functions satisfy $f(j) + h(j) = 0$ for $j \ne 0$ [G], or equivalently $|\mathbf{F}|^2 + |\mathbf{H}|^2 = 2g$ for any z on the unit circle $\mathbf{K}$ [Y4]. Golay complementary sequences $GCS(g)$ exist for $g = 2^a 10^b 26^c$ (Golay numbers), where a, b and c are nonnegative integers [T1]. For a given pair of Golay complementary sequences $\mathbf{F}$ and $\mathbf{G}$ of length g, $[\mathbf{F}, \mathbf{G}; 0, 0]$ is a regular δ-code $RD(g)$ in symbols i_1 and i_2, where $0 = 0_g$ is the sequence of zeros of length g. We shall use g for a Golay number.

A square matrix $\mathbf{S} = [s_{ij}]$ of order n is *circulant* if $s_{ij} = s_{k+1}$ for $k \equiv j - i \pmod{n}$. We note here that the first, second, …, nth rows of $\mathbf{S}$ are respectively, $(s_1, s_2, \dots, s_n)$, $(s_n, s_1, \dots, s_{n-1}), \dots, (s_2, s_3, \dots, s_1)$. Also

$\mathbf{S}^t\mathbf{S} = [p_{ij}]$ is symmetric circulant, i.e., $p_{ij} = p_{ji} = s^*(h)$, where $h = |i-j|$, and $s^*(h) = \sum_k s_k s_{k+h}$ $(1 \le k \le n)$, where the subscript $k+h$ is congruent modulo n.

$s^*(j)$ is called the *jth periodic auto-correlation function of* $S = (s_1, s_2, \dots, s_n)$. We note that $s^*(j) = s(j) + s(n-j) = s^*(n-j)$ and $s^*(0) = s(0)$. Also if $q(j) + r(j) + s(j) + t(j) = 0$ for $j \ne 0$ then $q^*(j) + r^*(j) + s^*(j) + t^*(j) = 0$ for $j \ne 0$.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ be four circulant square matrices of order m with entries ± 1 satisfying

$$\mathbf{A}^t\mathbf{A} + \mathbf{B}^t\mathbf{B} + \mathbf{C}^t\mathbf{C} + \mathbf{D}^t\mathbf{D} = 4m\mathbf{I_m}. \tag{5}$$

Then a *Goethals-Seidel (Hadamard) matrix* [Go], G of order $4m$ can be constructed, in which the first, second, third and fourth (block) rows are respectively $(\mathbf{A}, \mathbf{BR}, \mathbf{CR}, \mathbf{DR})$, $(-\mathbf{BR}, \mathbf{A}, -\mathbf{D}'\mathbf{R}, \mathbf{C}'\mathbf{R})$, $(-\mathbf{CR}, \mathbf{D}'\mathbf{R}, \mathbf{A}, -\mathbf{B}'\mathbf{R})$ and $(-\mathbf{DR}, -\mathbf{C}'\mathbf{R}, \mathbf{B}'\mathbf{R}, \mathbf{A})$, where $\mathbf{R} = [r_{ij}]$ is the matrix with $r_{ij} = 1$ for $i + j = m+1$, and $r_{ij} = 0$ otherwise, for $1 \le i,\ j \le m$ and $\mathbf{P}' = [p_{ji}]$ for $\mathbf{P} = [p_{ij}]$. We note that $\mathbf{P}^t \ne \mathbf{P}'$ if $p^t_{ij} \ne p_{ij}$, and $G^tG = (\mathbf{A}^t\mathbf{A} + \mathbf{B}^t\mathbf{B} + \mathbf{C}^t\mathbf{C} + \mathbf{D}^t\mathbf{D}) \times \mathbf{I_4} = (4m\mathbf{I_m}) \times \mathbf{I_4} = 4mI_{4m}$.

If Williamson matrices $\mathbf{W}, \mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ satisfying (1) and a 4-symbol δ-code $RD(n)$ of (4) exist, then we can construct a Goethals-Seidel (Hadamard) matrix G of order $4m$ $(m = nw)$ with entries from $\Omega = \{\pm\mathbf{W}, \pm\mathbf{X}, \pm\mathbf{Y}, \pm\mathbf{Z}\}$, by finding matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ satisfying (5), as follows. The first rows of circulant matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ of order $m = nw$ are respectively the corresponding component sequences A, B, C and D obtained by replacing the four orthogonal column vectors i_k, $1 \le k \le 4$, with the four orthogonal column vectors e_k of (2) in the δ-code $RD(n)$ of (4). Let $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$, $C = (c_1, \dots, c_n)$ and $D = (d_1, \dots, d_n)$, where a_k, b_k, c_k and d_k are from Ω. Then the following conditions corresponding to (3) hold:

$$a(j) + b(j) + c(j) + d(j) = 0 \quad \text{for } j \ne 0, \qquad \text{where } s(j) = \sum_k s^t_k \cdot s_{k+j}$$
$$(1 \le k \le n-j), s_k \in \Omega,$$

and

$$a(0) + b(0) + c(0) + d(0) = n(\mathbf{W}^t\mathbf{W} + \mathbf{X}^t\mathbf{X} + \mathbf{Y}^t\mathbf{Y} + \mathbf{Z}^t\mathbf{Z}) = 4nw\mathbf{I_w}.$$

