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On Concomitants of Order Statistics from Morgenstern Family — Source link

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CONCOMITANTS OF ORDER STATISTICS FROM MORGENSTERN FAMILY

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By

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CERTIFICATE

Certified that the thesis entitled **"Concomitants of Order Statistics from Morgenstern Family"** is a bonafide record of the work done by Johny Scaria under my guidance in the Department of Statistics, Cochin University of Science and Technology and no part of it has been included any where previously for the award of any degree or title.

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Chapter 1
INTRODUCTION

1.1 Concomitants of Order Statistics

Order statistics play a very important role in statistical theory and practice and accordingly a remarkably large body of literature has been devoted to its study. It helps to develop special methods of statistical inference, which are valid with respect to a broad class of distributions. Specific properties of order statistics are used to identify probability distributions in the form of characterizations, see Arnold, et al. (1992), Balakrishnan and Rao (1998). In spite of the established role of order statistics in statistical theory, discussions are largely confined to the univariate case and comparatively lesser volume of work is available in a multivariate setup. This is due to the fact that there is no straightforward way of extending the concept of order statistics from the univariate case to the multivariate case. A survey of different attempts of introducing multivariate order statistics can be found in Barnett (1976). Most of the theoretical development in this area of research is on concomitants of order statistics. The concept of concomitants, when bivariate data are ordered by one of its components, was first introduced by David (1973) and almost simultaneously under the name of induced order statistics by Bhattacharya (1974).

Let (X_i, Y_i), $i = 1,2 \ldots n$ be a random sample from a bivariate distribution with cumulative distribution function (cdf) $F(x,y)$ .If we order the values of X_i 's in the increasing order of magnitude , then the corresponding Y_i 's need not have a similar order among themselves . Unless X and Y are independent, the ordering of X's will affect the distribution of the associated Y's. The Y value associated with or paired with $X_{r:n}$, the r^{th} X order statistic is called the concomitant of $X_{r:n}$ and will be denoted by $Y_{[r:n]}$. The ordering of concomitants is similar to that of the marginal variable X if ρ =1 and completely reversed if ρ = -1.

The most important use of concomitants is identified in selection problems when k (<n) individuals are chosen on their X values. Then the corresponding Y values represent performance of an associated characteristic. For example, if the k out of n rams as judged by their genetic make up is selected for breeding, then $Y_{[n-k+1:n]},...Y_{[n:n]}$ might represent the quality of the wool of one of their female offspring. Or, X might be the score of a candidate on a screening test and Y the score on a later test. Concomitants have found a wide variety of applications in such applied fields as selection procedure (Yeo and David (1984)) ocean engineering (Castillo (1988)) inference problems (Do and Hall (1992), Yang (1981a,b)), prediction analysis (Gross (1973)) and double sampling plans (David (1996), O'Connell and David (1976)). An excellent review of work on concomitants of order statistics is available in David and Nagaraja (1998).

1.2 Basic Distribution Theory and a Brief Review

In a basic paper of concomitants, David (1973), considered the bivariate normal model in which the variable Y is linked with X through the regression model

$$Y = \mu_y + \rho\sigma_y\left(\frac{X-\mu_x}{\sigma_x}\right) + Z \qquad (1.2.1)$$

where Z ~N (0, $\sigma_y^2(1-\rho^2)$) and Z is independent of X . Under this model he has derived the finite and asymptotic distribution of the concomitants. Thus for r = 1, 2…n

$$Y_{[r:n]} = \mu_y + \rho\sigma_y(\frac{X_{r:n} - \mu_x}{\sigma_x}) + Z_{[r]}, \quad (1.2.2)$$

where $Z_{[r]}$ denotes the particular Z_r associated with $X_{r:n}$. In view of the independence of X_r and the Z_r we see that the set of $X_{r:n}$ is independent of $Z_{[r]}$. Moreover $Z_{[r]}$ are mutually independent and $Z_r \sim Z$. A more general model of (1.2.1) is discussed in Kim and David (1990). Let $Y_i = g(X_i, \varepsilon_i)$ represent a general model for the regression of Y on X, where neither the X_i nor the ε_i need be identically distributed (but still be independent). Then

$$Y_{[r:n]} = g(X_{r:n}, \varepsilon_{[r]}) \quad r = 1,2\ldots n \quad (1.2.3)$$

from the mutual independence of the X_i and the ε_i it follows that $\varepsilon_{[r]}$ has the same distribution as the ε_i accompanying $X_{r:n}$ and that the $\varepsilon_{[r]}$ are mutually independent. They have shown that concomitants are associated random variables. More over concomitants satisfy a stronger form of dependence, multivariate total positivity of order 2if each $\varepsilon_{[r]}$ in the general linear model has a Polya frequency of order two (Karlin and Rinott (1980)).

The general distribution of concomitants may be derived from the following Theorem due to Bhattacharya (1974).

Theorem 1.1

For $1 \le r_1 < r_2 \ldots < r_k \le n$, the $Y_{[r_i:n]}$ (i = 1, 2...n) are conditionally independent given $X_{r_i:n} = x_i$ (i =1, 2...k) with joint conditional density function $\prod_{i=1}^{k} f(y_i \mid x_i)$.

It follows from the above Theorem that the joint density function of the concomitants $Y_{[r_1:n]}, Y_{[r_2:n]}, \ldots Y_{[r_k:n]}$,

$$f_{Y_{[r_1:n]}\ldots Y_{[r_k:n]}}(y_1,\ldots y_k) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_k}\ldots\int_{-\infty}^{x_2} f_{r_1:n,\ldots r_k:n}(x_1,\ldots x_k)\prod_{h=1}^{k} f(y_h \mid x_h)dx_h \quad (1.2.4)$$

Yang (1977) has shown that

$$E[Y_{[r:n]}] = E[m(X_{r:n})]$$

$$Var(Y_{[r:n]}) = Var(m(X_{r:n})) + E(\sigma^2(X_{r:n}))$$

$$Cov(X_{r:n}, Y_{[s:n]}) = Cov(X_{r:n}, m(X_{s:n}))$$

$$Cov(Y_{[r:n]} Y_{[s:n]}) = Cov(m(X_{r:n}), m(X_{s:n})) \quad r \neq s$$

where

$$m(x) = E(Y|X = x)$$

and

$$\sigma^2(x) = Var(Y \mid X = x). \tag{1.2.5}$$

Jha and Hossein (1986) noted that (1.2.5) continues to hold when X is absolutely continuous but Y is discrete. They have derived the following important recurrence relations connecting the moments of concomitants for an arbitrary specified function h (.) such that E[h(Y)] exists.

$$(n-r)\,E(h(Y_{[r:n]})) + r\,E(h(Y_{[r+1:n]})) = n\,E(h(Y_{[r:n-1]})), \quad r = 1, 2 \ldots n-1 \tag{1.2.6}$$

$$E(h(Y_{[k:m]})) = \binom{m}{k}\sum_{s=0}^{i}\binom{k}{k-i}\frac{\binom{i}{s}}{\binom{m-i+s}{k-i}}E(h(Y_{[k-i:m-i+s]}))$$

$$i \leq k, \quad 1 \leq k \leq m \leq n, \tag{1.2.7}$$

$$E(h(Y_{[k:m]})) = \binom{m}{k}\sum_{s=0}^{j}(-1)^s\frac{k}{k+s}\frac{\binom{j}{s}}{\binom{m-j+s}{k+s}}E(h(Y_{[k+s:m-j+s]})),$$

$$0 \leq j \leq m-k \tag{1.2.8}$$

$$E(h(Y_{[r:n]})) = \sum_{i=r}^{n}\binom{i-1}{r-1}\binom{n}{i}(-1)^{i-r}E(h(Y_{[i:i]})), \quad i = 1, 2 \ldots n \tag{1.2.9}$$

$(r-1)\,E(Y_{[r:n]}\,Y_{[s:n]}) + (s-r)\,E(Y_{[r-1:n]}\,Y_{[s:n]}) + (n-s+1)\,E(Y_{[r-1:n]}\,Y_{[s-1:n]})$

$$= n\,E(Y_{[r-1:n-1]}\,Y_{[s-1:n-1]}\,,\ 1 \le r < s \le n \qquad (1.2.10)$$

A new recurrence relation emerges from the above results.

