On Computing the Minimum Distance for Faster than Nyquist Signaling

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Abstract —The degradation suffered when pulses satisfying the Nyquist criterion are used to transmit binary data at a rate faster than the Nyquist rate over the ideal band-limited (brick-wall) channel is studied. The minimum distance between received signals is used as a performance criterion. It is well-known that, when Nyquist pulses (i.e., pulses satisfying the Nyquist criterion) are sent at the Nyquist rate, the minimum distance between signal points is the same as the pulse energy. The main result is to show that the minimum distance between received signals is the same as the pulse energy for rates of transmission about 25 percent beyond the Nyquist rate, which is the best possible result. In fact, one can even identify the precise error event and signaling rate that causes the minimum distance to be no longer equal to the pulse energy. The mathematical formulation of the problem is to find the smallest value of δ, $0<\delta<0.5$, for which the best L_2 approximation to the constant 1 on the interval $(-\delta,\delta)$ is 1, when using a linear combination of the functions $\exp(i2\pi n\theta)$ to approximate, $n \geq 0$, and restricting the coefficients to be $0, \pm 1$. The smallest value of δ is $0.401\cdots$.

I. Introduction

We are concerned with the problem of determining the minimum distance between received signals, when data is sent faster than the Nyquist rate (the intersymbol interference free rate) over the ideal band-limited (brick-wall) channel using Nyquist pulses. The main result is to show that there is no degradation in the minimum distance for rates up to about 25 percent beyond the Nyquist rate, despite the presence of intersymbol interference. Furthermore, this is the best possible result, and the precise error event that causes degradation is identified. This problem is motivated by a number of considerations.

It has been known since the 1920's that Nyquist pulses can be used to send data at the Nyquist rate without intersymbol interference (ISI) over band-limited channels. This fact has played a major role in the design and implementation of data transmission over the telephone network. It is a natural question to ask as to the degradation suffered in the presence of intersymbol interference when Nyquist pulses are used to send data at a rate faster than the Nyquist rate. The minimum distance $d_{\min}$ between signal points is used as a performance criterion.

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The effects of intersymbol interference are a major problem for many data communication channels, the voiceband telephone channel being the main example. In this connection Forney's bound [1] and subsequent refinements [2], [7], [9], [10] have shown that, for large signal-to-noise ratios and when maximum likelihood sequence detection (MLSD) is employed, the computation of the minimum distance is fundamental in obtaining lower bounds on the bit error rate. However, computation of $d_{\min}$ is rather difficult for many problems. This is particularly true for the problem under consideration in this paper, since the ISI is not finite (in fact, it is not even L_1 summable), and in this case previous results in the literature [7], [8] are upper bounds on $d_{\min}$, obtained by minimizing the detection distance over a small set of error events.

Finally, it seems probable that the techniques of this paper can be used to compute the minimum distance for other channels that have a sharp drop at their band edges. The mathematical formulation of the problem is as follows [7]. In the classical case, Nyquist pulses

$$g(t)=\frac{\sin(\pi t/T)}{\pi t/T}$$

are used to send data without intersymbol interference over a channel of bandwidth $1/2T$. Thus we send pulse trains

$$\sum_{n=n_1}^{n_2} a_n g(t-nT)$$

where $a_n=\pm 1$ independently in the binary case, to which we shall restrict ourselves. Now we wish to use pulses,

$$g(t)=A\frac{\sin \pi t/T}{\pi t/T}$$

and send such pulses at intervals $R=2\delta T$ with $0<\delta<1/2$. We are interested in the minimum distance between received signals. Let E be the pulse energy. It is easily seen that the minimum distance satisfies [7],

$$\frac{d_{\min}}{2\sqrt{E}}=\inf_{P\in\pi}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}|P|^2\,d\theta\right)^{1/2}$$

where

$$\pi = \left\{ \sum_{k=0}^{n} \epsilon_k e^{2\pi i k \theta} | n \geq 0, \epsilon_k = 0, \pm 1; \epsilon_0 = 1 \right\}.$$

We denote this normalized distance $d_{\min}/2\sqrt{E}$ by $I(\delta)$.

Note that when we are signaling at the Nyquist rate, the minimum distance between signal points is equal to the pulse energy, that is $I(1/2) = 1$. It was conjectured in [7] that there was no degradation for rates somewhat faster than the Nyquist rate. Precisely, it was conjectured that there is a $\delta_0 < 1/2$ with $I(\delta) = 1$ for $\delta_0 \leq \delta \leq 1/2$. This conjecture was first proved in [3] (the conference proceedings version of this paper appeared earlier in [11]).