Also for any circulant matrix $\mathbf{S}$ of order n having the first row $(s_1, \dots, s_n)$ with entries $s_k \in \Omega$, we have $\mathbf{S}^t\mathbf{S} = [s^*(h)]$, where $h = |i-j|$, $s^*(h) = s(h) + s(n-h)$ and $s^*(0) = s(0)$, consequently

$$\begin{aligned}\mathbf{A}^t\mathbf{A} + \mathbf{B}^t\mathbf{B} + \mathbf{C}^t\mathbf{C} + \mathbf{D}^t\mathbf{D} &= [a^*(h) + b^*(h) + c^*(h) + d^*(h)] \\ &= (4nw\mathbf{I_w}) \times \mathbf{I_n} = 4nw\mathbf{I_{nw}}.\end{aligned}$$

A quadruple of $(1, -1)$-sequences $(A, B; C, D)$ respectively with lengths $m+p$ and m pairs, where

$$A = (a_1, \ldots, a_{m+p}), \quad B = (b_1, \ldots, b_{m+p}), \quad C = (c_1, \ldots, c_m) \quad \text{and} \quad D = (d_1, \ldots, d_m), \qquad p \geq 0,$$

is called a set of (Turyn) *base sequences for length* $t = 2m + p$ (abbreviated as $BS(t)$), if they have zero auto-correlation sum i.e., $a(j) + b(j) + c(j) + d(j) = 0$ for $j \neq 0$, or equivalently if $|A|^2 + |B|^2 + |C|^2 + |D|^2 = 2t$ for any z on $\mathbf{K}$. Sets of base sequences $BS(t)$ are known (published) for $t \in \{3, \ldots, 47, 51, 53, 59,$ and $2g + 1 : g = 2^a 10^b 26^c$ (Golay numbers), a, b, c nonnegative integers$\}$ $(t = 2m + 1)$ with lengths $m + 1$, m pairs and $t = g + g'$, where g and g' are any Golay numbers (e.g., we may take $g' = 1$). [See below for the cases $t = 37, 39, 43$ and 45.]

From the above $BS(t)$: $(A, B; C, D)$, $t = 2m+p$, we can obtain the following regular δ-codes $RD(t)$: $[Q, R; S, T]$, trivially.

$$Q = (A, 0), \quad R = (B, 0), \quad S = (\mathbf{0}, C) \quad \text{and} \quad T = (\mathbf{0}, D), \tag{6}$$

where $0 = 0_m =$ the sequence of zeros of length m, and $\mathbf{0} = 0_{m+p}$; and when $p = 1$ or 0, i.e., with lengths $m + 1$ and m pairs or with m and m pairs.

$$Q = (A/0), \quad R = (B/0), \quad S = (\mathbf{0}/C) \quad \text{and} \quad T = (\mathbf{0}/D), \tag{7}$$

where (X/Y) means the interleaved sequence $(x_1, y_1, \ldots, x_k, y_k, \cdots)$ for $X = (x_1, \ldots, x_k, \cdots)$ and $Y = (y_1, \ldots, y_k, \cdots)$. The δ-codes (6) and (7) are called *Turyn δ-codes of length t.*

The following *Lagrange identity for polynomials* is the key for composition of 4-symbol δ-codes and other related codes [Y1,Y2].

Theorem L (Lagrange identity for polynomials). *Let a, b, c, d, e, f, g and h be polynomials in z with real coefficients. Also let $p' = p(z^{-1})$ for $p = p(z)$ and*

$$\begin{aligned} \mathbf{q} &= -b'e + af' + cg + dh, \qquad & \mathbf{s} &= -d'e - cf + ag' - bh, \\ \mathbf{r} &= a'e + bf' + dg' - ch', \qquad & \mathbf{t} &= c'e - df + bg + ah'. \end{aligned} \tag{L}$$

Then $|\mathbf{q}|^2 + |\mathbf{r}|^2 + |\mathbf{s}|^2 + |\mathbf{t}|^2 = (|a|^2 + |b|^2 + |c|^2 + |d|^2)(|e|^2 + |f|^2 + |g|^2 + |h|^2)$ for any z on $\mathbf{K}$.

Let $P^* = (p_1^*, \ldots, p_n^*)$, where $p_k^* = p_{n+1-k}$, be the *reverse* of a sequence $P = (p_1, \ldots, p_n)$. We note that $P' = P(z^{-1}) = \sum p_k z^{1-k} = z^{1-n}(\sum p_k^* z^{k-1}) = z^{1-n} P^*(z)$; consequently $|P^*|^2 = |P|^2$.

A $(0, \pm 1)$-sequence of length $n, S = (s_k) = (s_1, \ldots, s_n)$ is symmetric if $s_k^* = s_k$ for each k, (i.e., $S^* = S$); it is skew if $s_k^* = -s_k$ for each k, (i.e., $S^* = -S$); and it is said to be *quasi-symmetric* if $s_k^* = e_k s_k$ for each k, where $e_k = 1$ or -1, i.e., zeros appear symmetrically in S. Consequenlty symmetric or skew sequences are quasi-symmetric. Two $(0, \pm 1)$-sequences of length $n, G = (g_k)$ and $H = (h_k)$ are said to be *supplementary* if $G + H = (g_k + h_k)$ is a $(1, -1)$-sequence; i.e., $|g_k| = 1$ and $h_k = 0$, or $g_k = 0$ and $|h_k| = 1$, for each k. We note here that Q, R and S, T are supplementary in a regular δ-code: $[Q, R; S, T]$; i.e., $G + H$ is a $(1, -1)$-sequence for $G = Q$ or R, and $H = S$ or T.

Definition 1. A triple $(F; G, H)$ of sequences is said to be a set of *normal sequences for length n* (abbreviated as $NS(n)$) if the following two conditions are satisfied.

(i) $F = (f_k)$ is a $(1, -1)$-sequence of length n; $G = (g_k)$ and $H = (h_k)$ are quasi-symmetric supplementary $(0, \pm 1)$-sequences of length n; i.e., $G + H = (g_k + h_k)$ is a $(1, -1)$-sequence and zeros appear symmetrically in G and H.