Theorem 1.2

For $2 \le i < j \le n$,

$$(i-1)\beta_{[i,j:n]} + (j-1)\beta_{[i-1,j:n]} + (n-j+1)\beta_{[i-1,j-1:n]} = \\ n\{\beta_{[i-1,j-1:n-1]} + [\mu_{[i-1:n-1]} - \mu_{[i-1:n]}][\mu_{[j-1:n-1]} - \mu_{[j:n]}]\} \qquad (1.2.11)$$

where

$$\beta_{[i,j:n]} = Cov(Y_{[i:n]}, Y_{[j:n]})$$

and

$$\mu_{[i:n]} = E(Y_{[i:n]})\,.$$

Proof: We have from (1.2.10)

$$(i-1)\beta_{[i,j:n]} + (j-1)\beta_{[i-1,j:n]} + (n-j+1)\beta_{[i-1,j-1:n]} = \\ n\beta_{[i-1,j-1:n-1]} - (i-1)\mu_{[i:n]}\mu_{[j:n]} - (j-1)\mu_{[i-1:n]}\mu_{[j:n]} - (n-j+1)\mu_{[i-1:n]}\mu_{[j-1:n]} + \\ n\mu_{[i-1:n-1]}\mu_{[j-1:n-1]}\}$$

(1.2.12)

Now consider,

$$(i-1)\mu_{[i:n]}\mu_{[j:n]} + (j-1)\mu_{[i-1:n]}\mu_{[j:n]} + (n-j+1)\mu_{[i-1:n]}\mu_{[j-1:n]} = \mu_{[j:n]}\{(i-1)\mu_{[i:n]} + \\ (n-i+1)\mu_{[i-1:n]}\} + (n-j+1)\mu_{[i-1:n]}(\mu_{[j-1:n]} - \mu_{[j:n]})$$

$$= n\mu_{[j:n]}\mu_{[i-1:n-1]} + n\mu_{[i-1:n]}\{\mu_{[j-1:n-1]} - \mu_{[j:n]}\} \qquad (1.2.13)$$

since $\qquad i\mu_{[i+1:n]} + (n-i)\mu_{[i:n]} = n\mu_{[i:n-1]}\,.$

Now using (1.2.13) on the right hand side of (1.2.12) and simplifying we get (1.2.11).

Some specific cases of distribution of concomitants relating to Gumbel's bivariate exponential, bivariate Weibull, bivariate Burr and new generalized Farlie-Gumbel-Morgenstern distributions, are discussed in, Balasubramanian and Beg (1997, 1998), Beg and Balasubramanian (1996), Begum and Khan (1997, 1998, 2000) and Baimarov, Kotz and Bekci (2001).

The asymptotic distribution of concomitants in the simple linear model (1.2.1) is discussed in David and Galambos (1974). They have shown that for all r, under the conditions $n \to \infty$ and $\lim \beta_{r:n}^2 = 0$, the asymptotic distribution of $Z_{[r]} = Y_{[r:n]} - E[Y_{[r:n]}]$ is normal, $N(0, \sigma_y^2(1-\rho^2))$, if $|\rho| < 1$. They have also proved two theorems concerning the asymptotic independence and asymptotic distribution of concomitants.

Theorem 1.3

For any fixed $k \geq 1$ and for any choice $1 \leq r_1 < r_2 < \ldots < r_k \leq n$ of subscripts,

$$\lim_{n \to \infty} P[\, Y^*_{[r_1:n]} < x_1, \ldots Y^*_{[r_k:n]} < x_k] = \prod_{i=1}^{k} \Phi(x_i \mid \sigma) \tag{1.2.14}$$

where

$$\sigma^2 = \sigma_y^2(1-\rho^2)$$

and

$$Y^*_{[r_i:n]} = Y_{[r_i:n]} - E[Y_{[r_i:n]}]$$

Φ is the cdf of a standard normal distribution.

Theorem 1.4

Let X and Z be independent random variables and let Z has a continuous distribution function F(x). Define Y= X+Z and let $Y_{[r:n]} = X_{r:n} + Z_{[r]}$.

Then if the distribution of X is such that the variance $\beta^2_{r:n}$ of $X_{r:n}$ tends to zero as n->∞, any fixed number of random variables $Y^*_{[r:n]} = Y_{[r:n]} - E[Y_{[r:n]}]$ are asymptotically independent, each with distribution function F(x). A corollary of Theorem 1.3 is also stated here.

Corollary 1.1

Let $y_{i,n}$,i =1,2...k real numbers such that, as n->∞ lim $y_{i,n} = x_i$ exist. Then, with the notations of Theorem1.3

$$\lim_{n->\infty} P[Y^*_{[r_1:n]} < y_{1,n}, ... Y^*_{[r_k:n]} < y_{k,n}] = \prod_{i=1}^{k} \Phi(x_i \mid \sigma) \qquad (1.2.15)$$

A more general case is discussed in Yang (1977). He has proved a powerful theorem on the asymptotic distribution of concomitants.

Theorem 1.5

Let $1 \le r_1 < r_2 < ... < r_k \le n$ be sequences of integers such that, as n->∞, $r_i / n \to \lambda_i$ with $0 < \lambda_i < 1$ (i =1, 2...k).

Then $\lim_{n->\infty} P[Y_{[r_1:n]} < y_1, ... Y_{[r_k:n]} < y_k] = \prod_{i=1}^{k} P[Y_i \le y_i \mid X_i = \lambda_i]$.

The extended version of the theorem in Galmbos (1978) by David (1994) gives a representation for the limit distribution of $Y_{[r:n]}$ for an arbitrary absolutely bivariate cdf F(x,y) .

Theorem 1.6

Let $F_X(x)$ satisfy one of the Von Mises condition and assume that the sequences of constants a_n, b_n >0, are such that as n->∞

$$\{ F_X(a_n+b_n x)\}^n \text{->} G(x) \quad \text{for all x} \qquad (1.2.16)$$

Further, suppose there exist constants A_n and B_n >0 such that

$$\{ F_Y(A_n+B_n y| a_n+b_n x)\} \to H(y|x) \text{ for all x and y .}$$

$$\text{Then } P[Y_{[n-k+1:n]} \le A_n+B_n y] \to \int_{-\infty}^{\infty} H(y|x)dG_{(k)}(x) \qquad (1.2.17)$$

where $G_{(k)}$, the kth lower record value from the extreme value cdf G. If (1.2.16) holds we say that F_X is in the domain of attraction of G it is well known that G must be one of the three extreme value cdf's which are of the following types, see Galambos (1987).

$$G_1(x,\alpha) = 0 \qquad x \le 0$$

$$= \exp(-x^{\alpha}) \qquad x>0;\ \alpha>0$$

$$G_2(x,\alpha) = \exp(-(-x^{\alpha})) \qquad x<0;\ \alpha>0$$

$$=1 \qquad x \ge 0$$

$$G_3(x) = \exp\{-\exp(-x)\} \qquad ;-\infty < x < \infty .$$

Suresh (1993) has shown that the central concomitants and extreme concomitants are asymptotically independent .In selection problems, we use a very important statistic called the rank of the concomitants denoted by $R_{[r:n]}$. Here $R_{[r:n]}$ is the rank of $Y_{[r:n]}$ among the n Y_i 's. David et al. (1977) have derived the pdf and expected value $R_{[r:n]}$.

