

On Concomitants of Order Statistics for Bivariate Pseudo Exponential Distribution

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Abstract: In this paper we have obtained the distribution of concomitant of order statistics for Bivariate Pseudo-Exponential distribution. The expression for moments has also been obtained. We have found that only the fractional moments of the distribution exist. The tables for survival function and hazard function has also been obtained.

Key words: Concomitants . order statistics . pseudo exponential distribution

INTRODUCTION

Let $x_{[1:n]} \leq x_{[2:n]} \leq \dots \leq x_{[n:n]}$ be the ordered sample from some distribution function $F(x)$. The r – th observation in this ordered sample is called the r – th order statistics. The density function of $x_{[r:n]}$ is given by David and Nagaraja [2] as:

$$f_{[r:n]}(x) = \frac{n!}{(r-1)!(n-r)!} f(x) [F(x)]^{r-1} [1-F(x)]^{n-r} \quad (1.1)$$

The joint distribution of two order statistics $x_{[r:n]} \leq x_{[s:n]}$ is obtained as:

$$f_{[r,s:n]}(x_1, x_2) = C_{r,s,n} f(x_1) f(x_2) [F(x_1)]^{r-1} [F(x_2) - F(x_1)]^{s-r-1} [1-F(x_2)]^{n-s} \quad (1.2)$$

where

$$C_{r,s,n} = \frac{n!}{(r-1)!(s-r)!(n-s)!} \quad (1.3)$$

It is often be the case that we have a random sample from a Bivariate distribution having distribution function $F(x, y)$ and the sample is arranged with respect to the variable X and the observations of second variable are automatically arranged. This second variable is called the concomitants of order statistics of variable X . The r -th concomitant is generally denoted as $y_{[r]}$. The concomitants of order statistics has been discussed by various authors. The distribution of r -th concomitant is given by [2] as:

$$g_{[r]}(y) = \int_{-\infty}^{\infty} f(y|x) f_{[r:n]}(x) dx \quad (1.4)$$

where $f_{[r:n]}(x)$ is defined in (1.1). The joint distribution of two concomitant is also given by David and Nagaraja (2003) as:

$$g_{[r,s]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f(y_1|x_1) f(y_2|x_2) f_{[r,s:n]}(x_1, x_2) dx_1 dx_2 \quad (1.5)$$

Where $f_{[r,s:n]}(x_1, x_2)$ is given in (1.2).

Many authors have discussed the concomitants of order statistics from time to time. A comprehensive review of the topic has been given by [2] alongside the recent developments of the topic. The distribution of rank of concomitant of order statistics has also been discussed by David *et al.* [3]. Nagaraja and David [6] has obtained the distribution of maximum of concomitant of order statistics. The asymptotic behavior of the distribution of concomitant of order statistics has been studied by Chu *et al.* [1]. Distribution of concomitant in case of multivariate distributions has been studied by Wang *et al* [7].

In the following section we have defined the Pseudo-Exponential distribution alongside some of its basic properties.

BIVARIATE PSEUDO-EXPONENTIAL DISTRIBUTION

The Pseudo distributions are new induction to the statistical probability distributions. The idea has been developed by [4] by introducing the Pseudo-Weibull and Pseudo-Gamma distributions as linear combinations of random variables. We have defined the Bivariate Pseudo-exponential Distribution as the compound distribution of two random variables. The distribution is given below:

Let the random variable X has an exponential distribution with parameter α . The density function of X is:

$$f(x; \alpha) = \alpha e^{-\alpha x}, \alpha > 0, x > 0 \quad (2.1)$$

Further, suppose that the random variable Y also has the exponential distribution with parameter $\phi(x)$, where $\phi(x)$ is some function of random variable X . The density function of Y is; therefore; given as:

$$f\{y; \phi(x)|x\} = \phi(x)e^{-\phi(x)y}; \phi(x) > 0, y > 0 \quad (2.2)$$

We define the compound distribution of (2.1) and (2.2) as the Bivariate Pseudo-Exponential distribution. The distribution has the density:

$$f(x, y) = \alpha \phi(x) \exp[-\{\alpha x + \phi(x) y\}]; \quad (2.3)$$

$$\alpha > 0, \phi(x) > 0, x > 0, y > 0$$

Different choices of $\phi(x)$ can lead us to different Bivariate distributions in (2.3). Using $\phi(x) = x$ in (2.3), the distribution becomes:

$$f(x, y) = \alpha x \exp[-x\{\alpha + y\}]; \quad (2.4)$$

$$\alpha > 0, x > 0, y > 0$$

The product moments of (2.4) are given as:

$$\begin{aligned} \mu'_{p,q} &= \int_0^\infty \int_0^\infty x^p y^q f(x, y) dx dy \\ &= \int_0^\infty \int_0^\infty x^p y^q \alpha x \exp[-x(\alpha + y)] dx dy \end{aligned}$$

which after simplification becomes:

$$\mu'_{p,q} = \alpha^{q-p} \Gamma(p-q+1) \Gamma(q+1) \quad (2.5)$$

The product moments exist for $q < p+1$.

