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On Concentration of Positive Bound States of Nonlinear Schrödinger Equations

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Abstract. We study the concentration behavior of positive bound states of the nonlinear Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = \frac{-h^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi .$$

Under certain condition on V, we show that positive ground state solutions must concentrate at global minimum points of V as $h \to 0^+$; moreover, a point at which a sequence of positive bound states concentrates must be a critical point of V. In case that V is radial, we prove that the positive radial solutions with least energy among all nontrivial radial solutions must concentrate at the origin as $h \to 0^+$.

Section 1. Introduction and Description of Main Results

Of concern are standing wave solutions of the following nonlinear Schrödinger equations:

$$ih\frac{\partial\psi}{\partial t} = \frac{-h^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \quad \text{with } x \in \mathbb{R}^n , \tag{1.1}$$

i.e., solutions of the form

$$\psi(x, t) = \exp(iEt/h)u(x) , \tag{1.2}$$

where h, m, γ and p are positive constants, $p > 1$, $E \in \mathbb{R}$, V is real and belongs to $C^1(\mathbb{R}^n)$ and u is real. In [FW], Floer and Weinstein proved for small $h > 0$ (and for $p = 3$, $n = 1$) the existence of standing wave solutions concentrating at each given nondegenerate critical point of the potential V, under the condition that V is bounded. In [O_1, O_3], Oh generalized this result and obtained for small $h > 0$ the existence of multi-lump standing wave solutions with u in (1.2) being positive and

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concentrating at any given finite collection of nondegenerate critical points of V, under the condition $n \geqq 1$, $1 < p < \frac{n+2}{(n-2)^+}$ $\left(\text{we use the convention: } \frac{n+2}{(n-2)^+} = \infty \text{ when } n = 1, 2\right)$, and $V \in (V)_a$ (namely, either $V \equiv a$ or $V(x) > a$ and $(V-a)^{-1/2} \in \mathrm{Lip}(\mathbb{R}^n)$). The arguments in these papers are based on the Lyapunov–Schmidt reduction.

Substituting (1.2) into (1.1) and assuming without loss of generality, $2m = 1$ and $\gamma = 1$, one has

$$h^2 \Delta u - (V(x) - E)u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n .$$

Throughout this paper, we shall assume $1 < p < \frac{n+2}{(n-2)^+}$ and V is bounded below. A suitable choice of E makes $V - E$ bounded below from zero. Thus, without loss of generality, we shall assume *throughout this paper that* $E = 0$ *and* V *is bounded below by a positive constant.* Now the equation for u can be rewritten as

$$h^2 \Delta u - V(x)u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n , \tag{1.3}$$

or

$$\Delta v - V(hx)v + |v|^{p-1}v = 0, \quad x \in \mathbb{R}^n , \tag{1.4}$$

where $v(x) = u(hx)$ and $\inf V > 0$. The existence of solutions of (1.3) (or (1.4)) and its various generalizations has long been studied extensively (mostly by variational methods). The interested readers may consult, in addition to the papers mentioned below, the survey articles [L_1] and [N] and references listed therein. Most of the results provide existence of solutions for *arbitrary* $h > 0$. Several papers deal with existence of *ground states*, i.e., in case of (1.3), solutions with least "energy"

$$\frac{1}{2} \int_{\mathbb{R}^n} (h^2 |\nabla u|^2 + Vu^2)\, dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}\, dx \tag{1.5}$$

among all nontrivial $H^1(\mathbb{R}^n)$ solutions of (1.3). Here we mention two papers that can be directly applied to (1.3). In [DN], Ding and Ni proved among other things, that if V is, roughly speaking, ultimately increasing along n independent directions and "almost symmetric", then for $h > 0$, (1.3) (or (1.4)) has a positive ground state (it was not stated in [DN] that the solution is a ground state, yet it is possible to check the solution is one). Another general result of [DN] implies that if $n \geqq 2$ and V is radial, then for every $h > 0$, (1.3) (or (1.4)) has a radial positive solution with least energy among all nontrivial $H^1(\mathbb{R}^n)$ radial solutions. (We shall call them *radial ground states.*) Recently, Rabinowitz [R] showed that (1.3) (or (1.4)) has a positive ground state for every $h > 0$ if $\liminf_{x \to \infty} V(x) = \sup V$ or for small $h > 0$ if $\liminf_{x \to \infty} V(x) > \inf V$. In both [DN] and [R], more general nonlinearity was treated, and the Mountain-Pass arguments were used.

In this paper, motivated by a question in [R], we study the concentration behavior of positive bound states (i.e., solutions with finite energy) of (1.3) as $h \to 0^+$. Concerning positive ground states, we obtain what can be roughly described as follows. *If* $\liminf_{x \to \infty} V(x) > \inf V$, *any sequence of positive ground states of* (1.3) *contains a subsequence concentrating at a global minimum point of* V *as* $h \to 0^+$. (*In particular, if the global minimum point of* V *is unique, then all positive* *ground states concentrate at that point as $h \to 0^+$.) On the other hand, if we only know a sequence of ground states u_{h_k} exists and there is a sequence of local maximum points moving toward a certain point x_0 as $h_k \to 0^+$, then x_0 is a global minimum point of V and $\{u_{h_k}\}$ concentrates at x_0 as $h_k \to 0^+$. (Thus if V has no global minimum, the positive ground states, if any, do not concentrate.)* As for radial positive ground states of (1.3), we prove that *if $n \geqq 2$, all of them concentrate at the origin as $h \to 0^+$.* For general positive bound states, we show that *under the condition $|\nabla V(x)| = O(e^{a|x|})$ near $x = \infty$ for some $a > 0$, a point at which a sequence of positive bound states of* (1.3) *concentrates must be a critical point of V* (this result is converse to those of Floer, Weinstein and Oh). For the precise statements of our results, see Theorem 2.1, Theorem 2.3, Theorem 2.5 and Theorem 3.1.