Based on numerical computation [7], the conjecture that was advanced was that $I(\delta) = 1$ for $\nu \leq \delta \leq 1/2$ where $\nu = 0.401 \cdots$ and is obtained by (1.1), where $R(\theta) = \sum_{j=0}^{7}(-1)^j e^{2\pi i j \theta}$:

$$\frac{1}{2\nu} \int_{-\nu}^{\nu} |R(\theta)|^2 \, d\theta = 1. \tag{1.1}$$

Thus it was conjectured that the particular seventh degree polynomial $R(\theta)$ was the unique error event responsible for degradation in the minimum distance. Motivated by our results [3], it was shown in [6] that $I(\delta) = 1$ for $0.4105 \leq \delta \leq 0.5$. Here we prove this conjecture in a stronger form (see Theorem 1.1 and the comment following it), based on our previous technique [3], the techniques in [6], and some new ideas. Thus we show the following best possible result.

Theorem 1.1: Let ν be the solution of (1.1). Then $I(\delta) = 1$ for all $\nu \leq \delta \leq 0.5$. In fact, for

$$R(\theta) = \sum_{j=0}^{7} (-1)^j e^{2\pi j \theta}$$

and for arbitrary

$$P(\theta) = 1 + \sum_{k=1}^{n} \epsilon_k e^{2\pi k \theta}, \qquad \epsilon_k = 0, \pm 1,$$

with $P(\theta)$ being neither 1 nor $R(\theta)$, the following assertions are true:

1) for all $\nu \leq \delta \leq 0.5$, $(1/2\delta)\int_{-\delta}^{\delta} |P(\theta)|^2 \, d\theta > 1$;
2) for all $\nu < \delta \leq 0.5$, $(1/2\delta)\int_{-\delta}^{\delta} |R(\theta)|^2 \, d\theta > 1$;
3) there are $\epsilon > 0$ and $\delta_0 < \nu$ (where values can be given for δ_0 and ϵ) such that for all $\delta_0 \leq \delta \leq 0.5$ and any $P(\theta) \neq R(\theta)$ or 1, $(1/2\delta)\int_{-\delta}^{\delta} |P(\theta)|^2 \, d\theta \geq 1 + \epsilon$.

Assertion 3) follows from an examination of the proof given here of 1) and 2) of Theorem 1.1. However, for the remainder of this paper, we shall concentrate on proving the first two parts of Theorem 1.1. The results in this paper were presented at the SITA 1987 Conference Proceedings [4].

Added in Proof: Using a different argument, Henry Landau and Jim Maza have extended upon their results in an early version of [6] and which were previously mentioned, to now reach the same conclusion as in parts 1) and 2) of Theorem 1.1.

Usage

$e(\theta)$	This means $e^{2\pi i \theta}$.
$c(a,\delta)$	For $0 < \delta \leq 1/2$ and $a = 1, 2, \cdots, c(a,\delta) = ((-1)^a/2\delta)\int_{-\delta}^{\delta}[\cos 2\pi a\theta/(1+\cos 2\pi\theta)]\, d\theta$.
$\hat{g}$	For a function $g(x)$, $\hat{g}(x) = \int_{-\infty}^{\infty} g(t) e(-tx)\, dt$.
$\|g\|_2$	For a function $g(x)$, $\|g\|_2 = (\int_{-\infty}^{\infty} \|g(t)\|^2 \, dt)^{1/2}$.
$R(\theta)$	$R(\theta) = \sum_{j=0}^{7}(-1)^j e^{2\pi i j \theta}$.
ν	ν satisfies (1.1) and $\nu = 0.401 \cdots$.
Polynomial	In this paper a polynomial means a trigonometric polynomial of the form $1 + \sum_{k=1}^{n} \epsilon_k e^{2\pi i k \theta}$ with $\epsilon_k = 0, \pm 1$, and $n \geq 1$.
$\bar{z}$	Complex conjugate of a complex number z.

II. Outline of the Proof and Reduction to Alternating Blocks with Gaps

A brief outline of the proof of Theorem 1.1 follows. The first step is to show that, given any $P(\theta)$ as in Theorem 1.1 with at least two consecutive nonzero ϵ_k having the same sign, then $1/2\delta\int_{-\delta}^{\delta}|P(\theta)|^2 \, d\theta > 1$ for $0.401 \leq \delta \leq 0.5$. The proof technique is to use suitable test functions as in [3]. It follows that, since we are interested in the value of $I(\delta)$ for only $0.401 \leq \delta \leq 0.5$, we may assume that $P(\theta)$ consists of blocks of alternating sign coefficients, with at least a gap of one between blocks. One can again use test functions and modify the argument in [6] to show that one needs to consider at most 13 blocks. It turns out that the idea in this particular argument (and in fact the test function used) can be traced back to some classical work of Ingham [5] on exponential sums. Ingham's theorem was also used in [3]. Now let N be the number of blocks in $P(\theta)$. If $4 \leq N \leq 13$, then one rewrites the integral $(1/2\delta)\int_{-\delta}^{\delta}|P(\theta)|^2 \, d\theta$ as a quadratic form and lower bounds the quadratic form using the result in [6]. This also works for $N = 3$ by appropriately tightening the argument. For $1 \leq N \leq 2$, if the length of any block is large enough then the integral in question approaches a limiting value larger than one. Finally, if the size of all the blocks is not too large, then an explicit numerical computation finishes the proof.

In this section we prove the following theorem.