(ii) $f(j) + g(j) + h(j) = 0$ for $j \neq 0$, i.e., they have zero auto-correlation sum.

Condition (ii) is also equivalent to

(iii) $|F|^2 + |G|^2 + |H|^2 = 2n$ for any z on $\mathbf{K}$.

It is known that if $J = L + M$ is a sequence of real numbers with symmetric L and skew M, then $|J|^2 = |L|^2 + |M|^2$ for any z on $\mathbf{K}$. From a given pair of Golay sequences $GCS(g): (\mathbf{F}, \mathbf{H})$, we can obtain trivially two sets of normal sequences, $(\mathbf{F}; \mathbf{H}, 0_g)$ and $(\mathbf{F}; L, M)$, where L and M are respectively the symmetric and skew parts of $\mathbf{H}$. Therefore, a set of normal sequences can be regarded as a generalization of a pair of Golay complementary sequences. We can also obtain sets of normal sequences $NS(2m+1): (A/C; B/0_m, 0_{m+1}/D)$ from *Turyn base sequences* $TBS(2m+1): (A, B; C, D)$ *with lengths* $m+1$ *and* m *pairs, for even* m *with symmetric* A *and skew* C, *and for odd* m *with skew* A *and symmetric* C. It is known that $TBS(2m+1)$ with such properties exists for $m \leq 7$, $m = 14$ [T1] and $m = 12$ [GS]. For example, from $TBS(3): (A, B; C, D) = (1-, 11; 1, 1)$, where $-$ stands for -1, we obtain $NS(3): F = A/C = 11-$; $G = B/0 = 101$, $H = 0_2/D = 010$, where $m = 1$ is odd, $A = 1-$ is skew and $C = 1$ is symmetric. Also from $TBS(5): (111, 11-; 1-, 1-)$, we obtain $NS(5): F = A/C = 111-1$; $G = B/0_2 = 1010-$, $H = 0_3/D = 010-0$, where $m = 2$ is even, $A = 111$ is symmetric and $C = 1-$ is skew. Also from $GCS(4): (\mathbf{F}, \mathbf{H}) = (111-, 11-1)$, we get $(\mathbf{F}; L, M) = (111-; 1001, 01-0)$, where $L = 1001$ and $M = 01-0$ are respectively symmetric and skew parts of $\mathbf{H}$.

The following are examples of $NS(n): (F; G, H)$, which are unobtainable from TBS and GCS.

$n = 7$: $1-111--$; $11010-1$, 0010100 and $11--1-1$; $101110-$, 0100010, as $F; G, H$.

$n = 9$: $1-11111--$; $11-0101-1$, $00010-000$.

$n = 11$: $1111-1--11-$; $1100-0100-1$, $00110101-00$.

$n = 12$: $111---1--1--$; $111000000-11$, $00011-1-1000$.

$n = 13$: $111----1---1-$; $111-0000011-1$, $00001-11-0000$.

Definition 2. A quadruple $(E, F; G, H)$ of $(0, \pm 1)$-sequences is said to be a set of *near normal sequences for length* $n = 4m + 1$ (abbreviated as $NN(n)$) if the following conditions are satisfied.

(i) $E = (X/0, 1)$, $F = (Y/0)$, where X and Y are $(1, -1)$-sequences of length m and $0 = 0_{m-1}$, the sequence of zeros of length $m - 1$, i.e., E and F are respectively of lengths $2m$ and $2m - 1$; G and H are quasi-symmetric supplementary $(0, \pm 1)$-sequences of length $2m$, i.e., $G + H$ is a $(1, -1)$-sequence of length $2m$ and zeros appear symmetrically in G and H.

(ii) $e(j) + f(j) + g(j) + h(j) = 0$ for $j \neq 0$, i.e., they have zero autocorrelation sum. Condition (ii) is also equivalent to

(iii) $|E|^2 + |F|^2 + |G|^2 + |H|^2 = 4m + 1$ for any z on $\mathbf{K}$.

The following are examples of sets of near normal sequences $NN(n)$: $(E, F; G, H)$. The cases for $n \geq 21$ were found by a computer search. (All $NN(n)$ for $n \leq 37$ have been found and classified in [Y5].

$n = 5$: $11, 1; 1-, 00$, as $E, F; G, H$, where $-$ stands for -1.

$n = 9$: $-011, 101; 1001, 01-0$.

$n = 13$: $10-011, 10101; 11-1--, 0_6$.

$n = 17$: $-0-01011, 10-010-; 111-1-11, 0_8$.

$n = 21$: $1010-01011, 101010-0-; 10101-010-, 010-0010-0$.

$n = 25$: $10101010-011, 1010-0-0-01; 110_8--, 001--11-1-00$.

$n = 29$: $10101010-010-1, 1010-0101010-; 1-0_{10}1-, 0011-1111---00$.

$n = 33$: $-0-010101010-011, 101010-010-0101; 1001--01-0---001, 01-0001001000-10$.

$n = 37$: $-010101010101010-1, 1010-0-01010-010-; 1101---001--110-1, 0010_5 1-0_5-00$.

$n = 41$: $1010-0-010101010-011, -0101010101010-010-; 110-10--1-1--10110--, 0010010_8-00-00$.

$n = 45$: $10-0-010-010-010101011, -01010-01010-0-010101; -0010_5 10010_5-001, 0-101-1--01101111-01-0$.