We should mention a recent work of Ni and Takagi [NT] concerning the asymptotic behavior of least energy solutions of the Neumann problem for (1.3) (with $V \equiv 1$ but more general nonlinear term) on a bounded domain Ω. They proved in [NT] that positive least energy solutions must exhibit "point-condensation" character on the boundary $\partial\Omega$. More recently, they announced that they have proved that these solutions concentrate at a point on $\partial\Omega$ where the maximum mean curvature of $\partial\Omega$ is achieved. Some of our arguments in Sect. 2 are inspired by [NT].

Finally, we wish to take this opportunity to point out that the main results in [FW] and [O_1, O_3] are still true *as long as V is bounded* – we *do not* need $V \in$ Kato class $(V)_a$ in this case. (So V can be highly oscillatory at ∞ as in the case of $V(x) = \sin|x|^2$ or $\sin e^{|x|}$.) The reason is that $V \in (V)_a$ is only needed in these papers to show

$$\|(-\Delta + V(hx) - E)u\|_{L^2(\mathbb{R}^n)} \geqq \lambda \|u\|_{H^2(\mathbb{R}^n)}, \tag{1.6}$$

where $\lambda > 0$ is independent of $h > 0$, $V(hx) - E \geqq \delta > 0$ in $\mathbb{R}^n$ (see (2.9) in [FW], (21) in [O_1] and the inequality following (38) in [O_3]). (We shall present the short proof of (1.6) in the Appendix.) This answers a question raised by Oh [O_1] after he found a technical error in an argument of Floer and Weinstein concerning (2.9) in [FW] (which is the reason that he required $V \in (V)_a$ even when V is bounded).

Recently, Gui [G] pointed out that the nondegeneracy condition on critical points of V in [FW, O_1, O_3] can be weakened.

Section 2. Positive Ground States

To study (1.3) and (1.4), following common practice, we define E_h to be the Hilbert subspace of $H^1(\mathbb{R}^n)$ consisting of real-valued functions v such that

$$\|v\|_{E_h}^2 = \int_{\mathbb{R}^n} (|\nabla v|^2 + V_h v^2)\,dx < +\infty,$$

where $V_h(x) = V(hx)$. Since $\inf V > 0$ (as we always assume), E_h is imbedded continuously into $H^1(\mathbb{R}^n)$. We also define the energy functional associated with (1.4),

$$I_h(v) = \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla v|^2 + V_h v^2)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^n} |v|^{p+1}\,dx. \tag{2.1}$$

Then it is well-known (see, e.g. [DN]) that I_h is well-defined on E_h, $I_h \in C^1(E_h, \mathbb{R})$, and any critical point of I_h gives a classical solution of (1.4). To study critical points of I_h, we further introduce

$$M_h = \left\{ v \in E_h \backslash \{0\} \mid \int_{\mathbb{R}^n} (|\nabla v|^2 + V_h v^2) dx = \int_{\mathbb{R}^n} |v|^{p+1} dx \right\}$$

(M_h is called the *solution manifold* because all $H^1(\mathbb{R}^n)$ solutions of (1.4) must belong to M_h),

$$\Gamma_h = \{\eta \in C([0, 1], E_h) \mid \eta(0) = 0, \eta(1) \not\equiv 0, I_h(\eta(1)) \leqq 0\}$$

and the mountain pass minimax value

$$c_h = \inf_{\eta \in \Gamma_h} \max_{t \in [0,1]} I_h(\eta(t)) .$$

Then for any $v \in E_h \backslash \{0\}$, there exists a unique $\theta > 0$ such that

$$I_h(\theta v) = \max_{t \geqq 0} I_h(tv), \quad \theta v \in M_h ; \tag{2.2}$$

furthermore,

$$0 < c_h = \inf_{v \in M_h} I_h(v) = \inf_{v \in E_h \backslash \{0\}} \max_{t \geqq 0} I_h(tv) \tag{2.3}$$

(see, e.g., [NT] for the case of Neumann problem).

As mentioned in the Introduction, under the condition

$$\liminf_{x \to \infty} V(x) > V^0 \equiv \inf V > 0 . \tag{2.4}$$

Rabinowitz [R] proved for small h the existence of a classical solution v_h of (1.4) with $I_h(v_h) = c_h$. In view of (2.3), v_h is a minimizer of I_h on M_h. We remark that each minimizer v_h of I_h on M_h is of one sign. Indeed, by (2.2), there exists an $\theta > 0$ such that $\theta |v_h| \in M_h$ and

$$\begin{aligned} \max_{t \geqq 0} I_h(t|v_h|) = I_h(\theta |v_h|) &= \left(\frac{1}{2} - \frac{1}{p+1} \right) \theta^{p+1} \int_{\mathbb{R}^n} |v_h|^{p+1} dx \\ &= \theta^{p+1} I_h(v_h) = \theta^{p+1} c_h . \end{aligned}$$

This and (2.3) imply $\theta \geqq 1$. But since

$$\int_{\mathbb{R}^n} (|\nabla |v_h||^2 + V_h v_h^2) dx \leqq \int_{\mathbb{R}^n} (|\nabla v_h|^2 + V_h v_h^2) dx = \int_{\mathbb{R}^n} |v_h|^{p+1}$$

and $\theta v_h \in M_h$, we have $\theta \leqq 1$. Thus $\theta = 1$ and $|v_h| \in M_h$. Hence $|v_h|$ is also a minimizer of I_h on M_h. A routine argument implies $|v_h|$ is a classical solution of (1.4). Now by the strong maximum principle, $|v_h|$ never vanish and hence v_h is of one sign.