Theorem 2.1: Let $P(\theta) = 1 + \sum_{k=1}^{n} \epsilon_k e(k\theta)$ with $\epsilon_k = 0, \pm 1$. If there are two consecutive (nonzero) ϵ_k with the same sign, then

$$\frac{1}{2\delta} \int_{-\delta}^{\delta} |P(\theta)|^2 \, d\theta \geq 1$$

for $0.401 \leq \delta \leq 0.5$.

The import of this theorem is that to prove Theorem 1.1, it suffices to consider only those polynomials $P(\theta)$ such that $P(\theta)$ consists of blocks of alternating sign coefficients, with at least a gap of one between blocks. This is by Theorem 2.1 and because we are interested in the values of $I(\delta)$ only for $0.401 \le \delta \le 0.5$ (recall from (1.1) that $\nu = 0.401\cdots$). The idea of the proof is that we may integrate $P(\theta)$ (after multiplying it, if needed, by a suitable exponential to put it into a more tractable form) against a suitable test function $h(\theta)$, over which we have control, and after applying the Cauchy–Schwartz inequality, we hope to get a useful lower bound on $(1/2\delta)\int_{-\delta}^{\delta}|P(\theta)|^2\, d\theta$.

Proof: Without loss of generality, assume that the first consecutive ϵ_k with the same signs are $+1$, by multiplying P if needed by -1 since this does not alter $|P|$. These two ϵ_k are, therefore, preceded by a coefficient of either 0 or -1. By multiplying P by any suitable exponential $e(a\theta)$ and calling the result Q, we may therefore lower-bound $(1/2\delta)\int_{-\delta}^{\delta}|Q|^2\, d\theta$ since $|P| = |Q|$. The exponential $e(a\theta)$ with which we want to multiply P is chosen so that (simply causing a shift in the exponentials in P) the $1 + e(\theta)$ in Q (see (2.1) and (2.2)) corresponds to the first consecutive ϵ_k in P with the same sign, which we assumed to be $+1$. Here,

$$Q(\theta) = 1 + e(\theta) + \sum_{|k| \ge 2} \delta_k e(k\theta) \tag{2.1}$$

or

$$Q(\theta) = 1 + e(\theta) - e(-\theta) + \sum_{|k| \ge 2} \delta_k e(k\theta) \tag{2.2}$$

and $\delta_k = 0, \pm 1$ (of course, only finitely many $\delta_k = \pm 1$ in (2.1) and (2.2)). By [3, Lemma 6 and Theorem 4(b)] for $Q(\theta)$ as in (2.1), for $0.38\cdots \le \delta \le 0.5$

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |Q(\theta)|^2\, d\theta \ge 1.$$

Actually, the polynomials in [3, Lemma 6] had a hypothesis requiring that there was a gap between consecutive negative k in (2.1); however, this hypothesis was never used in the proof of [3, Lemma 6] but only in a later theorem.

Now let $Q(\theta)$ be as in (2.2). Consider the coefficients δ_k in (2.2) for $k \le -2$. Let δ_{-l-1} be the first coefficient that is zero, for some $l \ge 1$. Then $\delta_{-k} = (-1)^k$ for $2 \le k \le l$. This is because we get Q by multiplying P by a suitable exponential, so that the $1 + e(\theta)$ in Q corresponds to the first consecutive ϵ_k in P with the same sign (which we assumed to be $+1$) and thus any consecutive signs in Q to the left of $1 + e(\theta)$ alternate. Let $T(\theta) = e(\theta)\overline{Q}(\theta)$. Then,

$$T(\theta) = 1 + e(\theta) + \sum_{k=1}^{l} (-1)^k e((k+1)\theta) + \sum_{k \ge l+3} \zeta_k e(k\theta) + \sum_{k \le -1} \zeta_k e(k\theta)$$

with $\zeta_k = 0, \pm 1$ and $l \ge 1$. Let,

$$g(x) = \frac{\sin \pi x}{\pi}\left(\frac{1}{x} - \frac{0.72}{x-1} - \frac{0.28}{x-2}\right) = -\frac{\sin \pi x}{25\pi}\frac{(32x - 50)}{(x^3 - 3x^2 + 2x)}.$$

One can check (see the Appendix) that (2.3) holds:

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |T(\theta)|^2\, d\theta\right)^{1/2} \ge \frac{1}{\sqrt{1.5968}}\left(\left|1 + g(2\delta) + \sum_{k=1}^{l} (-1)^k g(2\delta(k+1))\right| - \sum_{\substack{k \le -1 \\ k \ge l+3}} |g(2\delta k)|\right). \tag{2.3}$$

Call the right side of (2.3) $n(l, \delta)$. Since $|T(\theta)| = |P(\theta)|$ and because it is readily checked (see the Appendix) that $n(l, \delta) > 1$ for all $l \ge 1$ and $0.401 \le \delta \le 0.5$, the result follows. □

III. Bounding the Number of Alternating Blocks

By the previous section, if we are interested in the values of $I(\delta)$ for $0.401 \le \delta \le 0.5$, we may restrict ourselves to polynomials $P(\theta)$ where $P(\theta)$ consists of blocks of alternating sign coefficients, with at least a gap of one between blocks. For the remainder of this section, let $P(\theta)$ be such a polynomial. The next theorem bounds the number of blocks in such a polynomial.