We note that we can obtain base sequences $BS(2n+1)$: $(A, B; C, D)$ with lengths $n+1$, n pairs, from a set of normal sequences $NS(n)$: $(F; G, H)$ as follows: $A = (F, 1), B = (F, -1)$; $C = G+H$, $D = G-H$. And $BS(n)$ from a set of near normal sequences $NN(n)$: $(E, F; G, H)$, where $E = (X/0, 1)$ and $F = (Y/0)$, as follows: $A = F/E = (Y/X, 1)$, $B = (-F)/E = ((-Y)/X, 1)$ $C = G + H$, $D = G - H$. For example, from the above $NS(12)$, we obtain the following new construction for $BS(25)$: $A = 111---1--1--1$, $B = 111---1--1---$; $C = 11111-1-1-11$, $D = 111--1-1--11$. Also from the above $NN(n)$, we obtain the following new construction for $BS(n)$, $n = 29, 37$ and 45.

$n = 29$: 1111−1111−11−−1, −1−111−1−−−11−1; 1−11−1111−−−1−, 1−−−1−−−−1111−, as $(A, B; C, D)$.

$n = 37$: 1−11−1−11111−111−−1, −−−11111−1−111−11−1; 1111−−−1−1−−11, −−1, 11−1−−−−−−11−−111−1.

$n = 45$: −11−1−−11−11−−−11111111, 11−−−−11−−−11−11−1−1−11; −−111−1−−11111111−−1−1, −1−1−1−111−−1−−−−1−−11.

(For $n = 37$ also see [K].) The only unpublished among $BS(p)$ for all primes $p \leq 59$ is $p = 43$, so we give these as follows. $p = 43$: 11−1−1−−−−−−−−1−−11−1−−, 11−−−−1−1−11−−−1−111−1; 1−11−111111−−−11−−111, 11−1−1111−−1−−11−−−−1. We also give another previously unpublished case, $BS(39)$ as follows: 11−1−1−111−−1−−1−1−−, 11−−−−1−−−11−11111−1; 1−11−11−−111−1−1111, 11−−−−−−111−−1−−−−−−1.

3. General results

For given $BS(t)\colon (A, B; C, D)$ with lengths $m+p$ and m pairs, $t = 2m+p$, and $NS(n)\colon (F; G, H)$ with $F = (f_k)$ and $G + H = (g_k + h_k) = (j_k)$, let

(i*) when $j_k = g_k\colon \alpha j_k = Ag_k^*$, $\beta j_k = Bg_k$, $\gamma j_k = Cg_k$, $\delta j_k = Dg_k^*$;

(ii*) when $j_k = h_k\colon \alpha j_k = -Bh_k$, $\beta j_k = Ah_k^*$, $\gamma j_k = Dh_k$, $\delta j_k = -Ch_k^*$.

We also define five $2 \times t$ matrices as follows.

$$(*)\qquad \begin{gathered} \sigma_k = \begin{pmatrix} Af_k^*, & \gamma j_k \\ Bf_k^*, & \delta j_k \end{pmatrix}, \quad \tau_k = \begin{pmatrix} \alpha j_k, & -Cf_k^* \\ \beta j_k, & -Df_k^* \end{pmatrix}, \\ 0 = \begin{pmatrix} \mathbf{0}, \underline{0} \\ \mathbf{0}, \underline{0} \end{pmatrix}, \quad \varepsilon_1 = \begin{pmatrix} -B^*, & \underline{0} \\ A^*, & \underline{0} \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} \mathbf{0}, & -D^* \\ \mathbf{0}, & C^* \end{pmatrix}, \end{gathered}$$

where $\underline{0} = 0_m$ and $\mathbf{0} = 0_{m+p}$.

Theorem 1. *Let $(A, B; C, D)$ be base sequences $BS(t)$ with lengths $m+p$ and m pairs, $t = 2m+p$, and $(F; G, H)$ be normal sequences $NS(n)$. Then the following $4 \times (st)$ matrix $[Q, R; S, T]$ is a regular δ-code $RD(st)$, $s = 2n+1$, where $[Q, R]$ and $[S, T]$ are column vectors, i.e., $2 \times (st)$ matrices.*

$$(\mathrm{I})\qquad \begin{aligned} [Q, R] &= [(q_k), (r_k)] = (\sigma_1, 0, \sigma_2, 0, \ldots, \sigma_n, 0, \varepsilon_1), \\ [S, T] &= [(s_k), (t_k)] = (0, \tau_1, 0, \tau_2, \ldots, 0, \tau_n, \varepsilon_2), \end{aligned}$$

where $\sigma_k, \tau_k, \varepsilon_k$ and 0 are defined in $(*)$.

Proof. In (I), each column vector $[q_k, r_k, s_k, t_k]$ is obviously one of orthogonal vectors i_h or their negatives, $1 \leq h \leq 4$. In (L), let $a = A(z)$, $b = B(z)$, $c = C(z)z^M$, and $d = D(z)z^M$. Also let $f = F(z^{2t})z^{-x}$, $g = G(z^{2t})z^{-x}$, $h = H(z^{2t})z^{-x}$ and $e = z^y$, where $t = 2m+p$, $M = m+p$, $x = (n-1)t$ and $y = (n+1)t + M - 1$. Then we have $Q = \mathbf{q}z^x$, $R = \mathbf{r}z^x$, $S = \mathbf{s}z^{x+t}$ and $T = \mathbf{t}z^{x+t}$.

Consequently we obtain from Theorem L, $|Q|^2 + |R|^2 + |S|^2 + |T|^2 = |\mathbf{q}|^2 + |\mathbf{r}|^2 + |\mathbf{s}|^2 + |\mathbf{t}|^2 = (|A|^2 + |B|^2 + |C|^2 + |D|^2)(1^2 + |F|^2 + |G|^2 + |H|^2) = 2st$ for any z on $\mathbf{K}$. Thus the theorem is proved.