Throughout this section, we shall always assume v_h is chosen to be positive (note I_h is an even functional). Let

$$u_h(x) = v_h\left(\frac{x}{h} \right) . \tag{2.5}$$

Then u_h is a positive ground state of (1.3). Conversely, such a ground state u_h of (1.3) corresponds to a counterpart of (1.4). In this section, if not stated otherwise, u_h always stands for an arbitrary positive ground state of (1.3), u_h and v_h are always related by (2.5).

Theorem 2.1. *Suppose* (2.4) *holds. For each sequence* $h'_k \to 0$, *there exists a subsequence* $\{h_k\}$ *such that* $u_k \equiv u_{h_k}$ *concentrates at a global minimum point of* x_0 *of* V *in the following sense: For each large* $k > 0$, u_k *has only one local (hence global) maximum point* x_k, $x_k \to x_0$ *as* $k \to \infty$, *and for any* $\delta > 0$ *and large* k,

$$\max_{|x - x_0| \leqq \delta} u_k(x) > (V^0)^{\frac{1}{p-1}}, \tag{2.6}$$

$$u_k(x) \leqq C \left| \frac{x - x_k}{h_k} \right|^{\frac{1-n}{2}} \exp(-\sqrt{V^0}|x - x_k|/h_k) \quad \textit{for } |x - x_0| \geqq \delta, \tag{2.7}$$

where C *is independent of* k, δ, u_k *and* V *(but dependent of* V^0*).*

Remark 1. If the global minimum point of V is unique, then all ground states u_h concentrate at that point as $h \to 0^+$. In general, $V(x_h) \to V^0 \equiv \inf V$ as $h \to 0^+$, where x_h is the local maximum point of u_k whose uniqueness for small h is assured by this theorem.

Remark 2. By the proof of this theorem presented below,

$$u(h\cdot + x_h) \to u_0(\cdot) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n), L^\infty(\mathbb{R}^n) \text{ and } H^1(\mathbb{R}^n), \quad \text{as } h \to 0^+ ,$$

where u_0 is the unique solution of (2.8) below. In particular,

$$\| u_h \|_{L^\infty(\mathbb{R}^n)} \to \| u_0 \|_{L^\infty(\mathbb{R}^n)} = u_0(0) \quad \text{as } h \to 0^+ .$$

The proof of Theorem 2.1 will be lengthy, but the basic idea is to compare v_h to the positive solution u_0 of

$$\Delta u - V^0 u + u^p = 0, \quad u > 0, \quad u(\infty) = 0, \quad u(0) = \max u . \tag{2.8}$$

By [GNN], u_0 is radial, $u'_0(r) < 0$ for $r \neq 0$ and

$$u_0(r), |u'_0(r)| \leqq C r^{-\frac{(n-1)}{2}} e^{-\sqrt{V^0} r} . \tag{2.9}$$

By [K], u_0 is unique. Define E^0, $I^0(v)$ and M^0 by replacing V_h by the constant V^0 in the definitions of E_h, $I_h(v)$ and M_h. Define Γ^0 and c^0 by modifying the definitions of Γ_h and c_h in the obvious way. Then (2.2) and (2.3) with the corresponding modifications hold true. Moreover, $0 < c^0 = I^0(u_0)$, and u_0 and its translations are the only positive critical points of the functional I^0 (by [GNN] and [K]).

To prove Theorem 2.1, we shall need

Lemma 2.2. $\lim_{h \to 0^+} c_h = c^0$.

Remark. What we only need in proving this lemma is $V^0 > 0$ – no requirement on $\liminf_{x \to \infty} V$.

Proof of Lemma 2.2. For any $R>0$, take a $\varphi_R \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_R \equiv 1$ on $B_R(0)=\{|x|\leqq R\}$, $\varphi_R \equiv 0$ in $B_{R+1}^c(0)$, $0 \leqq \varphi_R \leqq 1$, $|\nabla \varphi_R| \leqq c(n)$. Let $v_R = \varphi_R u_0$. Take a sequence y_k such that $V(y_k)\to V^0$. Let $w(x) = v_R\left(x - \frac{y_k}{h}\right)$. Then there exists a unique $\theta > 0$ such that $\theta w \in M_h$, i.e.,

$$\theta^2 \int_{\mathbb{R}^n} (|\nabla w|^2 + V(hx)w^2)\,dx = \theta^{p+1} \int_{\mathbb{R}^n} w^{p+1}\,dx\,,$$

$$\int_{\mathbb{R}^n} (|\nabla v_R|^2 + V(hx+y_k)v_R^2)\,dx = \theta^{p-1} \int_{\mathbb{R}^n} v_R^{p+1}\,dx\,.$$