Theorem 3.1: Let $P(\theta)$ be as before, and let N be the number of blocks in it. If $N \ge 14$, then

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta \ge 1$$

for $0.401 \le \delta \le 0.5$.

The basic idea of the proof, as in the previous section, uses test functions, but this time in a slightly different way. The proof consists of modifying and strengthening some of the arguments in [6]. It turns out that the idea in this particular argument, and in fact the test function used, can be traced back to Ingham [5]. Ingham's basic estimate was also used in [3].

Proof: Let $g(x) = (1/2\delta)e(x/2)\cos(\pi x/2\delta)$ for $|x| \le \delta$ and $g(x) = 0$ for $|x| > \delta$. Then, letting $h(x) = \hat{g}(x)$,

$$h(x) = \frac{2}{\pi}\frac{\cos 2\delta\pi(x - 1/2)}{1 - 4(2\delta)^2(x - 1/2)^2}.$$

Thus by Plancherel's theorem and because $|g(x)| \le 1/2\delta$:

$$\begin{aligned}\left(\frac{1}{2\delta}\right)^2 \frac{1}{2\delta}\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta &\ge \frac{1}{2\delta}\int_{-\delta}^{\delta}|P(\theta)g(\theta)|^2\,d\theta\\ &= \frac{1}{2\delta}\int_{-1/2}^{1/2}|P(\theta)g(\theta)|^2\,d\theta\\ &= \frac{1}{2\delta}\sum_n|\widehat{Pg}(n)|^2\\ &= \frac{1}{2\delta}\sum_n\left|\sum_{k=0}^{m}\epsilon_k h(n-k)\right|^2\end{aligned}$$

for $P(\theta)=\sum_{k=0}^{m}\epsilon_k e(k\theta)$, $\epsilon_o=1$ and $\epsilon_m \ne 0$. Let $f(n)=\sum_{k=0}^{m}\epsilon_k h(n-k)$. Thus

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta \ge 2\delta\sum_n|f(n)|^2. \tag{3.1}$$

Let $\epsilon\sum_{k=a}^{b}(-1)^k e(k\theta)$, where $\epsilon=\pm1$, be a particular block in $P(\theta)$. Then $\epsilon_a=\epsilon(-1)^a$ and $\epsilon_{a-1}=0$, since there is a gap of at least one between blocks. First suppose that $\epsilon_{a-2}\ne 0$. Then, upon noting that $h(-k)=h(k+1)$ for $k\ge 1$, it follows by the triangle inequality:

$$\begin{aligned}|f(a)| &= |\epsilon_a h(0)+\epsilon_{a-1}h(1)+\epsilon_{a-2}h(2)\\ &\quad +\epsilon_{a+1}h(-1)+\cdots|\\ &= |\epsilon_a h(0)+\epsilon_{a-2}h(2)+\epsilon_{a+1}h(-1)+\cdots|\\ &\ge |h(0)|-\sum_{k\ge 2}|h(k)|-\sum_{k\le -1}|h(k)|\\ &= |h(1)|-2\sum_{k\ge 2}|h(k)|\end{aligned}$$

and

$$\begin{aligned}|f(a-1)| &= |\epsilon_{a-1}h(0)+\epsilon_a h(-1)+\epsilon_{a-2}h(1)+\cdots|\\ &\ge |h(1)|-\sum_{k\ne 0,1}|h(k)|\\ &= |h(1)|-2\sum_{k\ge 2}|h(k)|\\ &\ge 0.144\cdots\end{aligned}$$

for $\delta\ge 0.401$ by numerical computation. Then, for $\delta \ge 0.401$,

$$|f(a)|^2+|f(a-1)|^2 \ge 2(0.144)^2 = 0.041472.$$

On the other hand, if $\epsilon_{a-2}=0$, then

$$\begin{aligned}|f(a)| &= |\epsilon_a h(0)+\epsilon_{a-1}h(1)+\epsilon_{a-2}h(2)+\cdots|\\ &\ge |h(0)|-\sum_{k\le -1}|h(k)|-\sum_{k\ge 3}|h(k)|\\ &= |h(0)|-|h(2)|-2\sum_{k\ge 3}|h(k)|\end{aligned}$$

and so for $\delta\ge 0.401$, $|f(a)|^2 \ge 0.06290\cdots$ by numerical computation. In either case the contribution from the left side of the block, which we denote L, satisfies $L\ge 0.041472$. Similarly, upon noting that $\epsilon_b\ne 0$, $\epsilon_{b+1}=0$, we get