We have $$R_{[r:n]} = \sum_{i=1}^{n} I(Y_{[r:n]} - Y_i),$$

where

$$I(x) = 1 \text{ if } x \ge 0$$

$$= 0 \text{ if } x < 0.$$

Denote by

$$\Pi_{r,s} = P[R_{[r:n]} = s]$$

and

$$\pi_{rs} =$$

$$n \iint \sum_{k=u}^{t} C_k \theta_1^k \theta_2^{(r-1-k)} \theta_3^{(s-1-k)} \theta_4^{(n-r-s+k+1)} f(x,y)\,dx\,dy$$

where

$$u = \max(0, r+s-n-1) \qquad t = \min(r-1, s-1) \qquad (1.2.18)$$

$$\theta_1(x,y) = P[X \le x, Y \le y] = F_{X,Y}(x,y)$$

$$\theta_2(x,y) = P[X \le x, Y > y]$$

$$\theta_3(x,y) = P[X > x, Y \le y]$$

$$\theta_4(x,y) = P[X > x, Y > y]$$

and

$$C_k = \frac{(n-1)!}{k!(r-1-k)!(s-1-k)!(n-r-s+1+k)!}\,.$$

They have discussed in detail the bivariate normal case .The expected value of $R_{[r:n]}$ may be obtained directly by the characteristic order statistics argument. They have shown that

$$E[R_{[r:n]}] = 1 + n$$

$$\{\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} \theta_1(x,y) f(y \mid x)] f_{r-1:n-1}(x)\,dx\,dy + \int_{-\infty}^{\infty} [\theta_3(x,y) f(y \mid x)] f_{r:n-1}(x)\,dx\,dy\}\,.$$

(1.2.19)

In the bivariate normal case David et.al.(1977) have shown that

$$\lim_{n\to\infty} [\, E[R_{[r:n]}/n+1\,] = \Phi(\rho.\Phi^{-1}(\lambda)/(2-\rho^2)^{1/2}),$$

$$\lim P[R_{[r:n]} \le na] = \Phi([\Phi^{-1}(a) - \rho.\Phi^{-1}(\lambda)]/(1-\rho^2)^{1/2}) \qquad (1.1.20)$$

where

$$r / n+1 \to \lambda \qquad (0 < \lambda < 1).$$

For the general bivariate distribution Yang (1977) has shown that

$$\lim \mathrm{P}[R_{[r:n]} \leq nu] = P[Y \leq F_Y^{-1}(u) \mid X = F_X^{-1}(\lambda)].$$

He has presented an interesting application of concomitants in a prediction problem. Spruill and Gatsworth (1996) have applied the above results in connection with employment problems of a professional couple.

Yeo and David (1984) consider the problem of choosing best k objects out of n when, instead of measurements Y_i of primary interest, only associated measurements X_i (i = 1, 2...n) are available. For example Y_i could be very expensive measurements but X_i's are inexpensive measurements. It is assumed that the n pairs (X_i, Y_i) be a random sample from a continuous population. The actual values of Xi's are not required, only their ranks. A general expression is developed for the probability Π that the s objects with the largest X values include the k objects ($k \leq s$) with the largest Y values. They have applied the formula for Π in the bivariate normal case. They have also developed the formula for selecting the best object based on the actual values of X, instead of ranks by using a computer program.

Nagaraja and David (1994) have developed an important statistic for selection problem. In their approach the statistic $V_{k,n}$ = max ($Y_{[n-k+1:n]}$ ..., $Y_{[n:n]}$), k =1, 2, ..., n representing the best individual in a screening procedure with respect to the characteristic under study. They consider E[$V_{k,n}$] / E[$Y_{n:n}$] as a measure of effectiveness of the screening procedure . Both the finite and asymptotic theory of $V_{k,n}$ are discussed by them. They have shown that the cdf of $V_{k,n}$ is

$$F_{k,n}(y) = \mathrm{P}[V_{k,n} \leq y] = \int_{-\infty}^{\infty} \left[F_{Y|X}^{*}(y|x) \right]^{k} f_{X_{n-k:n}}(x)\, dx$$

where

$$F^{*}_{Y|X}(y|x) = P[Y \le y| X > x] \, . \tag{1.2.21}$$

They have also derived the limit distribution of $V_{k,n}$ in the extreme and quantile case. When k is held fixed, under some regularity conditions as n increases

$$F_{k,n}(A_n + B_n . y) = \int_{-\infty}^{\infty} \left[H_{Y|X}(y|x) \right]^k dG_{k+1}(x) \tag{1.2.22}$$

where

$$dG_{k+1}(x) = [-\log G(x)]^k \, \frac{1}{k!} g(x) \, .$$

In the quantile case, where k = [np], 0<p<1, under mild conditions, the limit distribution of $V_{k,n}$ coincides with the limit distribution of sample maximum from the cdf $F^{*}_{Y|X}(y|F_X^{-1}(1-p))$. They have applied their results to some interesting situations, including the bivariate normal population and the simple linear regression model. Joshi and Nagaraja (1995) have derived the joint distribution of $V_{k,n}$ and $V_{k,n}^{*}$ = max ($Y_{[1:n]}$ ..., $Y_{[n-k:n]}$). They used their result to study the joint distribution of $V_{k,n}$ and $Y_{n:n}$, since $Y_{n:n}$ = max ($V_{k,n}$,$V_{k,n}^{*}$). It can be used to choose k such that $V_{k,n}$ / $Y_{n:n}$ is close to 1. LiXiande (1999) has established a sufficient condition for the convergence of concomitants of selected order statistics .Let (X,Y) be the measurement of certain characteristic associated with the parent and offspring populations respectively. Suppose k parents, ranked highest on X, are selected and the average $Y'_{[k:n]} = \frac{1}{k}\sum_{i=1}^{k} Y_{[n-i+1:n]}$ of the y values associated with the offspring group due to the selection is the induced selection differential $D_{[k:n]} = (Y'_{[k:n]} - \mu_y)/\sigma_y$, also known as response to selection. $D_{[k:n]}$ measures the superiority of Y of the k individuals ranked highest on X. The asymptotic distribution this statistic, suitably standardized, is derived in extreme and quantile cases by Nagaraja (1982). Asymptotic properties of $D_{[k:n]}$ have also been investigated. Suresh and Kale (1994) discussed the induced selection percentiles and their properties. Yang (1981a, 1981b) and Sandstrom (1987) have studied the asymptotic properties of smooth linear functions of $Y_{[i:n]}$. Yang (1981a) has considered general linear functions of the form

$$L_{1n} = \frac{1}{n}\sum_{i=1}^{n} J(\frac{i}{n}) Y_{[i:n]}$$

and

$$L_{2n} = \frac{1}{n}\sum_{i=1}^{n} (\frac{i}{n}) \eta(X_{i:n}, Y_{[i:n]}) \quad ,$$

where J is a smooth function which may depend on n, and η is a real valued function. He has established the asymptotic normality of these statistics. These results are used to construct consistent estimators of quantiles associated with the conditional distribution of Y given X = x.

Do and Hall (1992) used the Effron's (1990) technique to estimate the percentiles of the bootstrap distribution based on concomitants.

Let $Y = X + \varepsilon$ where F_X is completely known and F_Y is to be estimated. The observation is

$$(X_{i:n}, \varepsilon_{[i:n]}), 1 \le i \le n, \ \varepsilon_{[i:n]} = Y_{[i:n]} - X_{i:n} .$$

They have suggested the estimator

$$\hat{F}_{n,Y}(y) = \frac{1}{n}\sum_{i=1}^{n} I(F_X^{-1}(i/n) + \varepsilon_{[i:n]} \le y),$$

where I(.) represents the indicator function and established that if ε's are sufficiently small $\hat{F}_{n,Y}(y)$ performs better than the classical estimator $F_{n,Y}(y)$, the empirical cdf. Application of concomitants in double sampling is discussed in O'Connell and David (1976).

They suggested the simple linear estimator of μ_y is

$$\bar{Y}_{[r:n]} = \mu_Y + \rho \frac{\sigma_y}{\sigma_x}(\bar{X}_{r:n} - \mu_X) + \bar{\varepsilon}_r \tag{1.2.23}$$

where $\bar{X}_{r:n}$ and $\bar{\varepsilon}_r$ are the means of $X_{r_i:n}$ and $\varepsilon_{[r_i]}$, i =1,2...k .

If X has a symmetric distribution and the ranks are symmetrically chosen

i.e $\quad r_{k+1-i} = n+1-r_i, i = 1,2...[\frac{k+1}{2}]$

then $\bar{Y}_{[r:n]}$ is unbiased for μ_Y .Also from (1.2.23)

$$\text{Var}\,(\bar{Y}_{[r:n]}/\sigma_Y) = \rho^2 Var(\bar{X}_{r:n}) + (1-\rho^2)/k\ . \tag{1.2.24}$$

Thus the ranks r_i minimizing $Var(\bar{X}_{r:n})$ also minimize Var ($\bar{Y}_{[r:n]}$) for all values of ρ.