In the following section the distribution of concomitant of record statistics has been derived for (2.4).

DISTRIBUTION OF r-th CONCOMITANT

The Bivariate Pseudo-Exponential distribution is given in (2.4). The distribution of r -th concomitant for (2.4) can be obtained by using (1.4). Now to obtain the distribution (1.4) we first find the distribution of r -th

order statistics for the marginal distribution of X given in (2.1). The distribution is given as:

$$\begin{aligned} f_{[rn]}(x) &= \frac{n!}{(r-1)!(n-r)!} \alpha e^{-\alpha x} [1 - e^{-\alpha x}]^{r-1} [e^{-\alpha x}]^{n-r} \\ &= \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} e^{-\alpha x(n-r+h+1)} \end{aligned}$$

Also the conditional distribution of Y given X ; for (2.4) is given as:

$$f(y|x) = xe^{-xy}; x, y > 0 \quad (3.2)$$

Now using (3.1) and (3.2) in (1.4), the distribution of r -th concomitant of order statistics for Bivariate Pseudo-Exponential distribution is obtained as:

$$\begin{aligned} f_r(y) &= \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \int_0^\infty xe^{-x\{y+\alpha(n-r+h+1)\}} dx \\ &= \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} [y + \alpha(n-r+h+1)]^{-2} \end{aligned}$$

Using Gradshteyn and Ryzhik [5], the density function can be written as:

$$f_r(y) = \frac{\Gamma(n+1) \Gamma(n-r+y/\alpha+1) [H_{(n+y/\alpha)} - H_{(n-r+y/\alpha)}]}{\alpha \Gamma(n-r+1) \Gamma(n+y/\alpha+1)} \quad (3.3)$$

where $H_{(n)}$ is Harmonic number of order n .

The expression for k -th moment of (3.3) is obtained as:

$$\mu'_k = \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \int_0^\infty y^k [y + \alpha(n-r+h+1)]^{-2} dy$$

After simplification, the k – th moment of r – th concomitant is:

$$\mu'_k = \frac{\alpha n! k \pi \text{Cosec}(\pi k)}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \left\{ \alpha(n-r+h+1) \right\}^{k-1} \quad (3.4)$$

The moments exist for $|k| < 1$. The following table contains the quantities μ'_k for $n = 10$ and $k = 0.5$. The table has been constructed for various values of α and r .

Table 1: The fractional moment of concomitants

α	R	1	2	3	4	5	6	7	8	9	10
0.1		1.5708	0.7647	0.5569	0.4492	0.3788	0.3266	0.2845	0.2479	0.2132	0.1746
0.2		2.2214	1.0815	0.7876	0.6352	0.5357	0.4619	0.4024	0.3506	0.3015	0.2469
0.3		2.7207	1.3245	0.9647	0.7780	0.6560	0.5657	0.4928	0.4294	0.3693	0.3024
0.4		3.1416	1.5294	1.1139	0.8984	0.7575	0.6533	0.5691	0.4959	0.4264	0.3491
0.5		3.5124	1.7100	1.2454	1.0044	0.8469	0.7304	0.6363	0.5544	0.4767	0.3903
0.6		3.8476	1.8732	1.3642	1.1003	0.9278	0.8001	0.6970	0.6073	0.5223	0.4276
0.7		4.1559	2.0232	1.4735	1.1884	1.0021	0.8642	0.7528	0.6560	0.5641	0.4619
0.8		4.4429	2.1629	1.5753	1.2705	1.0713	0.9238	0.8048	0.7013	0.6030	0.4937
0.9		4.7124	2.2941	1.6708	1.3476	1.1363	0.9799	0.8536	0.7438	0.6396	0.5237
1.0		4.9673	2.4182	1.7612	1.4205	1.1978	1.0329	0.8998	0.7841	0.6742	0.5520

From above table we can readily see that $E\left(Y_{[r]}^{0.5}\right)$ or $F(t) = 1 - \frac{\Gamma(n+1)\Gamma(n-r+t/\alpha+1)}{\Gamma(n-r+1)\Gamma(n+t/\alpha+1)}$ (4.1)

increases with the increase in the value of α whereas the value decreases with the increase in the value of r .