Whence,

$$\theta^{p-1} = \frac{\int_{\mathbb{R}^n} |\nabla v_R|^2 + V^0 v_R^2}{\int_{\mathbb{R}^n} v_R^{p+1}} + \frac{\int_{\mathbb{R}^n} (V(hx+y_k) - V^0) v_R^2}{\int_{\mathbb{R}^n} v_R^{p+1}}\,.$$

$$= I_1 + I_2\,.$$

Since u_0 is a solution of (2.8), it is easy to see $I_1 \to 1$ as $R\to\infty$. Also, for a fixed R, if we take a large k so that $V(y_k)$ is close to V^0 and fix such a k, then I_2 is small as $h \to 0^+$. Thus when R and k are taken large and fixed, θ is close to 1 as $h\to 0^+$. Observe

$$c_h = \inf_{M_k} I_h(v) \leqq I_h(\theta w)$$

$$= \theta^2\left[I_h(w) + \frac{1-\theta^{p-1}}{p+1} \int_{\mathbb{R}^n} w^{p+1}\,dx \right]$$

$$= \theta^2\left[I^0(w) + \frac{1}{2}\int_{\mathbb{R}^n} (V(hx) - V^0) w^2\,dx + \frac{1-\theta^{p-1}}{p+1} \int_{\mathbb{R}^n} w^{p+1}\,dx \right]$$

$$= \theta^2\left[I^0(v_R) + \frac{1}{2}\int_{\mathbb{R}^n} (V(hx+y_k) - V^0) v_R^2\,dx + \frac{1-\theta^{p-1}}{p+1} \int_{\mathbb{R}^n} v_R^{p+1}\,dx \right].$$

Obviously $I^0(v_R) \to I^0(u_0) = c^0$ and $R\to\infty$. This and the property of θ discussed above imply that the last quantity in the above inequalities is close to c^0 if R and k are taken large and fixed, and then we let $h\to 0^+$. Thus $\limsup_{h\to 0^+} c_h \leqq c^0$. On the other hand, since $I_h(v) \geqq I^0(v)$ for $v \in E_h$, we have $c_h \geqq c^0$. Now the desired conclusion follows. #

Proof of Theorem 2.1. First, we observe that since

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} (|\nabla v_h|^2 + V_h(x) v_h^2)\,dx = I_h(v_h) = c_h \to c^0$$

as $h\to 0^+$, $\|v_h\|_{H^1(\mathbb{R}^n)}$ is bounded as $h \to 0^+$.

Claim 1. There exists a sequence $\{y_h\}$ and positive constants R and β such that

$$\liminf_{h \to 0^+} \int_{B_R(y_h)} v_h^2(x)dx \geqq \beta > 0 . \tag{2.10}$$

For otherwise, for any $R > 0$, there exists a sequence $v_k \equiv v_{h_k}$ such that

$$\lim_{k \to \infty} \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} v_k^2(x)dx = 0 .$$

By Lemma I.1 in [L$_2$] or Lemma 2.18 in [CR], $v_k \to 0$ in $L^q(\mathbb{R}^n)$ for any $2 < q < \frac{2n}{(n-2)^+}$. This is impossible because

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} v_h^{p+1} dx = c_h \to c^0 \quad \text{as } h \to 0 .$$

Claim 1 is proved.

Now let $w_h(x) = v_h(x + y_h) = u_h(hx + hy_h)$. Then by (2.10),

$$\liminf_{h \to 0^+} \int_{B_R(0)} w_h^2(x)dx \geqq \beta > 0 . \tag{2.11}$$

Furthermore,

$$\Delta w_h - V(hx + hy_h)w_h + w_h^p = 0, \quad w_h > 0 \text{ in } \mathbb{R}^n , \tag{2.12}$$

$$c_h = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} (|\nabla w_h|^2 + V(hx + hy_h)w_h^2)dx$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} w_h^{p+1} dx . \tag{2.13}$$

Claim 2. hy_h is bounded for small $h > 0$. Otherewise, there exists a sequence $h_m \to 0^+$ such that $h_m y_{h_m} \to \infty$. By (2.13) and Lemma 2.2, $w_m \equiv w_{k_m}$ is bounded in $H^1(\mathbb{R}^n)$. Hence by passing to a subsequence if necessary, $w_m \to w_0 \geqq 0$ weakly in $H^1(\mathbb{R}^n)$, strongly in $L^q_{\text{loc}}(\mathbb{R}^n)$ $\left(2 < q < \frac{2n}{(n-2)^+}\right)$ and a.e. in $\mathbb{R}^n$. By (2.11), $w_0 \not\equiv 0$. So there exists a $\theta > 0$ such that $\theta w_0 \in M^0$. On the other hand, since $\liminf_{x \to \infty} V(x) > V^0$, it is easy to see from (2.12) that there exists $\varepsilon > 0$ such that

$$\Delta w_0 - (V^0 + \varepsilon)w_0 + w_0^p \geqq 0 \quad \text{in the } H^{-1}(\mathbb{R}^n) \text{ sense} .$$

In particular

$$\int_{\mathbb{R}^n} (|\nabla w_0|^2 + V^0 w_0^2)dx < \int_{\mathbb{R}^n} w_0^{p+1} dx .$$

Thus $0 < \theta < 1$. Now, we have

$$\begin{aligned} c^0 &= \inf_{M^0} I^0(v) \leqq I^0(\theta w_0) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} (\theta w_0)^{p+1} dx \\ &< \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} w_0^{p+1} dx \\ &\leqq \liminf_{m \to \infty} \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} w_m^{p+1} dx \qquad \text{(Fatou's Lemma)} \\ &= \lim_{m \to \infty} I_{h_m}(w_m) = c^0 \qquad \text{((2.13) and Lemma 2.2)}\,. \end{aligned}$$

This is impossible and hence Claim 2 is proved.

Now for any sequence $h'_k \to 0$, there exists a subsequence h_k such that $\bar{x}_k \equiv h_k y_{h_k} \to x_0$, $w_k \equiv w_{h_k} \to w_0 \geqq 0, \not\equiv 0$, weakly in $H^1(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$.

Claim 3. x_0 is a global minimum point of V.