Waterson (1959) has considered the linear estimation of the parameters of a bivariate normal population under various forms of censoring. Harrell and Sen (1979) have used the method of likelihood in one of these situations, namely when $X_{1:n}$,...$X_{k:n}$ and $Y_{[1:n]}$,...$Y_{[k:n]}$ are available. They derive the test of independence of X and Y.

An unbiased estimator of the regression coefficient is considered in Barton and Casley (1958). The estimator

$$B' = \frac{\bar{Y}'_{[k:n]} - \bar{Y}_{[k:n]}}{\bar{X}'_{k:n} - \bar{X}_{k:n}}$$

where

$$\bar{Y}'_{[k:n]} = \frac{1}{k}\sum_{i=1}^{k} Y_{[n-i+1:n]}\ , \quad \bar{Y}_{[k:n]} = \frac{1}{k}\sum_{i=1}^{k} Y_{[i:n]}$$

$$\bar{X}'_{k:n} = \frac{1}{k}\sum_{i=n}^{k} X_{n-i+1:n}\,,\quad \bar{X}_{k:n} = \frac{1}{k}\sum_{i=1}^{k} X_{i:n}\,,$$

does not use the i.i.d property and has efficiency of 75-80% when (X,Y) is bivariate normal, provided k is chosen about 0.27n. Tuskibayashi (1962) has suggested an estimator

$$\hat{\rho} = \frac{\bar{Y}_{[n:n]} - \bar{Y}_{[1:n]}}{\bar{Y}_{n:n} - \bar{Y}_{1:n}}$$

of ρ, the correlation coefficient. He points out that $\hat{\rho}$ can be calculated even if only the ranks of X_i's are available. Interesting results related to the distribution of $\hat{\rho}$ are developed in Tuskibayashi (1998). Barnett et.al.(1976) have discussed the estimation of ρ using concomitants.

Multivariate generalization of concomitants is first discussed in David (1973). Suppose that associated with each X there are t variates Y_j (j = 1, 2...t) the t+1variates following a multivariate normal distribution with covariance matrix

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

where

$$\Sigma_{11} = \sigma_x^2, \Sigma_{12} = (Cov(X, Y_j))_{1\times t} = \Sigma_{21}^T$$

and

$$\Sigma_{22} = (Cov(Y_j, Y_j'))_{t\times t}\,.$$

If $Y^*_{[r:n]}$ denotes the $t\times 1$ column vector of the $Y^*_{j,[r:n]}$,

where

$$Y^*_{j,[r:n]} = Y_{j,[r:n]} - E\,[Y_{j,[r:n]}],$$

the vectors $Y^{*}_{[r_i:n]}, i = 1,2...k$ are asymptotically independent, identically distributed $N(0, \Sigma_{22.1})$ variates,

and

$$\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\, \Sigma_{11}^{-1}\, \Sigma_{12}$$

$$= \Sigma_{22} - \frac{\Sigma_{21}\, \Sigma_{12}}{\sigma_x^2} \qquad (1.2.25)$$

The general multivariate case and some applications are discussed in Balakrishnan (1993). Suppose we have n sets of variates (X_i, $Y_{1i}, \ldots Y_{ti}$). Setting $m_j(x_i) = E(Y_{ji} \mid x_i)$ and writing $Y_{j[r:n]}$ for that Y_{ji} paired with $X_{r:n}$,we have

$$E(Y_{j[r:n]}) = E(m_j(X_{r:n}))$$

and

$$Cov(Y_{j[r:n]}, Y_{k[r:n]}) = Cov(m_j(X_{r:n}), m_k(X_{r:n})) + E\sigma_{jk}(X_{r:n}) \quad j \neq k$$

(1.2. 26)

where

$$\sigma_{jk}(x_i) = Cov(Y_{ji}, Y_{ki} \mid x_i).$$

In the multivariate normal case $\sigma_{jk}(x_i)$ does not depend on x_i and may obtained from standard theory

$$\sigma_{jk}(x_i) = \sigma_{jk} - \sigma_j \sigma_k / \sigma_x^2. \qquad (1.2.27)$$

Then $$Y_{j[r:n]} = \mu_j + \rho_j \sigma_j (X_{r:n} - \mu_x)/\sigma_x + \varepsilon_{j[r]} \qquad (1.2.28)$$

where

$$\mu_j = E[Y_j], \sigma_j^2 = Var(Y_j)$$

and $$\rho_j = Cov(X, Y_j).$$

Also noting that $\varepsilon_{j[r]} and \varepsilon_{k[s]}$ are independent unless r = s.

They have shown that

$$\text{Cov}\,(Y_{j[r:n]}\,,\,X_{k[r:n]}) = \rho_j\sigma_j\rho_k\sigma_k\beta_{rr:n} + \sigma_{jk}(x)$$

$$= \sigma_{jk} - \rho_j\sigma_j\rho_k\sigma_k(1-\beta_{rr:n}) \qquad (1.2.29)$$

$$\text{Cov}\,(Y_{j[r:n]}\,,\,Y_{k[s:n]}) = \rho_j\sigma_j\rho_k\sigma_k\beta_{rs:n} \qquad (1.2.30)$$

where,

$$\beta_{rs:n} = \text{Cov}\,(X_{r:n}\,,\,X_{s:n})\,.$$

Balakrishnan (1993) has introduced the multivariate order statistics induced by ordering linear combinations of the components observed in n independent observations from a multivariate normal distribution. The concept of induced bivariate order statistics is explained below.

Let (X_i, Yi), i = 1, 2…n be independents observations from the bivariate normal distribution BVN $(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho\sigma_x\sigma_y)$. Let a and b be non zero constants and

$$S_j = aX_j + bY_j\ , \qquad j = 1, 2\ldots n. \qquad (1.2.31)$$

Let $S_{1:n} \leq S_{2:n} \ldots \leq S_{n:n}$ be the order statistics of S_1 ,S_2 ,…S_n defined in (1.2.31). Then the bivariate order statistics induced by the order statistics $S_{k:n}$ as follows $X_{[k:n]} = X_j$ and $Y_{[k:n]} = Y_j$ whenever $S_{k:n} = S_j$ .In other words ($X_{[k:n]}$, $Y_{[k:n]}$) is that (X,Y) pair which corresponds to the smallest value among S_j's in (1.2.31). He has derived explicit expression for the means, variances and co-variances of the induced bivariate order statistics. He also extended the bivariate induced order statistics to the multivariate case and derived explicit expressions for means variances and co variances in the p-variate normal case.

Balasubramanian and Balakrishnan (1995) have provided a method of constructing a general class of distributions which is closed under marginal, conditional and concomitance of order statistics. They have constructed the bivariate member of the class defined by

$$h_2 [x_1 ,x_2 ;a_1 ,a_2 ,a_{12}] = f(x_1) f(x_2) \{1+ a_1 g(x_1) +a_2 g(x_2) +a_{12} g(x_1) g(x_2) \} \quad (1.2.32)$$

where f(x) is a density function and g(x) is an orthogonal function such that $E [g(X)] = 0$ and the parameters a_1, a_2 and a_{12} satisfy the conditions

$$1+a_1 +a_2 +a_{12} \geq 0 \qquad 1+a_1 -a_2 -a_{12} \geq 0$$

$$1-a_1 +a_2 -a_{12} \geq 0 \qquad 1-a_1 -a_2 +a_{12} \geq 0 . \quad (1.2.33)$$

They have shown that the concomitants belong to the univariate member of the family

$h (x ,a) = f(x) \{ 1+ a g (x) \} , a \in[-1,1]$, and have extended the method to the multivariate case and discussed some interesting properties of this class.