Using (4.1), the survival function is:

$$S(t) = \frac{\Gamma(n+1)\Gamma(n-r+t/\alpha+1)}{\Gamma(n-r+1)\Gamma(n+t/\alpha+1)} \quad (4.2)$$

Further, by using (3.3) and (4.2), the hazard function is:

$$h(t) = \alpha^{-1} \left[H_{(n+t/\alpha)} - H_{(n-r+t/\alpha)} \right] \quad (4.3)$$

We have obtained the numeric values of the survival function and hazard function for $n = 10$ and for various values of α . These values are obtained for $r = 1(1)10$ and for $y = 0.1(1.0)0.1$ and are given in the Appendix A.

Table A1 contains the values of survival function (4.2). Looking at these tables we can see that the survival probability of the concomitant increases with increase in the value of α holding y and r at a fixed level. Further, from the table we can see that; for fixed α and y ; the survival probability decreases with increase in r . Also the table shows that; for fixed α and r ; the survival probability decreases with increase in y .

Table A2 contains the values of hazard function (4.3). Looking at these tables we can see that the hazard rate of the concomitant decreases with increase in the value of α holding y and r at a fixed level. Further, from the table we can see that; for fixed α and y ; the hazard function increases with increase in r . Also the table shows that; for fixed α and r ; the hazard function decreases with increase in y .

These functions are useful for the survival analysis of any parent probability distribution. In this section we have obtained these functions for the distribution (3.3). For this, we first see that the distribution function of (3.3) is given as:

$$\begin{aligned} F(t) &= \int_0^t f_{[r]}(y) dy \\ &= \int_0^t \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} [y + \alpha(n-r+h+1)]^{-2} dy \\ &= \frac{\alpha n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \frac{y}{\alpha(n-r+h+1)} \end{aligned}$$

Using [5], the distribution function is:

Appendix A: Survival and hazard functions

Table A1: Survival function for concomitant of order statistics

$\alpha = 0.20$

r										
y	1	2	3	4	5	6	7	8	9	10
0.1	0.9524	0.9023	0.8492	0.7926	0.7316	0.6651	0.5912	0.5067	0.4054	0.2703
0.2	0.9091	0.8182	0.7273	0.6364	0.5455	0.4545	0.3636	0.2727	0.1818	0.0909
0.3	0.8696	0.7453	0.6277	0.5169	0.4135	0.3181	0.2313	0.1542	0.0881	0.0353
0.4	0.8333	0.6818	0.5455	0.4242	0.3182	0.2273	0.1515	0.0909	0.0455	0.0152
0.5	0.8000	0.6261	0.4770	0.3515	0.2481	0.1654	0.1018	0.0555	0.0247	0.0071
0.6	0.7692	0.5769	0.4196	0.2937	0.1958	0.1224	0.0699	0.0350	0.0140	0.0035
0.7	0.7407	0.5333	0.3710	0.2473	0.1562	0.0919	0.0490	0.0226	0.0082	0.0018
0.8	0.7143	0.4945	0.3297	0.2098	0.1259	0.0699	0.0350	0.0150	0.0050	0.0010
0.9	0.6897	0.4598	0.2943	0.1791	0.1023	0.0539	0.0253	0.0101	0.0031	0.0006
1.0	0.6667	0.4286	0.2637	0.1538	0.0839	0.0420	0.0186	0.0070	0.0020	0.0003

$\alpha = 0.50$

r										
y	1	2	3	4	5	6	7	8	9	10
0.1	0.9804	0.9591	0.9357	0.9097	0.8804	0.8465	0.8062	0.7558	0.6871	0.5726
0.2	0.9615	0.9206	0.8768	0.8294	0.7776	0.7200	0.6545	0.5775	0.4813	0.3438
0.3	0.9434	0.8844	0.8227	0.7578	0.6889	0.6151	0.5349	0.4457	0.3429	0.2143
0.4	0.9259	0.8503	0.7730	0.6938	0.6121	0.5277	0.4398	0.3472	0.2480	0.1378
0.5	0.9091	0.8182	0.7273	0.6364	0.5455	0.4545	0.3636	0.2727	0.1818	0.0909
0.6	0.8929	0.7878	0.6851	0.5848	0.4873	0.3930	0.3023	0.2159	0.1350	0.0613
0.7	0.8772	0.7591	0.6461	0.5384	0.4365	0.3410	0.2526	0.1722	0.1013	0.0422
0.8	0.8621	0.7319	0.6100	0.4965	0.3920	0.2969	0.2121	0.1383	0.0768	0.0296
0.9	0.8475	0.7062	0.5765	0.4586	0.3528	0.2594	0.1789	0.1118	0.0588	0.0210
1.0	0.8333	0.6818	0.5455	0.4242	0.3182	0.2273	0.1515	0.0909	0.0455	0.0152