By applying the elliptic regularity theory to (2.12), we have $w_k \to w_0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\Delta w_0 - V(x_0) w_0 + w_0^p = 0, \quad x \in \mathbb{R}^n\,. \tag{2.14}$$

So

$$\int_{\mathbb{R}^n} (|\nabla w_0|^2 + V^0 w_0^2)\, dx \leqq \int_{\mathbb{R}^n} (|\nabla w_0|^2 + V(x_0) w_0^2)\, dx = \int_{\mathbb{R}^n} w_0^{p+1} dx\,.$$

Therefore, there exists a $0 < \theta \leqq 1$ such that $\theta w_0 \in M^0$. Now

$$\begin{aligned} c^0 = \lim_{k \to \infty} c_{h_k} &= \lim_{k \to \infty} \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} w_k^{p+1} dx \\ &\geqq \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} w_0^{p+1} dx \\ &\geqq \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n} (\theta w_0)^{p+1} dx \\ &= I^0(\theta w_0) \geqq \inf_{M^0} I^0(v) \geqq c^0\,. \end{aligned}$$

Thus $\theta = 1$ and hence $V(x^0) = V^0 (= \inf V)$. The proof of Claim 3 is complete.

Now observe

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla w_0|^2 + V(x_0)w_0^2\,dx &\leqq \liminf_{k\to\infty} \int_{\mathbb{R}^n} (|\nabla w_k|^2 + V(x_0)w_k^2)\,dx\\ &\leqq \limsup_{k\to\infty} \int_{\mathbb{R}^n} (|\nabla w_k|^2 + V(x_0)w_k^2)\,dx\\ &\leqq \limsup_{k\to\infty} \int_{\mathbb{R}^n} (|\nabla w_k|^2 + V(h_k x + \bar{x}_k)w_k^2)\,dx\\ &\qquad\text{(by Claim 3)}\\ &= \lim_{k\to\infty} \frac{c_{h_k}}{\left(\frac{1}{2}-\frac{1}{p+1}\right)} = \frac{c^0}{\left(\frac{1}{2}-\frac{1}{p+1}\right)}\\ &= \int_{\mathbb{R}^n} (|\nabla w_0|^2 + V(x_0)w_0^2)\,dx\,.\end{aligned}$$

Thus $\int_{\mathbb{R}^n} |\nabla w_k|^2 + V(x_0)w_k^2\,dx \to \int_{\mathbb{R}^n} |\nabla w_0|^2 + V(x_0)w_0^2$ and hence $w_k \to w_0$ strongly in $H^1(\mathbb{R}^n)$. In particular,

$$\int_{|x|\geqq R} w_k^{2^*}\,dx \to 0 \text{ as } R\to\infty \text{ uniformly w.r.t. } k\,, \tag{2.15}$$

where $2^* = \frac{2n}{n-2}$ if $n \geqq 3$, and an arbitrary large number if $n = 1, 2$. Note w_k is a subsolution of $\Delta u + c(x)u = 0$ with $c(x) = w_k^{p-1}$. By the one-sided Harnack inequality (see [T]), we have

$$\max_{B_1(Q)} w_k \leqq c\left(\int_{B_2(Q)} w_k^{2^*}\,dx\right)^{\frac{1}{2^*}},$$

where Q is an arbitrary point in $\mathbb{R}^n$, c is a constant depending only on n and the bound of $\|w_k\|_{L^{2^*}(B_2(Q))}$. Thus by (2.15),

$$w_k(x) \to 0 \quad \text{as } |x|\to\infty \text{ uniformly w.r.t. } k\,.$$

Hence $u_k(x) \equiv u_{h_k}(x) = w_k\left(\frac{x-\bar{x}_k}{h_k}\right)$ decays to zero uniformly for x outside any fixed neighborhood of x_0 as $k\to\infty$. Let x_k be a local maximum point of u_k. By (1.3) and the strong maximum principle, $u_k(x_k) > (V^0)^{\frac{1}{p-1}}$. Therefore $x_k \to x_0$ as $k\to\infty$.

It remains to show the uniqueness of x_k and (2.7). Let $\bar{w}_k(x) = u_k(h_k x + x_k)$. Then

$$\Delta \bar{w}_k(x) - V(h_k x + x_k)\bar{w}_k + \bar{w}_k^p = 0, \quad \bar{w}_k > 0 \quad \text{in } \mathbb{R}^n\,, \tag{2.16}$$

O is a critical point of $\bar{w}_k$ and $\bar{w}_k(0) > (V^0)^{\frac{1}{p-1}}$. The arguments similar to those concerning w_k presented above show that after passing to a subsequence of $\{\bar{w}_k\}$,

$\bar{w}_k \to w_0$ in $C^2_{\text{loc}}(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ where $w_0 \not\equiv 0$ satisfies (2.14); furthermore, $\bar{w}_k \to 0$ as $x \to \infty$ uniformly w.r.t. k. Since O is a critical point of $\bar{w}_k$, it is also a critical point of w_0. By [GNN], w_0 is spherically symmetric about some point P, and $w_0'(s) < 0$ for $0 \neq s = |x - P|$. Thus $P =$ the origin, i.e., w_0 is radial. Since $\bar{w}_k$ decays to zero in x uniformly w.r.t. k, so does w_0. Now Kwong's result [K] says $w_0 \equiv u_0$ (see (2.8)).