1.3 Morgenstern Distributions

In modelling problems, one general approach is to first choose a family of distributions and then select a member that is appropriate to describe the observation. Of the desiderata for choice of the family, the most important one is that the family should be flexible, in the sense that it should contain a wide variety of models capable of representing any data situation. Another consideration is the sort of prior information available in the choice of the model. In problems involving several random variables, the analyst may make reasonable assumptions about the marginal distributions. Then the question is to construct a joint distribution function with a set of given marginals. The Morgenstern family of distributions assumes importance in such contexts as a highly flexible system. Accordingly in the present study we deal with the distribution theory and applications of concomitants from the Morgenstern family of bivariate distributions.

The Morgenstern system of bivariate distributions includes all cumulative distribution functions of the form

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y)[1+\alpha(1- F_X(x))(1- F_Y(y))], \quad -1 \leq \alpha \leq 1. \qquad (1.3.1)$$

The system provides a very general expression of a bivariate distribution from which members can be derived by substituting expressions of any desired set of marginal distributions. The joint density is given by

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)[1+\alpha(1-2\, F_X(x))(1-2\, F_Y(y))], \quad -1 \leq \alpha \leq 1. \qquad (1.3.2)$$

Since both the bivariate distribution function and density are given in terms of marginals, it is easy to generate a random sample from a Morgenstern distribution. Thus members of this family can be used in simulation studies, especially when weak dependence between variates is of interest. It follows that the conditional density of X given Y=y is

$$f_{X|Y}(x,y) = f_Y(y)[1+\alpha(1-2\, F_X(x))(1-2\, F_Y(y))], \quad -1 \leq \alpha \leq 1. \qquad (1.3.3)$$

When y = median (Y), the conditional density of X given Y = y is the same as the marginal density of X. The regression curve of X given Y = y is

$$E\,[X|Y=y] = E[X] + \alpha\,(1-2\, F_Y(y)) \int x\,(1-2\, F_X(x) f_X(x)\, dx \qquad (1.3.4)$$

which is linear in $F_Y(y)$.

A number of properties results from the simple analytic form of Morgenstern distributions. If the marginal distributions of X and Y are symmetric, the joint distribution is also symmetric. Random variables having a bivariate Morgenstern distribution are exchangeable whenever the marginal distributions are identical. The Morgenstern system is closed with respect to monotonic increasing functions of random variables. Also the system is closed with respect to mixtures of bivariate Morgenstern distributions having the same marginal distributions.

The Morgenstern family is characterized by its "closeness" to the distribution of independent random variables. The following characterization is discussed in Nelson (1994).

Theorem 1.5

If $|\rho_0| \leq 1/3$, the one whose joint density is closet to the product density of independent random variables (in the sense of minimizing Ψ^2-divergence) is the Morgenstern distribution with parameter $\alpha = 3\rho_0$, where ρ_0 is the Spearman's rank correlation coefficient.

The Morgenstern distributions are specially suited to data situations describing weak dependence between the random variables X and Y. Measures of dependence vary over a smaller range than for some other general classes of bivariate distributions. Schucany et al. showed that for this family the Pearson's correlation coefficient lies between -1/3 and 1/3 (see , Convey (1983)).

1.4 The Present Work

The present work is organized into five chapters. Chapter 1 contains a brief description of the basic distribution theory and a quick review of the existing literature. In this chapter we derive a new recurrence relation connecting the product moments of concomitants. We also introduce the concept of bivariate Morgenstern family of distributions and its basic properties.

Chapter 2 deals with the distribution theory of concomitants from the Morgenstern family. It also contains some interesting recurrence relations connecting the moments of concomitants. In this chapter we specialize the results to some well known members of the family, viz, bivariate exponential, bivariate uniform, bivariate logistic and bivariate gamma distributions. We also provide quick estimators for the parameters of the exponential, uniform and logistic models.

In Chapter 3 we deal with the distribution theory of the statistic $V_{k,n}$ discussed in the previous section from the Morgenstern family and obtain certain characteristics that could be useful in selection problems . We also derive the limiting distribution of $V_{k,n}$ and provide illustrative tables of the values of $e_{k,n}$ for the bivariate uniform ,bivariate exponential and bivariate logistic models.

Let $X_1, X_2 \ldots$ be an infinite sequence of independent and identically distributed random variables having the same absolutely continuous distribution function F(x) . An observation X_j will be called an upper record (or simply a record) if its value exceeds that of all previous observations. Thus X_j is a record if $X_j > X_i$ for every $i < j$. An analogous definition deals with lower record values. A comprehensive study on record values is presented in Arnold Balakrishnan and Nagaraja (1998).

Let (X_i, Y_i), i=1, 2... be a sequence of i.i.d random variable from an absolutely continuous distribution with distribution function F(x,y) and density function f(x,y). Let R_n denote the n^{th} record value in the sequence of the X's. The corresponding random variable Y, i.e. the Y-value paired with the X-value R_n is called the n^{th} record concomitant and will be denoted by $R_{[n]}$.The distribution theory of record concomitants from the Morgenstern family of bivariate distributions is discussed in Chapter 4. We also discuss the distribution theory of record concomitants from some important members of the family bivariate exponential, bivariate uniform and bivariate logistic distributions.

The procedure of ranked set sampling was suggested by McIntyre (1952) for improving the precision of $\overline{Y}$ as an estimator of the population mean. This method is applicable for situations where the primary variable of interest, Y, is difficult or expensive to measure, but where ranking in small sets is easy. The process involves selecting m samples, each of size m, and ordering each of the samples by eye or some relatively inexpensive means, without actual measurement of the individual, see David and Levine (1972) Stokes (1977). The smallest observation from the first sample is chosen for measurement, as is the second smallest observation from the second sample. The process continues in this way until the largest observation from the nth sample is measured, producing a total of n measured observations one from each order class.

Motivated from the ranked set sampling we use the following sampling method for selection of primary variable. Suppose there are two correlated variables Y and X, where Y is difficult to measure or to rank. Consider a bivariate sample of size n = mk, where k is an integer. Randomly subdivide the sample in to k sub samples (groups) each of size m .In each sub sample we measured only the Y-value corresponding to the r^{th} order statistic $X_{r:m}$. Then the Y- value measured in the i^{th} sample is the r^{th} concomitant will be denoted by $Y_{[r:m],i}$ $i=1,2,\ldots k$. The $Y_{[r:m],i}$ are independent random variables having the same marginal distribution as $Y_{[r:m]}$.

Let $M_{k,[r:m]} = \max[Y_{[r:m],1}, Y_{[r:m],2}, \ldots Y_{[r:m],k}]$

and

$$m_{k,[r:m]} = \min[Y_{[r:m],1}, Y_{[r:m],2}, \ldots Y_{[r:m],k}]$$

denote the largest and smallest among the selected concomitants. Thus $M_{k,[r:m]}$ and $m_{k,[r:m]}$ are the extremes of the selected expensive measurements in the samples . In particular $M_{k,[m:m]}$ is the largest observation of the concomitants of maximum of order statistics in the sub samples . Then the ratio $\frac{E[M_{k,[m:m]}]}{E[Y_{n:n}]}$ which clearly increases to 1 with k, is a measure of effectiveness of the selection procedure. One may wish to choose the value of the number of subdivisions (populations), k, to make this ratio sufficiently close to 1. In Chapter 5 we discuss the general distribution theory of $M_{k,[r:m]}$ and $m_{k,[r:m]}$ from the Morgenstern family of distributions and discuss some applications in inference, estimation of the parameter of the marginal variable Y in the Morgenstern type uniform distributions. We also apply the results to the selection problem discussed earlier. The work concludes with the distribution theory of the rank of the r^{th} concomitant $R_{[r:n]}$. We also provide illustrative tables for values of $\Pi_{r,s} = P[R_{[r:n]} = s]$ and $E[R_{[r:n]}]$.