We note here that $p = p(z) = P(z^{2t})z^{-x}$ is symmetrized, i.e., $p' = p(z^{-1}) = P^*(z^{2t})z^{-x}$ for $p = f, g$ and h. And in $\mathbf{q}, \mathbf{r}, \mathbf{s}$ and $\mathbf{t}$, the quasi-symmetry of sequence $P = (p_k)$ is required for $P = F$, G and H, since p_k and p_k^* of the reverse P^* determine the kth block of length $2t$ and align its nonzero and zero parts. Similarly G and H must be supplementary. Therefore quasi-symmetry and supplementary are essential for the construction of a δ-code in Theorem 1.

A set of four $(1, -1)$-sequences (U, W, X, Y) of length m is said to be *complementary*, if $u(j) + w(j) + x(j) + y(j) = 0$ for $j \neq 0$, or equivalently, if $|U|^2 + |W|^2 + |X|^2 + |Y|^2 = 4m$ for any z on $\mathbf{K}$. The sequences U, W, X and Y can be regarded as respectively the first rows of circulant matrices A, B, C and D of order m satisfying condition (5), i.e., $A^tA + B^tB + C^tC + D^tD = 4mI_m$, thus a Goethals-Seidel (Hadamard) matrix of order $4m$ can be constructed. Also we note here that the above $(U, W; X, Y)$ can be regarded as $BS(2m)$ with lengths m and m pairs.

Theorem 2. *Let $BS(t)$ and $NS(n)$ be given as in Theorem* 1. *Then the following (Q, R, S, T) is a set of four complementary sequences of length nt (i.e., $BS(2nt)$ with lengths nt and nt pairs).*

(II) $$[Q, R] = (\sigma_1, \sigma_2, \dots, \sigma_n); \qquad [S, T] = (\tau_1, \tau_2, \dots, \tau_n),$$

where σ_k and τ_k are defined in $(*)$.

The proof of Theorem 2 is similar to that of Theorem 1, therefore we omit it. We note here that z^t should be used instead of z^{2t} in F, G and H; and $e = 0$, consequently there are no A^*, B^*, C^* and D^* in (II). Also there are no shifting for S and T, i.e., $S = \mathbf{s}z^x$ and $T = \mathbf{t}z^x$, $x = (n-1)t/2$.

Let the four components of a regular δ-code $RD(u)$: $[\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}]$ be

(II*) $$\mathbf{A} = (\underline{A}, \mathbf{0}), \quad \mathbf{B} = (\underline{B}, \mathbf{0}); \quad \mathbf{C} = (\mathbf{0}, \underline{C}) \quad \text{and} \quad \mathbf{D} = (\mathbf{0}, \underline{D}),$$

where $\underline{P}$ is the part of $\mathbf{P} = (p_k)$ in which $p_k \neq 0$ and $\mathbf{0}$ is that in which $p_k = 0$, for $\mathbf{P} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. Thus $\mathbf{X} + \mathbf{Y} = (\underline{X}, \underline{Y})$ for $\mathbf{X} \in \{\pm\mathbf{A}, \pm\mathbf{B}\}$ and $\mathbf{Y} \in \{\pm\mathbf{C}, \pm\mathbf{D}\}$, since $\mathbf{X}$ and $\mathbf{Y}$ are supplementary.

For example, in $RD(3t)$: $\mathbf{A} = (A, C; 0, 0; -B^*, 0)$, $\mathbf{B} = (B, D; 0, 0; A^*, 0)$; $\mathbf{C} = (0, 0; A, -C; 0, -D^*)$ and $\mathbf{D} = (0, 0; B, -D; 0, C^*)$, we have $(\mathbf{A}, \mathbf{C}) = \mathbf{A} + \mathbf{C} = (A, C; A, -C; -B^*, -D^*) = (\underline{A}, \underline{C})$, etc.

By observing that in Theorem 2 our argument is still valid if we replace $BS(t)$: $(A, B; C, D)$ by $RD(u)$ of (II*), i.e., by replacing P of BS with $\underline{P}$ of RD for $\mathbf{P} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ and t with u.

Consequently we have the following.

Theorem 2*. *Let* $[\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}]$ *be a regular δ-code $RD(u)$ of* (II*), *then (Q, R, S, T) of* (II) *is a set of four complementary sequences of length nu (or equivalently $BS(2nu)$).*

We note here that, from condition (iii) of Definition 1, a set of normal sequences $NS(2m)$ exists only if $4m$ is the sum of three squares of even integers, consequently m must be the sum of three squares of integers. Therefore

$NS(2m)$ does not exist for $m \equiv 7 \pmod 8$. It is known that $NS(6)$ does not exist. Although $RD(nt)$ cannot be composed by Theorem 1 for $n = 2k+1$, $k = 6$ or 14 (since $NS(k)$ does not exist for these cases), $RD(nt)$ can be constructed for $n = 4m+1$, which includes $13, 29, \ldots$, by the following theorem with a set of near normal sequences $NN(n)$.