To show the uniqueness of the local maximum point of u_k, it suffices to do so for $\bar{w}_k$. Observe at each local maximum point of $\bar{w}_k$, $\bar{w}_k > (V^0)^{\frac{1}{p-1}}$. Since $\bar{w}_k$ decays to zero uniformly w.r.t. k as $x \to \infty$, all the local maximum points of $\bar{w}_k$ stay in a finite ball in $\mathbb{R}^n$. Since $\bar{w}_k \to w_0 = u_0$ in $C^2_{\text{loc}}(\mathbb{R}^n)$ and O is the only critical point of u_0, these points must approach the origin and hence stay in a small ball $B_\varepsilon(0)$ as $k \to \infty$. As can be easily seen, we can take ε so small that $u_0''(r) < 0$ for $0 \leqq r \leqq \varepsilon$. Now by Lemma 4.2 in [NT], for large k, $\bar{w}_k$ has no critical points other than the origin.

To show (2.7), we recall that $\bar{w}_k(x)$ decays to zero at $x = \infty$ uniformly w.r.t. k. Applying the argument in the proof of Proposition 4.1 in [GNN] (see also Kato [Ka]), we have

$$\bar{w}_k(x) \leqq C|x|^{\frac{1-n}{2}} \exp(-\sqrt{V^0}|x|) \quad \text{for } |x| \geqq 1 \text{ and large } k\,,$$

where C is independent of k, $\bar{w}_k$ and V (but dependent of V^0). This implies (2.7). #

Our next result deals with the situation when we do not know if $\liminf_{x \to \infty} V(x) > \inf V$. In this case, (1.3) may or may not have a ground state: in the trivial case $V \equiv \text{constant} > 0$, (1.3) has infinitely many positive ground states, while when for some direction, the directional derivative of V is nonnegative but not identically zero, then it is possible to prove by integration by parts that (1.3) does not even possess an $H^1(\mathbb{R}^n)$ solution $\not\equiv 0$ (see the proof of Theorem 3.1 in Sect. 3). However, we have

Theorem 2.3. *Suppose there exists a sequence $h_k \to 0^+$ such that* (1.3) *with $h = h_k$ has a positive ground state u_k and u_k has a local maximum point x_k which converges to some point x_0 as $k \to \infty$. Then x_0 must be a global minimum point of V, x_k is the only local (hence global) maximum point of u_k for k large. Moreover,* (2.6) *and* (2.7) *hold true.*

Proof. Since the details of this proof are similar to those of the proof of Theorem 2.1, we shall only sketch the main steps. Let $w_k(x) = u_k(h_k x + x_k)$.

Step 1. Show x_0 is a global minimum point of V by arguing as in the proof of Claim 3 in the proof of Theorem 2.1.

Step 2. By using the conclusion in *Step 1*, show $w_k \to u_0$ in $H^1(\mathbb{R}^n)$ as $k \to \infty$, where u_0 satisfies (2.8).

Step 3. By using the conclusion in *Step 2* and the one-sided Harnack inequality, show w_k decays to zero at $x = \infty$ uniformly w.r.t. k. Then (2.7) follows.

Step 4. Show the uniqueness of x_k. #

In the remaining part of this section, we deal with the case when the potential V is radial and is bounded below from zero. Define E_h^r and M_h^r to be the subset of E_h and M_h consisting of radial functions. Define

$$c_h^r = \inf_{M_h^r} I_h(v) .$$

It is routine to see $c_h^r > 0$. Then

Proposition 2.4. *Assume $n \geqq 2$. Then for each $h > 0$, c_h^r is assumed by a minimizer v_h of I_h over M_h^r.*

Remark. As we mentioned in Sect. 1, this follows essentially from a general result in [DN]. However, in the present case, we can prove this result by showing that a minimizing sequence converges to some $v_h \in M_h^r$ by virtue of the standard fact that $E_h^r \subsetneq L^q$ compactly with $2 < q < \frac{2n}{(n-2)^+}$. An alternative is to apply the Mountain Pass type argument as in [R] and the well-known Radial Lemma of Strauss.

It is easy to see v_h is also a (radial) solution of (1.4). As before, we can show that v_h is of one sign and hence we shall always assume that $v_h > 0$. Let $u_h(r) = v_h\left(\frac{r}{h}\right)$ $(r = |x|)$. Then u_h is a positive radial solution of (1.3) with least energy (1.5) among all nontrivial radial $H^1(\mathbb{R}^n)$ solutions. Conversely, each such radial ground state of (1.3) corresponds to a v_h in the statement of Proposition 2.4.

Theorem 2.5. *Suppose $n \geqq 2$ and V is radial. Let u_h be a positive radial ground state of* (1.3) (*whose existence is assured by Proposition* 2.4). *Then u_h concentrates at the origin in the following sense: For small $h > 0$, u_h has only one local* (*hence global*) *maximum point x_h, $x_h \to 0$ as $h \to 0$; moreover* (2.6) *and* (2.7) *with $x_0 = x_k = 0$, $h_k = h$ and $u_k = u_h$ hold true.*

Remark. If $V(0)$ is not a global minimum of V, by combining Theorem 2.3 and Theorem 2.5, we see that a ground state, i.e., least energy solution of (1.3), if any, must not be radial for small $h > 0$.

Proof of Theorem 2.5. By the proof of Lemma 2.2, c_h^r is bounded for $1 \geqq h > 0$. Thus $\|v_h\|_{H^1(\mathbb{R}^n)}$ is also bounded for $1 \geqq h > 0$. Recall the Radial Lemma of Strauss [S]:

$$|u(r)| \leqq C r^{\frac{1-n}{2}} \|u\|_{H^1(\mathbb{R}^n)} \quad \text{for } r \geqq 1 ,$$

where C depends on n. Using this, we have that v_h decays at $r = \infty$ uniformly w.r.t. small $h > 0$. Now applying the arguments in the proof of Proposition 4.1 in [GNN] to (1.4), one has

$$v_h(r) \leqq C r^{\frac{1-n}{2}} \exp(-\sqrt{V^0} r) \quad \text{for } r \geqq 1 , \tag{2.17}$$

from which (2.7) with the modification follows.

