On Computing Backbones of Propositional Theories

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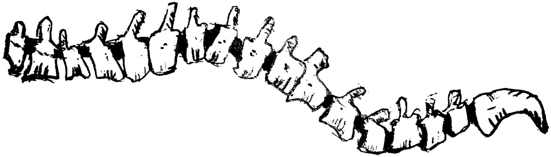

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$\phi \models x_j$ $\quad$ $\phi \models \neg x_k$

Motivation

- backbones tell us more about the formula, e.g.
  - upper bound for number of models

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Motivation

- Can we compute backbones for large instances?
- How many backbone literals do real-world instances have?

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$$\texttt{SAT}(x \wedge \neg x) = (\text{false}, -)$$

Model Enumeration

Input : CNF formula φ
Output: Backbone of φ, ν_R

$\nu_R \leftarrow \{\neg x, x \mid x \in X\}$ // Initial backbone estimate
repeat
  $(\text{outc}, \nu) \leftarrow \texttt{SAT}(\varphi)$ // SAT solver call
  if outc $=$ false **then**
    return ν_R // Terminate if unsatisfiable
  $\nu_R \leftarrow \nu_R \cap \nu$ // Update backbone estimate
  $\omega_B \leftarrow \texttt{BlockClause}(\nu)$ // Block model
  $\varphi \leftarrow \varphi \cup \omega_B$
until $\nu_R = \emptyset$
return $\emptyset$

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Iterative SAT Testing

Input : CNF formula φ, with variables X
Output: Backbone of φ, ν_R

```
ν_R ← ∅
foreach l ∈ {¬x, x | x ∈ X} do
    (outc, ν) ← SAT(φ ∪ {l̄})
    if outc = false then
        ν_R ← ν_R ∪ {l}          // l is backbone
        φ ← φ ∪ {l}
return ν_R
```

Iterative SAT Testing

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$\nu_R \leftarrow \emptyset$
foreach $l \in \{\neg x, x \mid x \in X\}$ **do**
  $(\text{outc}, \nu) \leftarrow \texttt{SAT}(\varphi \cup \{\bar{l}\})$
  if outc = false **then**
    $\nu_R \leftarrow \nu_R \cup \{l\}$ // l is backbone
    $\varphi \leftarrow \varphi \cup \{l\}$
return ν_R

- SAT is called twice per variable

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$\phi \not\models x_i$

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- Implicant $\nu = l_1 \wedge \ldots \wedge l_k$ is a conjunction of literals such that

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- Implicants are like "models with wild cards"

$$\nu \quad * \quad * \quad * \quad *$$

Implicants and CNF

- Any model of a formula is an implicant

$$x \wedge y \wedge \neg z \quad \Rightarrow \quad \begin{array}{ccccc} \mathbf{x} & \vee & \mathbf{y} & \vee & z \\ \mathbf{x} & \vee & \neg y & \vee & \neg \mathbf{z} \\ \mathbf{x} & \vee & \mathbf{y} & \vee & \neg \mathbf{z} \end{array}$$

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- A model can be reduced

$$y \wedge \neg z \quad \Rightarrow \quad \begin{array}{ccccc} x & \vee & \mathbf{y} & \vee & z \\ x & \vee & \neg y & \vee & \mathbf{\neg z} \\ x & \vee & \mathbf{y} & \vee & \mathbf{\neg z} \end{array}$$

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- Multiple reductions may exist

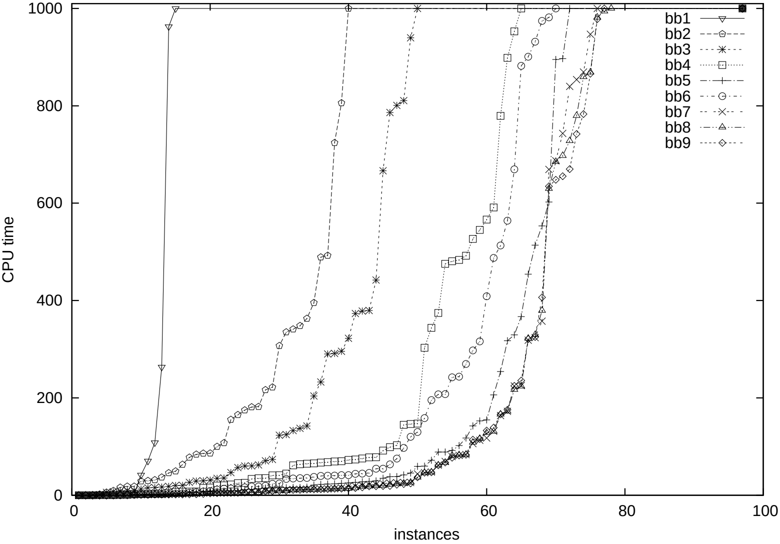
$$x \;\Rightarrow\; \begin{array}{ccccc} \mathbf{x} & \vee & y & \vee & z \\ \mathbf{x} & \vee & \neg y & \vee & \neg z \\ \mathbf{x} & \vee & y & \vee & \neg z \end{array}$$

Improving Iterative Testing

Input : CNF formula φ, with variables X
Output: Backbone of φ, ν_R

$\Lambda \leftarrow \{x, \neg x \mid x \in X\}$ `// candidates for backbone`
$\nu_R \leftarrow \emptyset$ `// initial backbone estimate`
foreach $l \in \Lambda$ **do**
  $(\text{outc}, \nu) \leftarrow \texttt{SAT}(\varphi \cup \{\bar{l}\})$
  if outc = false **then**
    $\nu_R \leftarrow \nu_R \cup \{l\}$ `// Backbone identified`
    $\varphi \leftarrow \varphi \cup \{l\}$
  else
    $\nu \leftarrow \texttt{ReduceModel}(\nu)$ `// Simplify model`
    $\Lambda \leftarrow \Lambda \cap \nu$
return ν_R

Results



CPU time
instances
bb1
bb2
bb3
bb4
bb5
bb6
bb7
bb8
bb9
0
200
400
600
800
1000
0
20
40
60
80
100

Results

Instance	#vars	%bb	best time [s]
narain-sat07-clauses-2	75528	89.3	869.3
2dlx_cc_mc_ex_bp_f2_bug001	4821	36.6	14.8
2dlx_cc_mc_ex_bp_f2_bug005	4824	44.7	17.9
2dlx_cc_mc_ex_bp_f2_bug009	4824	34.8	12.1
grieu-vmpc-s05-25	625	100.0	92.1
grieu-vmpc-s05-27	729	92.9	591.2
IBM_FV_03_SAT_dat.k35	34174	59.8	320.8
IBM_FV_04_SAT_dat.k25	27670	78.4	163.6
IBM_FV_06_SAT_dat.k35	42801	50.8	655.4
IBM_FV_1_02_3_SAT_dat.k20	15775	17.4	36.8
AProVE09-03	59231	51.7	743.3
AProVE09-05	14685	76.3	41.7
AProVE09-17	33894	65.4	629.8
AProVE09-24	61164	18.0	648.0

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- Backbones can be computed for formulas with dozens of thousands of variables.
- Large number off backbones appeared in real-world examples.