Chapter 2

DISTRIBUTION OF CONCOMITANTS OF ORDER STATISTICS FROM MORGENSTERN FAMILY

2.1 Introduction

In recent years modelling has become a convenient technique in many scientific studies to understand the basic characteristics of the phenomenon under consideration. Situations that exhibit uncertainty require the use of probability models in which the prime consideration is often the distribution followed by the observations. When more than one variable is involved in the data generating process, multivariate distributions come in to play. Identification of the appropriate distribution can be accomplished in more than one way. The only exact method is to locate a characteristic property of the process and then derive the distributions possessing such a property. All other methods lead to appropriate solutions and depend largely on the prior information one has about the mechanism that generate the observations. A generally accepted practice is to start with a family of probability distributions that has different types of members capable of accommodating a variety of patterns of uncertainty and then choose one member that befits the data adequately. Therefore multivariate distribution theory is abundant with methods of construction of such families. One method is to extend the defining equation of the univariate family (differential, difference or functional) in a multivariate set-up and solve it to obtain the corresponding law. Bivariate Pearson family, Ord family and Burr family belong to this category. Another way is to generalize physical characteristic of univariate distribution in the multivariate form and seek the distribution possessing the extended version. Several multivariate exponential distributions have been derived in this fashion in literature. Thirdly there are systems based on the form of the marginal and conditional distributions. The Morgenstern family considered in the present study provides a very general expression of a bivariate distribution from which several members can be derived by substituting expressions of any desired set of marginal distributions. In modelling bivariate data, when the prior information is in the form of marginal distributions, it is advantage to consider families of bivariate distributions with specified marginals. The Morgenstern system discussed in Johnson and Kotz (1972) provides a flexible family that can be used in such contexts. It provides a general technique by which a bivariate distribution can be constructed direct from the specified marginal distributions and the correlation between the variables.

The system is capable of accommodating any functional form of the marginals and is specified by the distribution function F(x,y) of a continuous two dimensional random variable (X,Y) through the equation

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y)[1+\alpha(1- F_X(x))(1- F_Y(y))], \quad -1 \leq \alpha \leq 1 \qquad (2.1.1)$$

where $F_X(x)$ and $F_Y(y)$ denote respectively the distribution functions of the component variables and α is the parameter.

The conditional distribution function of Y given X = x is

$$F_{Y|X}(y|x) = F_Y(y)\,[1+\alpha(1-2\, F_X(x))(1- F_Y(y))]. \qquad (2.1.2)$$

In this chapter we discuss some aspects of the distribution of the concomitants from family (2.1.1) and then specialise these results to some well-known members of the family. We also point out some applications of our results in inference on the parameters of the important members of the family. The remaining part of this chapter is organised as follows.

In Section 2.2 we derive the distribution function and probability density function of the concomitant $Y_{[r:n]}$ of the Morgenstern family. The joint cumulative distribution function and probability density function of $Y_{[r:n]}$ and $Y_{[s:n]}$,$(r \neq s)$ are obtained in Section 2.4. In view of the importance of the results given in Section 2.2 we calculate the k^{th} moment of the r^{th} concomitant in Section 2.3 and use the resulting expressions in arriving at useful recurrence relations connecting the successive moments of the concomitants. The discussions on the properties of family need an examination how it is shared and made use of by the various members. Accordingly the rest of the Chapter concentrate on such aspects with reference to some well known constituents of the Morgenstern system. In Section 2.5 we consider the Gumbel's bivariate exponential distribution in some detail and provide some quick estimates for the parameter of that model. Section 2.6 attempts a similar treatment on Morgenstern type uniform distribution and explore the application of the results in estimating the parameters of that model. Section 2.7 is devoted to the Gumbel's type II Logistic distribution. We present the distribution theory of concomitants and derive a number of recurrence relations connecting the moments of concomitants and see how a quick estimator of the location parameter of the marginal variable Y can be developed as a bye product. Finally in Section 2.8 the distribution of the concomitants from bivariate gamma distribution is discussed.

2.2 Distribution of the r^{th} Concomitant $Y_{[r:n]}$

From David (1981), the distribution function and density function of $Y_{[r:n]}$ are given by

$$F_{Y_{[r:n]}}(y) = \int F_{Y|X}(y|x) f_{r:n}(x) dx \tag{2.2.1}$$

and

$$f_{Y_{[r:n]}}(y) = \int f_{Y|X}(y|x) f_{r:n}(x)dx \tag{2.2.2}$$

where $f_{r:n}(x)$ is the density of $X_{r:n}$.

Using

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} [F_X(x)]^{r-1} [1 - F_X(x)]^{n-r} f_X(x)$$

we get

$$F_{Y_{[r:n]}}(y) = F_Y(y)[1 + \alpha\{\int (1 - F_X(x)) f_{r:n}(x)dx - \int F_X(x) f_{r:n}(x)dx\}(1 - F_Y(y))]$$

$$= F_Y(y)[1 + \alpha\{B(r,n-r+2) - B(r+1,n-r+1)\}\frac{(1 - F_Y(y))}{B(r,n-r+1)}]$$

$$= F_Y(y)\left[1 + \frac{n-2r+1}{n+1}\alpha\ [1 - F_Y(y)]\right] \tag{2.2.3}$$

and

$$f_{Y[r:n]}(y) = f_Y(y)[1 + \frac{n-2r+1}{n+1}\alpha\ [1 - 2F_Y(y)]]. \tag{2.2.4}$$

In particular, for r = n

$$f_{Y[n:n]}(y) = f_Y(y)[1 - \frac{n-1}{n+1}\alpha\ [1 - 2F_Y(y)]] \tag{2.2.5}$$

and for r = 1

$$f_{Y[1:n]}(y) = f_Y(y)[1 + \frac{n-1}{n+1}\alpha\ [1 - 2F_Y(y)]]. \tag{2.2.6}$$

Two interesting relations that derive from the above equations are

$$f_{Y[r:n]}(y) + f_{Y[n-r+1:n]}(y) = 2 f_Y(y) \tag{2.2.7}$$

and

$$\sum_{r=1}^{n} f_{Y[r:n]}(y) = n f_Y(y) \tag{2.2.8}$$

in which, (2.2.8) is valid for all density functions. Writing 1-2*F* as 1-*F*-*F* in (2.2.4) and using the formula for the density of order statistics, we find

$$f_{Y_{[r:n]}}(y) = f_Y(y) + \alpha\frac{n-2r+1}{n+1}\{(1-F_Y(y))f_Y(y) - F_Y(y)f_Y(y)\}$$

$$= f_Y(y) + \alpha/2\frac{n-2r+1}{n+1}\{2(1-F_Y(y))f_Y(y) - 2F_Y(y)f_Y(y)\}$$

$$= f_Y(y) + \frac{\alpha}{2}\ \frac{n-2r+1}{n+1}[f_{1:2}(y)-f_{2:2}(y)], \qquad (2.2.9)$$

where $f_{r:2}(y)$ is the density function of $Y_{r:2}$, r = 1, 2.

Since

$$f_{Y_{[1:2]}}(y) - f_{Y_{[2:2]}}(y) = \frac{\alpha}{2}.\frac{2}{3}[f_{1:2}(y) - f_{2:2}(y)]$$

$$= \frac{\alpha}{3}[f_{1:2}(y) - f_{2:2}(y)],$$

(2.2.9) can be rewritten as

$$f_{Y_{[r:n]}}(y) = f_Y(y) + \frac{3}{2}\ \frac{n-2r+1}{n+1}[f_{[1:2]}(y)-f_{[2:2]}(y)]. \qquad (2.2.10)$$

Equation (2.2.9) reveals that the distribution of the rth concomitant depends only on the marginal distribution of Y and the distribution of the order statistics $Y_{1:2}$ and $Y_{2:2}$.