Theorem 3. *Let $(A, B; C, D)$ be a set of base sequences $BS(t)$ with lengths $s+p$ and s pairs, and $(E, F; G, H)$ be a set of near normal sequences $NN(n)$ of Definition 2, where $n = 4m+1$ and $t = 2s+p$. Then the following $4 \times (nt)$ matrix $[Q, R; S, T]$ is a regular δ-code $RD(nt)$.*

$$\text{(III)} \qquad \begin{aligned} [Q, R] &= (\lambda_1, \mu_1, \ldots, \lambda_m, \mu_m, \varepsilon_2, 0, 0, \ldots, 0, 0), \\ [S, T] &= (0, 0, \ldots, 0, 0, \varepsilon_3, \nu_m, \pi_m, \ldots, \nu_1, \pi_1), \end{aligned}$$

where

$$\lambda_k = \begin{pmatrix} \alpha j_{2k-1}, & -Cy_k \\ \beta j_{2k-1}, & -Dy_k \end{pmatrix}, \qquad \mu_k = \begin{pmatrix} \alpha j_{2k}, & -D^* x_k \\ \beta j_{2k}, & C^* x_k \end{pmatrix},$$

$$\nu_k = \begin{pmatrix} -Bx_k, & \gamma^* j_{2k} \\ Ax_k, & \delta^* j_{2k} \end{pmatrix}, \qquad \pi_k = \begin{pmatrix} A^* y_k^*, & \gamma^* j_{2k-1} \\ B^* y_k^*, & \delta^* j_{2k-1} \end{pmatrix},$$

$$\varepsilon_2 = \begin{pmatrix} \mathbf{0}, & -D^* \\ \mathbf{0}, & C^* \end{pmatrix} \quad \text{and} \quad \varepsilon_3 = \begin{pmatrix} -B, & \underline{0} \\ A, & \underline{0} \end{pmatrix}.$$

Also ξj_i, for $\xi = \alpha$, β, γ and δ, are defined as in (i*) *and* (ii*), *where $\xi^* j_i = (\xi j_i)^*$ for $1 \le i \le 2m$, and $y_k^* = y_{m+1-k}$ for $1 \le k \le m$; we note that 0 and ε_2 are similar to those defined in $(*)$, i.e., $\mathbf{0} = 0_{s+p}$ and $\underline{0} = 0_s$.*

Proof. Obviously each column vector in (III) is one of orthogonal vectors or their negatives $\pm i_k$. In (L), let $p = P(z)$ for $p = a, b, c$ and d. Also let $f = F(z^t)z^{-u}$, $e = E(z^t)z^{-w}$, $g = G(z^t)z^{-x}$ and $h = H(z^t)z^{-x}$, where $t = 2s+p$, $u = (m-1)t - (p/2)$, $w = (2m-5)t/2 + 1$, and $x = (2m-1)t/2$. Then $Q = \mathbf{s}z^x$, $R = \mathbf{t}z^x$, $S = \mathbf{q}'z^y$ and $T = \mathbf{r}'z^y$, where $y = 3(2m+1)t/2 - 1$. Consequently, $|Q|^2 + |R|^2 + |S|^2 + |T|^2 = |\mathbf{s}|^2 + |\mathbf{t}|^2 + |\mathbf{q}|^2 + |\mathbf{r}|^2 = (|A|^2 + |B|^2 + |C|^2 + |D|^2)(|E|^2 + |F|^2 + |G|^2 + |H|^2) = 2nt$ for any z on $\mathbf{K}$.

We note here that because of the quasi-symmetry of sequences F, G and H, in $\mathbf{q}, \mathbf{r}, \mathbf{s}$ and $\mathbf{t}$, the first $2m$ nonzero blocks of length t are aligned and in the $(2m+1)$th (last) block, the nonzero (or zero) part in $\mathbf{q}, \mathbf{r}$ and the zero (or nonzero) part in $\mathbf{s}, \mathbf{t}$ are of the same length, thus $\mathbf{q}', \mathbf{r}'$ and $\mathbf{s}$, $\mathbf{t}$ can be fitted to from a δ-code.

From given two sets of base sequences with lengths $m+1$, m pairs and $n+1$, n pairs, we can also construct four complementary $(1, -1)$-sequences of length $(2m+1)(2n+1)$ as follows.

Theorem 4. *Let $(A, B; C, D)$ and $(F, G; H, E)$ be two sets of base sequences respectively with lengths $m+1$, m pairs and $n+1$, n pairs. Then the following (Q, R, S, T) are four complementary $(1, -1)$-sequences of length $(2m+1)(2n+1)$.*

$$[Q, R] = (\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n, \alpha_{n+1}),$$
$$[S, T] = (\gamma_1, \delta_1, \gamma_2, \delta_2, \ldots, \gamma_n, \delta_n, \gamma_{n+1}),$$

where

$$\alpha_k = \begin{pmatrix} Af_k^*/Cg_k \\ Bf_k^*/Dg_k^* \end{pmatrix}, \quad \beta_k = \begin{pmatrix} -B^*e_k/Dh_k \\ Ae_k/-Ch_k^* \end{pmatrix},$$
$$\gamma_k = \begin{pmatrix} Ag_k^*/-Cf_k \\ Bg_k/-Df_k \end{pmatrix} \quad \text{and} \quad \delta_k = \begin{pmatrix} -Bh_k/-D^*e_k \\ Ah_k^*/C^*e_k \end{pmatrix}.$$

Proof. Obviously (Q, R, S, T) are four $(1, -1)$-sequences of length $(2m+1)(2n+1)$. By letting $a = A(z^2)$, $b = B(z^2)$, $c = zC(z^2)$ and $d = zD(z^2)$, also $f = z^{-nM}F(z^{2M})$, $g = z^{-nM}G(z^{2M})$, $h = z^x H(z^{2M})$ and $e = z^y E(z^{2M})$, where $M = 2m+1$, $x = (1-n)M$ and $y = 2m + (1-n)M$, in (L), and by observing that $Q = \mathbf{q}w$, $R = \mathbf{r}w$, $S = \mathbf{s}w$ and $T = \mathbf{t}w$, where $w = z^{nM}$, we obtain

$$\begin{aligned} |Q|^2 + |R|^2 + |S|^2 + |T|^2 &= |\mathbf{s}|^2 + |\mathbf{t}|^2 + |\mathbf{q}|^2 + |\mathbf{r}|^2 \\ &= (|A|^2 + |Bq|^2 + |C|^2 + |D|^2)(|E|^2 + |F|^2 + |G|^2 + |H|^2) \\ &= 4(2m+1)(2n+1) \quad \text{for any } z \text{ on } \mathbf{K}. \end{aligned}$$

$BS(t)$ with $m+1$ and m pairs $(t = 2m+1)$ exist for $t \le 59$ (except 49, 57) and $t = 2g+1$ [T1,T2].