$$|f(b+1)|^2+|f(b+2)|^2 \ge 0.041472$$

for $\delta\ge 0.401$ provided $\epsilon_{b+2}\ne 0$ and $|f(b+1)|^2 \ge 0.0629\cdots$ for $\delta\ge 0.401$ if $\epsilon_{b+2}=0$. Thus the contribution from the right side of the block, which we denote R, satisfies $R\ge 0.041472$. Finally, since $\epsilon_k=0$ for $k<0$, the contribution from the left side of the leftmost block, which we denote L_s, satisfies (for $\delta\ge 0.401$) $L_s \ge (|h(1)|-\sum_{k\ge 2}|h(k)|)^2 \ge 0.119025$ and, similarly, the right side of the rightmost block, which we denote R_s, also satisfies $R_s\ge 0.119025$. Let N be the number of blocks in $P(\theta)$. Then by (3.1), for $\delta\ge 0.401$

$$\begin{aligned}\frac{1}{2\delta}\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta &\ge 2\delta((N-1)(L+R)+L_s+R_s)\\ &\ge 0.802(2(N-1)(0.041472)\\ &\quad +2(0.119025))\\ &= 0.066521088(N-1)+0.1909161\\ &\ge 1\end{aligned}$$

for $N\ge 14$. □

IV. The Case of Three or More Alternating Blocks

Since we are interested in the values of $I(\delta)$ only for $0.401\le\delta\le 0.5$, by the results of the last section we need only consider polynomials that consist of at most 13 blocks, separated by gaps, with the blocks consisting of alternating coefficients. Let N be the number of blocks in the polynomial. In this section $3\le N\le 13$. The basic idea is to rewrite the integral $(1/2\delta)\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta$ as a quadratic form and then lower-bound the quadratic form by using a "telescoping" argument as in [6]. For $4\le N\le 13$ we can use the result from [6]. For $N=3$ one has to tighten the argument in [6].

To be precise, let $P(\theta)$ have N blocks. Because

$$\sum_{k=a}^{b}(-1)^k z^k = \frac{(-z)^a\left(1-(-z)^{b+1}\right)}{1+z},$$

we may conclude as in [6]:

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta = \frac{1}{2\delta}\int_{-\delta}^{\delta}\frac{|Q(\theta)|^2}{|1+e(\theta)|^2}\,d\theta$$

with Q having at most $2N$ nonzero coefficients consisting of ±1, with each pair of consecutive nonzero coefficients coming from a block. We only get $\pm 1,0$ coefficients because of the gaps between blocks. Because $|1+e(\theta)|^2\le 4$, it suffices to show that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}|Q(\theta)|^2 \ge 4. \tag{4.1}$$

For $4\le N\le 13$ the result follows at once from [6]. In fact from [6, (29)] (taking $\epsilon=0.2032$ there), we have that for $\delta\ge(1-0.2032)/2=0.3984$. In particular, part a) of Theorem 4.1 follows.

Theorem 4.1: a) For $4 \le N \le 13$ and for Q as in (4.1)

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}|Q(\theta)|^2\,d\theta \ge 4$$

for $0.3984 \le \delta \le 0.5$. b) For $N=3$, (4.1) holds for $0.40072 \le \delta \le 0.5$.

Proof: b) We may write as in [6] that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}|Q|^2\,d\theta = q^TMq \tag{4.2}$$

with $q^T=(q_1,q_2,\cdots)$, q_i being $0,\pm1$ and M being the real symmetric Toeplitz matrix whose first row is

$$\left(1,\frac{\sin 2\pi\delta}{2\pi\delta},\cdots,\frac{\sin 2\pi n\delta}{2\pi n\delta},\cdots\right) = \left(1,\frac{\epsilon}{p}\frac{\sin\pi\epsilon}{\pi\epsilon},\cdots,\frac{\epsilon}{p}(-1)^{n+1}\frac{\sin n\pi\epsilon}{n\pi\epsilon},\cdots\right)$$

where $\epsilon = 1-p$ and $p=2\delta$. We now refine the technique in [6] to lower-bound the contribution of each row of M to (4.2).

Let $\operatorname{sinc}(x)=\sin\pi x/\pi x$. The values of $\operatorname{sinc}(x)$ for $n\le x<n+1$ are called the positive nth lobe and for $-n-1<x\le -n$ the negative nth lobe. The values of $\operatorname{sinc}(x)$ for $|x|\le 1$ are called the main lobe. The maximum of $|\operatorname{sinc}(x)|$ in the nth lobe is denoted $s^*(n)$ as in [6]. We need to lower-bound the inner product of a row j of M with q, if $q_j\ne 0$. There are six nonzero q_j: first consider the pair of q_j with one pair above them and one pair below them in q. Call the pair q_{j_0},q_{j_1}. Call the pair above them in q (going from top to bottom in q), q_{j_2}, q_{j_3} and the pair below them q_{j_4},q_{j_5}.

To obtain a lower bound to the inner product of a row j (where $q_j\ne 0$) with q, consider the sum of the individual terms in the inner product corresponding to a particular lobe. We recall the following from [6].