$\alpha = 0.75$

r										
y	1	2	3	4	5	6	7	8	9	10
0.1	0.9868	0.9724	0.9565	0.9386	0.9182	0.8944	0.8655	0.8287	0.7769	0.6855
0.2	0.9740	0.9460	0.9155	0.8819	0.8444	0.8016	0.7515	0.6902	0.6090	0.4808
0.3	0.9615	0.9206	0.8768	0.8294	0.7776	0.7200	0.6545	0.5775	0.4813	0.3438
0.4	0.9494	0.8963	0.8402	0.7808	0.7170	0.6479	0.5717	0.4854	0.3832	0.2499
0.5	0.9375	0.8728	0.8057	0.7356	0.6621	0.5842	0.5007	0.4097	0.3073	0.1844
0.6	0.9259	0.8503	0.7730	0.6938	0.6121	0.5277	0.4398	0.3472	0.2480	0.1378
0.7	0.9146	0.8287	0.7421	0.6548	0.5667	0.4775	0.3872	0.2953	0.2013	0.1041
0.8	0.9036	0.8079	0.7128	0.6186	0.5252	0.4329	0.3417	0.2521	0.1644	0.0796
0.9	0.8929	0.7878	0.6851	0.5848	0.4873	0.3930	0.3023	0.2159	0.1350	0.0613
1.0	0.8824	0.7685	0.6587	0.5533	0.4527	0.3574	0.2681	0.1856	0.1113	0.0477

$\alpha = 1.00$

r										
y	1	2	3	4	5	6	7	8	9	10
0.1	0.9901	0.9792	0.9671	0.9535	0.9379	0.9195	0.8971	0.8681	0.8268	0.7516
0.2	0.9804	0.9591	0.9357	0.9097	0.8804	0.8465	0.8062	0.7558	0.6871	0.5726
0.3	0.9709	0.9396	0.9056	0.8684	0.8270	0.7802	0.7258	0.6598	0.5737	0.4413
0.4	0.9615	0.9206	0.8768	0.8294	0.7776	0.7200	0.6545	0.5775	0.4813	0.3438
0.5	0.9524	0.9023	0.8492	0.7926	0.7316	0.6651	0.5912	0.5067	0.4054	0.2703
0.6	0.9434	0.8844	0.8227	0.7578	0.6889	0.6151	0.5349	0.4457	0.3429	0.2143
0.7	0.9346	0.8671	0.7974	0.7249	0.6491	0.5694	0.4846	0.3929	0.2911	0.1712
0.8	0.9259	0.8503	0.7730	0.6938	0.6121	0.5277	0.4398	0.3472	0.2480	0.1378
0.9	0.9174	0.8340	0.7497	0.6643	0.5776	0.4895	0.3996	0.3074	0.2120	0.1116
1.0	0.9091	0.8182	0.7273	0.6364	0.5455	0.4545	0.3636	0.2727	0.1818	0.0909

Table A2: Hazard function for concomitant of order statistics

$\alpha = 0.20$

r		1	2	3	4	5	6	7	8	9	10
y											
0.1	0.4762	1.0025	1.5907	2.2574	3.0266	3.9357	5.0468	6.4754	8.4754	11.8087	
0.2	0.4545	0.9545	1.5101	2.1351	2.8494	3.6827	4.6827	5.9327	7.5994	10.0994	
0.3	0.4348	0.9110	1.4373	2.0255	2.6922	3.4614	4.3705	5.4816	6.9102	8.9102	
0.4	0.4167	0.8712	1.3712	1.9268	2.5518	3.2661	4.0994	5.0994	6.3494	8.0161	
0.5	0.4000	0.8348	1.3110	1.8373	2.4255	3.0922	3.8614	4.7705	5.8816	7.3102	
0.6	0.3846	0.8013	1.2558	1.7558	2.3114	2.9364	3.6507	4.4840	5.4840	6.7340	
0.7	0.3704	0.7704	1.2052	1.6813	2.2077	2.7959	3.4626	4.2318	5.1409	6.2520	
0.8	0.3571	0.7418	1.1584	1.6130	2.1130	2.6685	3.2935	4.0078	4.8411	5.8411	
0.9	0.3448	0.7152	1.1152	1.5500	2.0262	2.5525	3.1407	3.8074	4.5766	5.4857	
1.0	0.3333	0.6905	1.0751	1.4918	1.9463	2.4463	3.0019	3.6269	4.3411	5.1745	