If we change n to 2n+1 and r to 2r, then $\frac{n-2r+1}{n+1}$ has to be replaced by $\frac{2n+1-4r+1}{2n+2}$ and hence, from (2.2.4),

$$f_{Y_{[r:n]}}(y) = f_{Y_{[2r:2n+1]}}(y).$$

Which in turn implies

$$f_{Y_{[r:n]}}(y) = f_{Y_{[2r:2n+1]}}(y) = \ldots = f_{Y[2^k r:2^k n+2^k-1]}(y)$$

$$\text{for } k = 0, 1, 2, \ldots \qquad (2.2.11)$$

In general, if λ is some rational number such that $r\lambda$ and $(n+1)\lambda$ are integers, then

$$f_{Y_{[r\lambda:(n+1)\lambda-1]}}(y) = f_{Y_{[r:n]}}(y). \tag{2.2.12}$$

2.3 Moments of $Y_{[r:n]}$

From (2.2.4) we have the k^{th} moment of $Y_{[r:n]}$ can be derived as

$$\mu^{(k)}_{[r:n]} = E[Y^k_{[r:n]}]$$

$$= \int y^k f_{Y_{[r:n]}}(y)\,dy$$

$$= \int y^k [f_Y(y) + \frac{n-2r+1}{n+1}\frac{\alpha}{2}(f_{1:2}(y) - f_{2:2}(y))]dy$$

$$= \mu^{(k)} + \frac{\alpha}{2}\frac{n-2r+1}{n+1}[\mu^{(k)}{}_{1:2} - \mu^{(k)}{}_{2:2}] \tag{2.3.1}$$

where,

$$\mu^{(k)} = E(Y^k)$$

and

$$\mu^{(k)}_{1:2} = E[Y^k_{r:2}] \quad r = 1, 2.$$

By direct substitution it can be seen from (2.3.1) that for all $r = 1, 2, \ldots, n-2$ the recurrence relation

$$\mu^{(k)}_{[r+2:n]} - 2\,\mu^{(k)}_{[r+1:n]} + \mu^{(k)}_{[r:n]} = 0 \tag{2.3.2}$$

holds. A physical interpretation of this result is that $\mu^{(k)}_{[r+1:n]}$ is the arithmetic mean between $\mu^{(k)}_{[r+2:n]}$ and $\mu^{(k)}_{[r:n]}$ for all the above designated values of r. In other words the sequence of the k^{th} moments of the concomitants are in arithmetic progression. Using (2.3.1) and (2.3.2) we further find that

$$\mu^{(k)}_{[r:n]} = \mu^{(k)}_{[1:n]} + (r-1)\,d \tag{2.3.3}$$

where d is the common difference of the said arithmetic progression with value

$$d = \mu^{(k)}_{2:n} - \mu^{(k)}_{1:n}$$

$$= \mu^{(k)} + \frac{\alpha}{2}\frac{n-3}{n+1}[\mu^{(k)}{}_{1:2} - \mu^{(k)}{}_{2:2}] - [\mu^{(k)} + \frac{\alpha}{2}\frac{(n-1)}{n+1}[\mu^{(k)}{}_{1:2} - \mu^{(k)}{}_{2:2}]]$$

$$= \frac{\alpha}{n+1}[\mu^{(k)}{}_{2:2} - \mu^{(k)}{}_{1:2}]. \tag{2.3.4}$$

This gives

$$\mu^{(k)}_{[r:n]} = \mu^{(k)}_{[1:n]} + \frac{(r-1)\alpha}{n+1}[\mu^{(k)}{}_{2:2} - \mu^{(k)}{}_{1:2}],\ n > 2. \tag{2.3.5}$$

Several observations on the utility of the above results seems to be in order at this juncture

1. The kth moment of the rth concomitant is a linear function in r. Hence the graph (r, $\mu_{[r:n]}$) should exhibit a straight line for r = 1,2,...n-2. If the sample plots of the values of $\mu_{[r:n]}$ is approximately on a line, it is indicative of the fact that Morgenstern family is a likely candidate for modelling.

2. Theoretically, the second forward differences of the $\mu_{[r:n]}$ values should be zero by virtue of them being in arithmetic progression. Hence a check alternative to using a graph as pointed out above is to see whether the second differences of the values of moments are approximately zero or the first differences are nearly the same.

3. The moments of the concomitants of various order statistics behave in a systematic way. Once the kth moment of the first concomitant and the increment factor is estimated, it is straightforward to predict the value of the rth concomitant. Estimates of the first and second of these factors are measured in terms of the intercept and slope of the graph indicated at 1 above or by direct calculation from the differences pointed out at 2.

From (2.2.4) we further have

$$f_{[r:n]}(y) - f_{[r-1:n]}(y) = -\frac{2\alpha}{n+1}(1 - 2F_Y(y))f_Y(y) \tag{2.3.6}$$

and hence by iteration to lower values of r,

$$f_{[r:n]}(y) = f_{[1:n]}(y) - \frac{2(r-1)}{n+1}(1-2F_Y(y))f_Y(y).$$

Thus the densities are also in arithmetic progression, this to be decreasing with a common difference of (2.3.5).

The moment generating function of $Y_{[r:n]}$ is

$$M_{[r:n]}(t) = E[\exp(t\,Y_{[r:n]})] = \int \exp(ty) f_{Y_{[r:n]}}(y)dy$$

$$= \int \exp(ty)\,[f_Y(y) + \frac{\alpha}{2}\,\frac{n-2r+1}{n+1}\,[f_{1:2}(y)-f_{2:2}(y)]]dy$$

$$= M_Y(t) + \frac{n-2r+1}{n+1}\frac{\alpha}{2}(M_{1:2}(t) - M_{2:2}(t)) \qquad (2.3.7)$$

where, $M_{r:2}(t)$ is the moment generating function of $Y_{r:2}$ r = 1, 2 .

The moment generating function also satisfies the recurrence relation

$$M_{[r:n]}(t) = M_{[r-1::n]}(t) - \frac{2\alpha}{n+1}(M_{1:2}(t) - M_{2:2}(t))$$

and from this relationship (2.3.4) can be obtained.

From (2.2.4) we have

$$f_{Y_{[r:n]}}(y) - f_{Y_{[r:n-1]}}(y) = \alpha[1-2F_Y(y)]f_Y(y)\{\frac{n-2r+1}{n+1} - \frac{(n-2r)}{n}\}$$

$$= \alpha\frac{2r}{n(n+1)}[1-2F_Y(y)]f_Y(y) \qquad (2.3.8)$$

and hence we obtain the following recurrence relation connecting the moments of concomitants

$$\mu^{(k)}_{[r:n]} - \mu^{(k)}_{[r:n-1]} = \frac{2r\alpha}{n(n+1)}[\mu^{(k)} - \mu^{(k)}_{2:2}]. \qquad (2.3.9)$$

Moreover from (2.3.9) by adding telescopic sum we get the following identity

$$\mu^{(k)}_{[r:n]} = \mu^{(k)}_{[r:r]} + \frac{2r(n-r)}{(r+1)(n+1)}\alpha(\mu^{(k)} - \mu^{(k)}_{2:2}) \qquad (2.3.10)$$

see Beg and Balasubramanian (1998).

2.4 Joint Distribution of $Y_{[r:n]}$ and $Y_{[s:n]}$

In order to understand the behaviour of one concomitant with respect to the others we need the joint probability distribution of any two concomitants. Accordingly in this section, we derive the joint distribution of rth and s^{th} concomitants. From David (1981), the joint distribution function of $(Y_{[r:n]}, Y_{[s:n]})$ is

$$F_{[r,s:n]}(y_1,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} F_{Y|X}(y_1|x_1)\, F_{Y|X}(y_2|x_2) f_{r,s:n}(x_1,x_2)\, dx_1\, dx_2,\quad 1 \le r < s \le n \tag{2.4.1}$$

where

$$f_{r,s:n}(x_1,x_2) = [B(r,s-r,n-s+1)]^{-1}\, [F_X(x_1)]^{r-1}\, [F_X(x_2) - F_X(x_1)]^{s-r-1}$$
$$[1 - F_X(x_2)]^{n-s} f(x_1) f(x_2). \tag{2.4.2}$$