1) The main lobe contributes at least $1-(\epsilon/p)$ with the 1 coming from the diagonal term q_j^2.
2) An upper bound to the possible contribution of half of any lobe (other than the main lobe), say the positive nth lobe, is $s^*(n)$, and thus for the whole lobe is $2s^*(n)$.
3) The bound $s^*(n)$ additionally holds for the whole nth lobe if there is a single point in one of the halves of the nth lobe.
4) $s^*(1)=0.21723\cdots$, $s^*(2)=0.12836\cdots$, $s^*(n)=2/\pi(2n+2)$ for $n\ge 3$ with sufficient accuracy.

It follows that the contribution of row j_0 is at least

$$\left(1-\frac{\epsilon}{p}\right)-\frac{\epsilon}{p}\left(s^*(1)+s^*(2)+s^*(1)+s^*(2)\right). \tag{4.3}$$

Here the $(1-(\epsilon/p))$ comes from the main lobe while the rest of (4.3) represents a worst case upper bound to the possible contributions of other lobes from the pair q_{j_2},q_{j_3} lying to the left of the peak of the main lobe and q_{j_4},q_{j_5} lying to the right. The same contribution as in (4.3) also arises from row j_1. For row j_3, one can again use the foregoing facts to see that it contributes at least

$$\left(1-\frac{\epsilon}{p}\right)-\frac{\epsilon}{p}\left(2s^*(1)+s^*(2)\right). \tag{4.4}$$

Here $2s^*(1)+s^*(2)$ is a worst case upper bound to the possible contribution of the lobes (other than the main lobe) from the pairs $q_{j_0},q_{j_1};q_{j_4},q_{j_5}$. In the same way the bound (4.4) also holds for row j_4. Finally, from [6,(28)] we see that the first row of M (i.e., j_2) and the last row of M (i.e., j_5) each contribute at least

$$1-\frac{\epsilon}{p}\left(2s^*(1)+2s^*(2)+s^*(3)\right). \tag{4.5}$$

Adding up all the contributions from (4.3)–(4.5) one obtains s, where s is

$$6-\frac{\epsilon}{p}\left(12s^*(1)+10s^*(2)+2s^*(3)+4\right). \tag{4.6}$$

From (4.6) we see that $s\ge 4$ and thus so is q^TMq for $2\delta=p\ge 0.801435$, which proves the result. □

By the results so far, to prove Theorem 1.1, it remains to consider polynomials which consist of one or two blocks, with gaps between the blocks and with the blocks consisting of alternating sign coefficients. This is done in the next two sections.

V. The Case of One Alternating Block

The key to treating polynomials consisting of one or two alternating blocks is the following lemma in which the proof is in the Appendix. (Recall the notation $c(a,\delta)$ from Section I.)

Lemma 5.1: For any $a\ge 1$, $0<\delta<1/2$,

$$|c(a,\delta)|\le \frac{1}{\pi\delta(1+\cos 2\pi\delta)a} \tag{5.1}$$

and thus

$$\max_{0.4\le\delta\le 0.4105}|c(a,\delta)|\le 5.036/a. \tag{5.2}$$

The basic idea of the proof is that (5.2) allows us to show that $(1/2\delta)\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta$ approaches a limiting value larger than one, if the length of the block of alternating coefficients that makes up $P(\theta)$ is large enough. On the other hand, if the length of the block is not too large, then a numerical computation finishes the proof.

To see this note that

$$\min_{0.4\le\delta\le 0.5}\frac{\tan\pi\delta}{2\pi\delta}\ge 1.22457. \tag{5.3}$$

Let $P(\theta)=\sum_{k=0}^{l}(-1)^k e(k\theta)$ be the polynomial consisting of one alternating block. Since

$$|P(\theta)|^2=\left(1+(-1)^l\cos(l+1)2\pi\theta\right)/1+\cos 2\pi\theta$$

by summing the geometric series that represents $P(\theta)$,

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta=\frac{1}{2\delta}\int_{-\delta}^{\delta}\frac{d\theta}{1+\cos 2\pi\theta}-c(l+1,\delta) = \frac{\tan\pi\delta}{2\pi\delta}-c(l+1,\delta). \tag{5.4}$$

By (5.4), (5.3), and (5.2) it follows that for $l \geq 22$ and $0.4 \leq \delta \leq 0.4105$, $(1/2\delta)\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta \geq 1$. Since it is shown in [6] that $(1/2\delta)\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta \geq 1$ for $0.4105 \leq \delta \leq 0.5$ and any P, numerical computation for $1 \leq l \leq 21$ now proves the following that settles the single block case.

Theorem 5.2: Let $P(\theta) = \sum_{k=0}^{l}(-1)^k e(k\theta)$, and let v be such that $(1/2v)\int_{-v}^{v}|\sum_{k=0}^{7}(-1)^k e(k\theta)|^2\,d\theta = 1$. Then $v = 0.401\cdots$ and for $v \leq \delta \leq 0.5$, $(1/2\delta)\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta \geq 1$.