From given sets of base sequences $BS(t){:}\,(A, B; C, D)$ and normal sequences $NS(m)$ (or near normal sequences $NN(n)$), we can construct a regular δ-code $RD((2m+1)t)$ by Theorem 1 (or $RD(nt)$ by Theorem 3). I.e., we can construct $RD(ut)$, where $u \in \mathbf{U} =$ (the set of $2m+1$ and n such that $NS(m)$ or $NN(n)$ exists) $= \{n{:}\, n \le 33, n = 37, 41, 45, 51, 53, 59, 65, 81, \ldots, \text{ and } n = 2g+1\}$.

Thus, we can construct $BS(2mut)$ by Theorem 2^* with $NS(m)$ and $RD(ut)$, where $u \in \mathbf{U}$ and $m \in \mathbf{M} =$ (the set of all m such that $NS(m)$ exists) $= \{n{:}\, \text{odd } n \le 15, n = 25, 29; n = 12 \text{ or } n = g\}$. We note also that from given $RD(ut)$ and Golay sequences $GCS(g)$, four complementary $(1, -1)$-sequences of length $(2g+1)ut$, equivalently $BS(2(2g+1)ut)$ can be constructed [Y2].

Consequently from the above, we can obtain $BS(t_1)$, with $t_1 = 2s_1 t$, $s_1 \in \mathbf{S}$, where

(IV) $$\mathbf{S} = \{mu{:}\, m \in \mathbf{M} \text{ or } m = 2g+1, u \in \mathbf{U}\},$$

which contains all positive integers ≤ 100 (except 43, 47 and primes ≥ 61).

By repeating the above process, we obtain $BS(t_2)$, where $t_2 = 2s_2 t_1 = 2^2 s_1 s_2 t$, $s_2 \in \mathbf{S}$.

After repeating the process n times, we have the following:

Theorem 5. *Four-symbol δ-codes of length r exist for $r = 2^n t \prod_j s_j$ $(1 \le j \le n)$, where $s_j \in \mathbf{S}$, $t \in T = \{k: k \le 59$ (except 49, 57) or $k = g + g'\}$, g and g' are Golay numbers; and $\mathbf{S}$ is defined in* (IV).

We note that instead of the above, s_1 and t may belong to $T_0 = \{h: h \le 59$ (except 49, 57) or $h = 2g+1\}$, since $BS(2s_1 t)$ can be constructed by Theorem 4; also $k = g + g'$ includes odd $g+1$, when $g' = 1$.

4. Examples and remark

From given $BS(t): (A, B; C, D)$ (or $RD(u)$), and $NS(3): (-11; 101, 010)$, we obtain the following $[Q, R; S, T]$ as $RD(7t)$ from Theorem 1 and as four complementary sequences of length $3t$ (or $3u$) from Theorem 2 (or Theorem 2^*) respectively.

$$\begin{aligned}
Q &= (A, C; 0, 0; A, D; 0, 0; -A, C; 0, 0; -B^*, 0)\\
R &= (B, D; 0, 0; B, -C; 0, 0; -B, D; 0, 0; A^*, 0)\\
S &= (0, 0; A, C; 0, 0; -B, -C; 0, 0; A, -C; 0, -D^*)\\
T &= (0, 0; B, D; 0, 0; A, -D; 0, 0; B, -D; 0, C^*),
\end{aligned}$$

and

$$\begin{aligned}
Q &= (A, C; A, D; -A, C)\\
R &= (B, D; B, -C; -B, D)\\
S &= (A, C; -B, -C; A, -C)\\
T &= (B, D; A, -D; B - D).
\end{aligned}$$

Similarly from $NS(5): (111-1; 1010-, 010-0)$, we obtain the following $RD(11t)$.

$$\begin{aligned}
Q &= (A, C; 0, 0; -A, D; 0, 0; A, C; 0, 0; A, -D; 0, 0; A, -C; 0, 0; -B^*, 0),\\
R &= (B, -D; 0, 0; -B, C; 0, 0; B, D; 0, 0; B, -C; 0, 0; B, D; 0, 0; A^*, 0),\\
S &= (0, 0; -A, -C; 0, 0; -B, -C; 0, 0; A, -C; 0, 0; B, C; 0, 0; A, -C;\\
&\quad 0, -D^*)\\
T &= (0, 0; B, -D; 0, 0; -A, -D; 0, 0; B, -D; 0, 0; A, D; 0, 0; -B, -D;\\
&\quad 0, C^*).
\end{aligned}$$

Also from given Golay sequences $GCS(g): (F, H)$ regarded as $NS(g)$: $(F: H, 0_g)$, we obtain the following $RD((2g+1)t)$ from Theorem 1, i.e., $2g + 1 = 3, 5, 9, 17, 21, 33, 41, 53, 65, 81, \cdots$.

$$\begin{aligned}
Q &= (Af_1^*, Ch_1; 0, 0; Af_2^*, Ch_2; 0, 0; \cdots; Af_g^*; Ch_g; 0, 0; -B^*, 0),\\
R &= (Bf_1^*, Dh_1^*; 0, 0; Bf_2^*, Dh_2^*; 0, 0; \cdots; Bf_g^*, Dh_g^*; 0, 0; A^*, 0),\\
S &= (0, 0; Ah_1^*, -Cf_1; 0, 0; Ah_2^*, -Cf_2; 0, 0; \cdots; Ah_g^*, -Cf_g; 0, -D^*),\\
T &= (0, 0; Bh_1, -Df_1; 0, 0; Bh_2, -Df_2; 0, 0; \cdots; Bh_g, -Df_g; 0, C^*).
\end{aligned}$$

Thus Theorem 1 is a generalization of [Y1, Y2, Theorem 2] and we can construct $RD(ut)$ for $u = 2n + 1 = 15, 19, 23, 27, 31, 51$ and 59, which are not published before. Although sets of $NS(n)$ have not been found for $n \geq 17$ (except $n = 25, 29$ and g), we can construct $RD((2n+1)t)$ for any n, by Theorem 1, if a new set of $NS(n)$ can be found in the future.