If x_h is a local maximum point of u_h, then by the maximum principle, $u_h(x_h) > (V^0)^{\frac{1}{p-1}}$. This and (2.7) imply $x_h \to 0$ as $h \to 0$. To show x_h is unique, observe that since $\|v_h\|_{H^1(\mathbb{R}^n)}$ is bounded for small h and v_h satisfies (1.4), each sequence of $\{v_h\}$ contains a subsequence $\{v_k\}$ such that $v_k \to$ some $\bar{u}_0$ weakly in $H^1(\mathbb{R}^n)$ and strongly in $C^2_{\text{loc}}(\mathbb{R}^n)$, where $\bar{u}_0$ is a radial solution of

$$\Delta u - V(0)u - u^p = 0, \quad u > 0, \quad u(\infty) = 0 . \tag{2.18}$$

By Kwong [K], $\bar{u}_0$ is unique. Thus $v_h \to \bar{u}_0$ in $C^2_{\text{loc}}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ as $h \to 0^+$. Now we can show the uniqueness of x_h as in the proof of Theorem 2.1. #

Remark. By the proof above, $\|u_h\|_{L^\infty(\mathbb{R}^n)} \to \|\bar{u}_0\|_{L^\infty} = \bar{u}_0(0)$, where $\bar{u}_0$ is the unique radial solution of (2.18).

Section 3. Positive Bound States

The purpose of this section is to show that a point at which a sequence of positive bound states concentrates must be a critical point of V. Recall Floer, Weinstein and Oh have obtained results converse to this. Recall also by a bound state, we mean an $H^1(\mathbb{R}^n)$ solution of (1.3) with finite energy (1.5). (It is well-known that a (positive) bound state decays exponentially at ∞, as may be proved by the one-sided Harnack inequality and then using a result of [GNN].)

Theorem 3.1. *Assume* $|\nabla V(x)| = O(e^{a|x|})$ *at* $x = \infty$ *for some* $a > 0$. *Let* $u_k \equiv u_{h_k}$ *be a sequence of positive bound states of* (1.3) *with* $h = h_k$. *Suppose* u_k *concentrates at a point* x_0 *in the following sense:* $\forall \varepsilon > 0$, $\exists$ *constants* R *and* $K > 0$ *such that*

$$u_k(x) \leqq \varepsilon \quad \text{for } k \geqq K \text{ and } |x - x_0| \geqq h_k R . \tag{3.1}$$

Then $\nabla V(x_0) = 0$. *Moreover, for large* k, u_k *has only one local maximum point* x_k, $x_k \to x_0$ *as* $k \to \infty$, *and* (2.6)–(2.7) *with* x_k *replaced by* x_0 *hold true.*

Remark 1. It is possible to check the solutions obtained by Floer, Weinstein and Oh [FW, O_1] satisfy (3.1).

Remark 2. We suspect that the condition $|\nabla V(x)| = O(e^{a|x|})$ could be removed.

Proof of Theorem 3.1. We shall break up this proof into 3 steps; In Step 1, we show $\|u_k\|_{L^\infty(\mathbb{R}^n)}$ is bounded. In Step 2, we show the second part of this theorem; In Step 3, we prove $\nabla V(x_0) = 0$.

Step 1. Suppose $\exists$ a sequence $h_m \to 0$ so that the $L^\infty(\mathbb{R}^n)$ norm of $u_m = u_{h_m}$ tends to ∞ as $m \to \infty$. Let $\alpha_m = \max u_m$ and $\beta_m = \alpha_m^{-(p-1)/2}$. Define

$$v_m(x) = \frac{1}{\alpha_m} u_m(x_m + h_m \beta_m x) ,$$

where x_m is a global maximum point of u_m. Then

$$\Delta v_m - \beta_m^2 V(x_m + h_m \beta_m x) v_m + v_m^p = 0, \quad x \in \mathbb{R}^n ,$$

and

$$v_m(0) = 1, \quad 0 \leqq v_m \leqq 1 .$$

Therefore, by the elliptic regularity theory, we have

$$v_m \to v_0 \text{ is } C^2_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } m \to \infty ,$$

where

$$\Delta v_0 + v_0^p = 0 \quad \text{in } \mathbb{R}^n, \quad v_0(0) = 1 .$$

This is impossible according to [CL] or [CGS]. Step 1 is finished.

Step 2. Let $w_k(x) = u_k(x_0 + h_k x)$. Then

$$\Delta w_k - V(x_0 + h_k x) w_k + w_k^p = 0, \quad x \in \mathbb{R}^n , \tag{3.2}$$

and by (3.1), w_k decays to zero uniformly w.r.t. k. Now, as before, by the arguments in the proof of Proposition 4.1 in [GNN],

$$w_k(x) \leqq C|x|^{\frac{1-n}{2}} \exp(-\sqrt{V^0}|x|) \quad \text{for } |x| \geqq 1 . \tag{3.3}$$

This implies (2.7) with x_k there replaced by x_0. Observe by the conclusion in Step 1 and the elliptic regularity, for any subsequence of $\{w_k\}$, there exists a subsequence $\{w'_k\}$ of the subsequence such that $w'_k \to$ some w_0 in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, where w_0 satisfies

$$\Delta w_0 - V(x_0) w_0 + w_0^p = 0, \quad x \in \mathbb{R}^n . \tag{3.4}$$

By (3.3) and the fact that local maximum values of w_k are larger than $(V^0)^{\frac{1}{p-1}}$ we see local maximum points of w_k must stay in a fixed ball for all k. This and the fact that $w'_k \to w_0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ imply $w_0 \not\equiv 0$ and hence positive by the strong maximum principle. Now we can show the uniqueness of local maximum point of w'_k for k large as in the proof of Theorem 2.1. From this the uniqueness of x_k follows. By (3.1), $x_k \to x_0$. Equation (2.6) follows from the strong maximum principle. Now Step 2 is complete.