Note that one requires only that $I(\delta) = 1$ for $\delta_0 \leq \delta \leq 1/2$ for some $\delta_0 < 1/2$ (as in [3]) for the preceding argument to work, instead of $I(\delta) = 1$ for $0.4105 \leq \delta \leq 0.5$. The only difference is that one requires a more extensive numerical computation, though using the result of [3] does not result in significant extra numerical computation. The foregoing remarks also apply to the next section.

VI. The Case of Two Alternating Blocks

In this section $P(\theta)$ consists of two alternating blocks:

$$P(\theta) = \sum_{k=0}^{l}(-1)^k e(k\theta) + \epsilon \sum_{k=a}^{b}(-1)^k e(k\theta) \quad (6.1)$$

with $0 \leq l$, $l+2 \leq a \leq b$ and $\epsilon = \pm 1$. We now prove the following theorem.

Theorem 6.1: For $P(\theta)$ as in (6.1) and for $v \leq \delta \leq 0.5$, with v as in Theorem 5.2

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta \geq 1. \quad (6.2)$$

The basic idea of the proof of Theorem 6.1 is exactly the same as that of Theorem 5.2. The main difference is that several technical difficulties arise. It suffices to prove (6.2) for $v \leq \delta \leq 0.4105$ by [6], and we shall restrict ourselves to these δ. Again by using (5.2) and numerical computation, it is easy to see that

$$\max_{a,\, v \leq \delta \leq 0.4105}|c(a,\delta)| = |c(3, 0.4105)| \leq 0.480997 \quad (6.3)$$

$$\max_{a \geq 32,\, v \leq \delta \leq 0.4105}|c(a,\delta)| \leq 0.0667737. \quad (6.4)$$

Let $V(\delta) = (1/2\delta)\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta$. It is easily seen by summing the geometric series in (6.1) and integrating that

$$V(\delta) = \frac{\tan \pi\delta}{\pi\delta} - c(l+1,\delta) - c(b+1-a,\delta) + \epsilon(c(a,\delta) - c(a-l-1,\delta) - c(b+1,\delta) + c(b-l,\delta)). \quad (6.5)$$

Call a term $c(m,\delta)$ in (6.5) "dominant" if $1 \leq m \leq 31$ and "negligible" otherwise. Let $d_1 = l$, $d_2 = a - l$ and $d_3 = b - a$. There are eight cases to consider corresponding to whether or not a given $d_i \leq 32$ or $d_i \geq 33$ for $i = 1,2,3$. The case $d_i \leq 32$ for $i = 1,2,3$ is done numerically, and one checks that in this case, $V(\delta) \geq 1$ for $v \leq \delta \leq 0.4105$. Now assume that at least one of the $d_i \geq 33$, and let μ be the number of dominant terms in (6.5). If $\mu = 0$, then by (6.5), (6.4), and (5.3),

$$V(\delta) \geq 2(1.22457) - 6(0.0667737) = 2.0484978.$$

If $\mu = 1$, then by (6.5), (6.4), and (5.3),

$$V(\delta) \geq 2(1.22457) - 0.480997 - 5(0.0667737) = 1.6342745.$$

If $\mu = 2$, then $d_1 \leq 32$, $d_2 \geq 33$, $d_3 \leq 32$, and in this case the dominant terms are $c(l+1,\delta)$ and $c(b+1-a,\delta)$. By (5.4) and Theorem 5.2 we know that, for $v \leq \delta \leq 0.4105$,

$$\frac{\tan \pi\delta}{2\pi\delta} - c(l+1,\delta) \geq 1 \quad (6.6)$$

$$\frac{\tan \pi\delta}{2\pi\delta} - c(b+1-a,\delta) \geq 1. \quad (6.7)$$

Thus using (6.5), (6.4), (6.6), and (6.7),

$$V(\delta) \geq 2 - 4(0.0667737) = 1.7329052.$$

The case $\mu = 3$ (and it is clear that this is the largest μ can be) can be handled the same way as $\mu = 2$ (see the Appendix). This completes the proof of Theorem 6.1. □

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Appendix

Proof of (2.3)

Set $h(x) = g(2\delta x)$. It is readily checked that $\hat{h}$ has support in $[-\delta,\delta]$ and $\|g\|_2 = (1 + (0.72)^2 + (0.28)^2)^{1/2} = (1.5968)^{1/2}$. By the Cauchy–Schwartz inequality, the inversion theorem, the triangle inequality and Plancherel's theorem therefore:

$$\|g\|_2\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}|P|^2\,d\theta\right)^{1/2} = \|g\|_2\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}|T|^2\,d\theta\right)^{1/2} = \|\hat{h}\|_2\left(\int_{-\delta}^{\delta}|T|^2\,d\theta\right)^{1/2}$$

$$\geq \left|\int_{-\delta}^{\delta}\left(\hat{h}(\theta) + \hat{h}(\theta)e(\theta) + \sum_{k=1}^{l}\hat{h}(\theta)(-1)^k e((k+1)\theta) + \sum_{\substack{k \leq -1 \\ k \geq l+3}}\hat{h}(\theta)\zeta_k e(k\theta)\right)d\theta\right|$$

$$\geq \left|h(0) + h(1) + \sum_{k=1}^{l}(-1)^k h(k+1)\right| - \sum_{k \leq -1}|h(k)| - \sum_{k \geq l+3}|h(k)|.$$

□

Proof that $n(l,\delta) > 1$ for all $l > 1$ and $0.401 \le \delta \le 0.5$.