Theorem 3 can be regarded as a generalization of [Y2, Theorem 3]. For Example, from $NN(5): (11, 1; 1-, 00)$ and $NN(13): (10-011, 10101; 11-1--\}$, we obtain respectively the following $RD(5t)$ and $RD(13t)$.

$$\begin{aligned}
Q &= (-A, -C; A, -D^*; 0, -D^*; 0, 0; 0, 0),\\
R &= (0, 0; 0, 0; -B, 0; -B, -C^*; A^*, C^*),\\
S &= (B, -D; -B, C^*; 0, C^*; 0, 0; 0, 0),\\
T &= (0, 0; 0, 0; A, 0; A, D^*; B^*, -D^*);
\end{aligned}$$

and

$$\begin{aligned}
Q = {} & (-A, -C; -A, -D^*; A, -C; -A, D^*; A, -C; A, -D^*; 0, -D^*;\\
& 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0),\\
R = {} & (B, -D; B, C^*; -B, -D; B, -C^*; -B, -D; -B, C^*; 0, C^*; 0, 0;\\
& 0, 0; 0, 0; 0, 0; 0, 0; 0, 0),\\
S = {} & (0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; -B, 0; -B, -C^*; A^*, -C^*; B, C^*;\\
& A^*, -C^*; -B, C^*; A^*, C^*)\\
T = {} & (0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; A, 0; A, D^*; B^*, D^*; -A, -D^*; B^*, D^*;\\
& A, -D^*; B^*, -D^*).
\end{aligned}$$

We can also construct regular δ-codes $RD(nt)$ by Theorem 3 in a similar way for $n = 9, 17, 21, 25, 29, 33, 37, 41, 45$ and any n for which a set of $NN(n)$ exists.

Finally we remark that in order to compose 4-symbol δ-codes successfully by application of the Lagrange identity, when $e \neq 0$ all terms $a, b, \ldots, h$ have to be quasi-symmetric except possibly e, since e' is not involved in (L) as in Theorems 1, 3 and 4; and when $e = 0$ f, g and h have to be quasi-symmetric but not a, b, c and d, since there are no a', b', c' and d' in (L) as in Theorems 2 and 2^*.

References

[A] S. S. Agaian, *Hadamard matrices and their applications*, Springer-Verlag, Berlin, 1985.

[GS] A. V. Geramita and J. Seberry, *Orthogonal designs*, Dekker, New York, 1979.

[Go] J. M. Goethals and J. J. Seidel, *A skew Hadamard matrix of order* 36, J. Aust. Math. Soc. **11** (1970), 343–344.

[G] M. J. E. Golay, *Complementary series*, IRE Trans. Information Theory, IT-7 (1961), 82–87.

[H.] M. Hall, Jr., *Combinatorial theory*, 2nd ed., Wiley and Sons, New York, 1986.

[HS] M. Harwit and N. J. A. Sloane, *Hadamard transform optics* , Academic Press, New York, 1979.

[HW] A. Hedayat and W. D. Wallis, *Hadamard matrices and their applications*, Ann. Math. Stat. **6** (1978), 1184–1238.

[K] C. Koukouvinos, S. Kounias and J. Seberry, *Further results on base sequences, disjoint complementary sequences, $OD(4t; t, t, t)$ and the excess of Hadamard matrices*, (to appear).

[M] A. C. Mukhopadyay, *Some infinite classes of Hadamard matrices*, J. Combin. Theory (A) **25** (1978), 128–141.

[S] E. Spence, *An infinite family of Williamson matrices*, J. Aust. Math. Soc. (A) **24** (1977), 252–256.

[T1] R. J. Turyn, *Hadamard matrices, Baumert-Hall units, four symbol sequences, pulse compression, and surface wave encodings*, J. Combin. Theory (A) **16** (1974), 313–333.

[T2] ——, Personal communication, 1980.

[T3] ——, *An infinite class of Williamson matrices*, J. Combin. Theory (A) **12** (1972), 319–321.

[W] J. Seberry Wallis, *Construction of Williamson type matrices*, J. Linear and Multilinear Alg. **3** (1975), 197–207.

[Wi] J. Williamson, *Hadamard's determinant theorem and the sum of four squares*, Duke Math. J. **11** (1944), 65–81.

[Y1] C. H. Yang, *A composition theorem for δ-codes*, Proc. Amer. Math. Soc. **89** (1983), 375–378.

[Y2] ——, *Lagrange identity for polynomials and δ-codes of lengths $7t$ and $13t$*, Proc. Amer. Math. Soc. **88** (1983), 746–750.

[Y3] ——, *Hadamard matrices and δ-codes of length $3n$*, Proc. Amer. Math. Soc. **85** (1982),

[Y4] ——, *Hadamard matrices, finite sequences, and polynomials defined on the unit circle*, Math. Comp. **33** (1979), 688–693.

[Y5] ——, *On Golay sequences and near normal sequences*, (to appear).

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