Step 3. Without loss of any generality, assume $w_k \to w_0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. Multiplying (3.2) by ∇w_k and integrating on $B_R(0)$, we have

$$0 = \int_{B_R} \Delta w_k \nabla w_k - \frac{1}{2} \nabla(V(x_0 + h_k x) w_k^2) + \frac{1}{2} h_k \nabla V(x_0 + h_k x) w_k^2 + \frac{\nabla w_k^{p+1}}{p+1} dx ,$$

$$\frac{h_k}{2} \int_{B_R} \nabla V(x_0 + h_k x) w_k^2 \, dx = \int_{\partial B_R} \left(\frac{1}{2} V(x_0 + h_k x) w_k^2 v - \frac{1}{p+1} w_k^{p+1} v \right) ds - \int_{B_R} \Delta w_k \nabla w_k \, dx , \tag{3.5}$$

where ν is the exterior normal field on ∂B_R. We compute the second integral on the right-hand side as follows (write w_k as w):

$$\int_{B_R} \Delta w \frac{\partial w}{\partial x_i} dx = \int_B \left(\operatorname{div}\left(\nabla w \frac{\partial w}{\partial x_i} \right) - \nabla w \frac{\partial}{\partial x_i} \nabla w \right) dx$$

$$= \int_{\partial B} \left(\frac{\partial w}{\partial x_i} \frac{\partial w}{\partial \nu} - \nu_i |\nabla w|^2/2 \right) dS .$$

Therefore (3.5) becomes

$$\frac{h_k}{2} \int_{B_R} \nabla V(x_0 + h_k x) w_k^2 \, dx = \int_{\partial B_R} \left(\frac{1}{2} V(x_0 + h_k x) w_k^2 \nu - \frac{1}{p+1} w_k^{p+1} \nu - \nabla w_k \frac{\partial w_k}{\partial \nu} + \nu |\nabla w_k|^2/2 \right) dS \equiv I_R . \tag{3.6}$$

Observe

$$\int_0^\infty |I_R| \, dR \leqq \int_0^\infty dR \int_{\partial B_R} \left(\frac{3}{2} (|\nabla w_k|^2 + V(x_0 + h_k x)) w_k^2 + \frac{1}{p+1} w_k^{p+1} \right) dS$$

$$\leqq \frac{3}{2} \int_{\mathbb{R}^n} ((|\nabla w_k|^2 + V(x_0 + h_k x) w_k^2 + w_k^{p+1}) dx$$

$$< +\infty \text{ for each } k ,$$

by the assumption that u_k is a bound state. Thus for each fixed k there exists a sequence $R_m \to \infty$ such that $I_{R_m} \to 0$ as $m \to \infty$. Now letting $R = R_m \to \infty$ in (3.6), by virtue of (3.3), the growth condition on $|\nabla V|$ and the Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^n} \nabla V(x_0 + h_k x) w_k^2 \, dx = 0 .$$

Now letting $h_k \to 0$ and by the Dominated Convergence Theorem again, we have

$$\int_{\mathbb{R}^n} \nabla V(x_0) w_0^2 \, dx = 0 .$$

Thus $\nabla V(x_0) = 0$. #

Remark. Equation (3.6) may be deduced from the dilated Pohozaev identity which appeared in, e.g., [H].

Appendix

We shall prove (1.6) when V is bounded. Let $\bar{V}(x) = V(hx) - E$. For $u \in H^2(\mathbb{R}^n)$, let

$$f = -\Delta u + \bar{V}(x) u . \tag{A.1}$$

Multiplying (A.1) by u and integrating on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |\nabla u|^2 + \bar{V}(x)u^2\,dx = \int_{\mathbb{R}^n} fu\,dx \leqq \|f\|_{L^2(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)}\,.$$

Recalling $\bar{V} \geqq \delta > 0$, we then have

$$\|u\|_{L^2(\mathbb{R}^n)} \leqq \frac{1}{\delta}\|f\|_{L^2(\mathbb{R}^n)}\,.$$

By this and (A.1), we obtain

$$\begin{aligned}\|\Delta u\|_{L^2(\mathbb{R}^n)} &= \|f - \bar{V}u\|_{L^2(\mathbb{R}^n)} \leqq \|f\|_{L^2(\mathbb{R}^n)} + \|\bar{V}\|_{L^\infty(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)}\\ &\leqq (1 + \|\bar{V}\|_{L^\infty(\mathbb{R}^n)}/\delta)\|f\|_{L^2(\mathbb{R}^n)}\,.\end{aligned}$$

So

$$\begin{aligned}\|u\|_{H^2(\mathbb{R}^n)} &= \|u\|_{L^2(\mathbb{R}^n)} + \|\Delta u\|_{L^2(\mathbb{R}^n)}\\ &\leqq (1 + (\|\bar{V}\|_{L^\infty(\mathbb{R}^n)} + 1)/\delta)\|f\|_{L^2(\mathbb{R}^n)}\,.\end{aligned}$$

Choosing λ in the obvious way, we obtain (1.6). Note the dependence of λ on $\|\bar{V}\|_{L^\infty(\mathbb{R}^n)}$ does not change the validity of the arguments in [FW], [O_1] and [O_3].

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