For $l = 1,2,4,5$ let

$$v(l,\delta) = \left|1 + g(2\delta) + \sum_{k=1}^{l} (-1)^k g(2\delta(k+1))\right| - \sum_{-50 \le k \le -1} |g(2\delta k)| - \sum_{l+3 \le k \le 50} |g(2\delta k)|$$

$$a(\delta) = \sum_{|k| \ge 51} |g(2\delta k)|.$$

Now,

$$|g(x)| \le \frac{32|x|}{25\pi|x|^3} \frac{(1 + 1.5625/|x|)}{\left(1 - \frac{3}{|x|} - \frac{2}{|x|^2}\right)}. \quad (2.4)$$

For $|k| \ge 51$ and $\delta \ge 0.401$, $2\delta|k| \ge 40.902$ and so from (2.4), $|g(2\delta k)| \le 0.710617491/k^2$ and thus

$$a(\delta) \le 2(0.710617491)\int_{50}^{\infty} \frac{dx}{x^2} = 0.0284247. \quad (2.5)$$

Now, by numerical computation, for $\delta \ge 0.401$, $v(1,\delta) > v(2,\delta) \ge 1.326 \cdots$, $v(4,\delta) = 1.293388 \cdots$, $v(5,\delta) = 1.32 \cdots$ and thus by (2.5), $n(l,\delta) > 1$ for $l = 1,2,4,5$. For $l \ge 5$, by the triangle inequality and numerical computation $n(l,\delta) \ge n(5,\delta) - (|g(14\delta)|/(1.5968)^{1/2}) > 1$ for $\delta \ge 0.401$. Finally, for $l = 3$, let

$$v(3,\delta) = |1 + g(2\delta) - g(4\delta) + g(6\delta) - g(8\delta)| - \sum_{6 \le k \le 700} |g(2\delta k)| - \sum_{-700 \le k \le -1} |g(2\delta k)|$$

and $a(\delta) = \sum_{|k| \ge 701} |g(2\delta k)|$. Then $v(3,\delta) \ge 1.265532128 \cdots$ and $a(\delta) \le 0.001824652$ (by again using (2.4)), for $\delta \ge 0.401$. Then $n(3,\delta) \ge (1.263707 \cdots)/(1.26364 \cdots) > 1$ for $\delta \ge 0.401$. Thus $n(l,\delta) > 1$ for all $l \ge 1$ and $0.401 \le \delta \le .5$. □

Proof of Lemma 5.1

Equation (5.2) is obvious from (5.1). Let $g(\theta) = 1/(1 + \cos 2\pi\theta)$, which is increasing on $[0,\delta]$. By the mean value theorem for integrals, and since $(\cos 2\pi a\theta) g(\theta)$ is an even function,

$$c(a,\delta) = \frac{(-1)^a}{2\delta}\int_{-\delta}^{\delta} (\cos 2\pi a\theta) g(\theta)\, d\theta$$

$$= \frac{g(\delta)}{\delta}(-1)^a \int_{\zeta}^{\delta} \cos 2\pi a\theta\, d\theta$$

for some $\zeta \in (0,\delta)$. Since

$$\left|\int_{\zeta}^{\delta} \cos 2\pi a\theta\, d\theta\right| = \left|\frac{1}{2\pi a}(\sin 2\pi a\theta)\Big|_{\zeta}^{\delta}\right| \le \frac{1}{\pi a},$$

(5.1) is proved. □

Proof in the Case $\mu = 3$

The two cases that arise are $d_1 \ge 33$, $d_2 \le 32$, $d_3 \le 32$, and $d_1 \le 32$, $d_2 \le 32$, $d_3 \ge 33$. Clearly, one can go from one case to the other by conjugating P and by multiplying by a suitable exponential, so we need consider only $d_1 \le 32$, $d_2 \le 32$, $d_3 \ge 33$. In this case the dominant terms are $c(a,\delta)$, $c(a-l-1,\delta)$ and $c(l+1,\delta)$. Then, by (5.4), Theorem 5.2, and (6.5),

$$\frac{\tan \pi\delta}{2\pi\delta} - c(l+1,\delta) \ge 1 \quad (6.8)$$

$$\frac{\tan \pi\delta}{2\pi\delta} - c(a-l-1,\delta) \ge 1, \quad \text{if } \epsilon = 1 \quad (6.9)$$

or

$$\frac{\tan \pi\delta}{2\pi\delta} - c(a,\delta) \ge 1, \quad \text{if } \epsilon = -1. \quad (6.10)$$

Thus by (6.5), (6.3), (6.4), (6.8)–(6.10) one has

$$V(\delta) \ge 2 - 0.480997 - 3(0.0667737) = 1.3186819.$$

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