$\alpha = 0.50$

r		1	2	3	4	5	6	7	8	9	10
y											
0.1	0.1961	0.4135	0.6574	0.9351	1.2577	1.6423	2.1185	2.7435	3.6526	5.3193	
0.2	0.1923	0.4051	0.6432	0.9134	1.2259	1.5963	2.0509	2.6391	3.4724	4.9010	
0.3	0.1887	0.3970	0.6296	0.8927	1.1958	1.5529	1.9877	2.5432	3.3125	4.5625	
0.4	0.1852	0.3893	0.6165	0.8729	1.1671	1.5119	1.9286	2.4549	3.1692	4.2803	
0.5	0.1818	0.3818	0.6040	0.8540	1.1398	1.4731	1.8731	2.3731	3.0398	4.0398	
0.6	0.1786	0.3746	0.5920	0.8359	1.1137	1.4363	1.8209	2.2971	2.9221	3.8312	
0.7	0.1754	0.3677	0.5805	0.8186	1.0889	1.4014	1.7717	2.2263	2.8145	3.6479	
0.8	0.1724	0.3611	0.5694	0.8020	1.0651	1.3682	1.7253	2.1601	2.7157	3.4849	
0.9	0.1695	0.3547	0.5588	0.7860	1.0424	1.3366	1.6814	2.0981	2.6244	3.3387	
1.0	0.1667	0.3485	0.5485	0.7707	1.0207	1.3064	1.6398	2.0398	2.5398	3.2064	

$\alpha = 0.75$

r		1	2	3	4	5	6	7	8	9	10
y											
0.1	0.1316	0.2776	0.4415	0.6284	0.8458	1.1055	1.4281	1.8537	2.4787	3.6551	
0.2	0.1299	0.2738	0.4350	0.6185	0.8313	1.0845	1.3970	1.8051	2.3934	3.4460	
0.3	0.1282	0.2700	0.4288	0.6090	0.8173	1.0642	1.3672	1.7594	2.3149	3.2673	
0.4	0.1266	0.2664	0.4227	0.5997	0.8038	1.0447	1.3388	1.7162	2.2425	3.1121	
0.5	0.1250	0.2629	0.4168	0.5907	0.7907	1.0260	1.3117	1.6753	2.1753	2.9753	
0.6	0.1235	0.2595	0.4110	0.5820	0.7780	1.0079	1.2857	1.6366	2.1128	2.8535	
0.7	0.1220	0.2562	0.4054	0.5735	0.7658	0.9905	1.2608	1.5998	2.0543	2.7440	
0.8	0.1205	0.2529	0.4000	0.5653	0.7540	0.9737	1.2369	1.5648	1.9995	2.6447	
0.9	0.1190	0.2498	0.3947	0.5573	0.7425	0.9575	1.2139	1.5314	1.9481	2.5541	
1.0	0.1176	0.2467	0.3895	0.5495	0.7314	0.9419	1.1919	1.4996	1.8996	2.4710	

$\alpha = 1.00$

r		1	2	3	4	5	6	7	8	9	10
y											
0.1	0.0990	0.2089	0.3324	0.4732	0.6371	0.8332	1.0771	1.3997	1.8759	2.7850	
0.2	0.0980	0.2067	0.3287	0.4676	0.6289	0.8212	1.0593	1.3718	1.8263	2.6596	
0.3	0.0971	0.2046	0.3251	0.4621	0.6208	0.8095	1.0421	1.3451	1.7799	2.5491	
0.4	0.0962	0.2025	0.3216	0.4567	0.6130	0.7982	1.0254	1.3195	1.7362	2.4505	
0.5	0.0952	0.2005	0.3181	0.4515	0.6053	0.7871	1.0094	1.2951	1.6951	2.3617	
0.6	0.0943	0.1985	0.3148	0.4464	0.5979	0.7765	0.9938	1.2716	1.6562	2.2812	
0.7	0.0935	0.1966	0.3115	0.4414	0.5906	0.7661	0.9788	1.2491	1.6195	2.2077	
0.8	0.0926	0.1946	0.3083	0.4365	0.5835	0.7559	0.9643	1.2274	1.5846	2.1401	
0.9	0.0917	0.1928	0.3051	0.4317	0.5766	0.7461	0.9502	1.2066	1.5514	2.0777	
1.0	0.0909	0.1909	0.3020	0.4270	0.5699	0.7365	0.9365	1.1865	1.5199	2.0199	

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