Using (2.1.2) in (2.4.1)

$$F_{[r,s:n]}(y_1,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} F_Y(y_1)F_Y(y_2)[1+\alpha(1-F_Y(y_2))(1-2F_X(x_2))]$$
$$[1+\alpha(1-F_Y(y_1))(1-2F_X(x_1))]\, f_{r,s:n}(x_1,x_2)dx_1dx_2$$
$$= F_Y(y_1)F_Y(y_2)[\int_{-\infty}^{\infty}(\int_{-\infty}^{x_2} f_{r,s:n}(x_1,x_2)dx_1)dx_2 +$$
$$\alpha[1-F_Y(y_1)]\int_{-\infty}^{\infty}[1-2F_X(x_1)]\{\int_{x_1}^{\infty} f_{r,s:n}(x_1,x_2)dx_2\}dx_1 +$$
$$\alpha[1-F_y(y_2)]\int_{-\infty}^{\infty}[1-2F_X(x_2)]\int_{-\infty}^{x_2} f_{r,s:n}(x_1,x_2)dx_1dx_2 +$$
$$\alpha^2(1-F_Y(y_1))(1-F_Y(y_2))\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}(1-2F_X(x_1))(1-2F_X(x_2)f_{r,s:n}(x_1,x_2)dx_1dx_2$$

$$= F_Y(y_1)\, F_Y(y_2)[1+\alpha\,(1-F_Y(y_2))\left\{2\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}(1-F_x(x_2))f_{r,s:n}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 - 1\right\}$$

$$+\,\alpha\,[1-F_Y(y_1)]\left\{1-2\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}F_x(x_1)f_{r,s:n}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2\right\} + \alpha^2\,[1-F_Y(y_1)]\,[1-F_Y(y_2)]$$

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}(1-2F_x(x_1))[2(1-F_x(x_2))-1]f_{r,s:n}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2\right\}]$$

$$= F_Y(y_1)\, F_Y(y_2)[1+\alpha\,(1-F_Y(y_1))\,\frac{n-2r+1}{n+1} + \alpha\,[1-F_Y(y_2)]\,\frac{n-2s+1}{n+1} + \alpha^2$$

$$[1-F_Y(y_1)]\,[1-F_Y(y_2)]\left\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\right\}]. \qquad (2.4.3)$$

The density function corresponding to (2.4.3) is

$$f_{[r,s:n]}(y_1,y_2) = f_Y(y_1)\, f_Y(y_2)[1+\alpha(1-2F_Y(y_1))\frac{n-2r+1}{n+1} + \alpha\,[1-2F_Y(y_2)]\,\frac{n-2s+1}{n+1} +$$

$$\alpha^2\,[1-2F_Y(y_1)]\,[1-2F_Y(y_2)]\left\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\right\}]. \qquad (2.4.4)$$

Using 1-2F = 1-F-F we write (2.4.4) as

$$f_{[r,s:n]}(y_1,y_2) = f_Y(y_1)f_Y(y_2) + \frac{n-2r+1}{n+1}\,\frac{\alpha}{2}[f_{1:2}(y_1)-f_{2:2}(y_1)]\; f_y(y_2) +$$

$$\frac{n-2s+1}{n+1}\,\frac{\alpha}{2}[f_{1:2}(y_2)-f_{2:2}(y_2)]\;\; f_y(y_1) \;+\left\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\right\}$$

$$\frac{\alpha^2}{4}[f_{1:2}(y_1)-f_{2:2}(y_1)]\,[f_{1:2}(y_2)-f_{2:2}(y_2)] \qquad (2.4.5)$$

With the joint density of $Y_{[r:n]}$ and $Y_{[s:n]}$ as given in (2.4.5), the product moments $E(Y^l_{[r:n]}Y^m_{[s:n]})$, denoted by $\mu^{(l,m)}_{[r,s:n]}$, l, m > 0, are given by

$$\mu^{(l,m)}_{[r,s:n]} = \iint y_1^l y_2^m\, [f_Y(y_1)f_Y(y_2) + \frac{\alpha}{2}[f_{1:2}(y_1)-f_{2:2}(y_1)]\; f_y(y_2)\,\frac{n-2r+1}{n+1}$$

$$+\frac{n-2s+1}{n+1}\,\frac{\alpha}{2}[f_{1:2}(y_2)-f_{2:2}(y_2)]\;\; f_y(y_1) \;+\left\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\right\}$$

$$\frac{\alpha^2}{4}[f_{1:2}(y_1)-f_{2:2}(y_1)]\,[f_{1:2}(y_2)-f_{2:2}(y_2)]]\; dy_1dy_2.$$

Performing the integration we find that

$$\mu_{[r,s:n]}^{(l,m)} = \mu^{(l)}\mu^{(m)} + \frac{n-2r+1}{n+1}\frac{\alpha}{2}[\mu_{1:2}^{(l)} - \mu_{2:2}^{(l)}]\mu^{(m)} + \frac{n-2s+1}{n+1}\frac{\alpha}{2}[\mu_{1:2}^{(m)} - \mu_{2:2}^{(m)}]\mu^{(l)} +$$

$$\left\{\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right\}\frac{\alpha^2}{4}[\mu_{1:2}^{(l)} - \mu_{2:2}^{(l)}][\mu_{1:2}^{(m)} - \mu_{2:2}^{(m)}]. \qquad (2.4.6)$$

Changing r to (r+1) in (2.4.6) and from the resulting expression subtracting (2.4.6)

$$\mu_{[r+1,s:n]}^{(l,m)} - \mu_{[r,s:n]}^{(l,m)} = -\frac{\alpha}{n+1}[\mu_{1:2}^{(l)} - \mu_{2:2}^{(l)}]\mu^{(m)} - \left\{\frac{(n-2s)}{(n+1)(n+2)}\right\}\frac{\alpha^2}{2}$$

$$[\mu_{1:2}^{(l)} - \mu_{2:2}^{(l)}][\mu_{1:2}^{(m)} - \mu_{2:2}^{(m)}]. \qquad (2.4.7)$$

Now subtracting (2.4.7) from the expression obtained by increasing s to s+1 in (2.4.7),

$$\mu_{[r+1,s+1:n]}^{(l,m)} - \mu_{[r+1,s:n]}^{(l,m)} - \mu_{[r,s+1:n]}^{(l,m)} + \mu_{[r,s:n]}^{(l,m)} = \alpha^2/(n+1)(n+2)\ [\mu_{1:2}^{(l)} - \mu_{2:2}^{(l)}][\mu_{1:2}^{(m)} - \mu_{2:2}^{(m)}]. \qquad (2.4.8)$$

We immediately see that right hand expression in (2.4.8) is independent of both r and s. This leads to the recurrence relations

$$\{\mu_{[r+2,s+1n]}^{(l,m)} - \mu_{[r+2,s:n]}^{(l,m)}\} - 2\{\mu_{[r+1,s+1n]}^{(l,m)} - \mu_{[r+1,s:n]}^{(l,m)}\} + \{\mu_{[r,s+1n]}^{(l,m)} - \mu_{[r,s:n]}^{(l,m)}\} = 0$$

$$\{\mu_{[r+1,s+2n]}^{(l,m)} - \mu_{[r,s+2:n]}^{(l,m)}\} - 2\{\mu_{[r+1,s+1n]}^{(l,m)} - \mu_{[r,s+1:n]}^{(l,m)}\} + \{\mu_{[r+1,s:n]}^{(l,m)} - \mu_{[r,s:n]}^{(l,m)}\} = 0$$

It may also be noted that all the observations made with regard to first order moments hold good here also with the change that instead of first order forward difference mentioned there, the second order differences have to be reckoned with. Thus the property of arithmetic progressions in the extended sense, is shared by the product moments of two concomitants in the Morgenstern family.

Using (2.4.5), the joint moment generating function of $Y_{[r:n]}$ and $Y_{[s:n]}$ is

$$M_{[r,s:n]}(t_1,t_2) = E\{\exp(t_1 Y_{[r:n]} + t_2 Y_{[s:n]})\}$$

$$= \iint \exp\{t_1 y_1 + t_2 y_2\} f_{[r,s:n]}(y_1,y_2)dy_1 dy_2$$

$$= M_Y(t_1)M_Y(t_2) + \frac{n-2r+1}{n+1}\frac{\alpha}{2}[M_{1:2}(t_1) - M_{2:2}(t_1)]M_Y(t_2) +$$

$$\frac{n-2s+1}{n+1}\frac{\alpha}{2}[M_{1:2}(t_2) - M_{2:2}(t_2)]M_Y(t_1) +$$

$$\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\}\alpha^2 / 4$$

$$[M_{1:2}(t_1) - M_{2:2}(t_1)][M_{1:2}(t_2) - M_{2:2}(t_2)] \qquad (2.4.9)$$

Furthermore, using (2.3.10) and (2.4.6), the covariance between $Y_{[r:n]}$and $Y_{[s:n]}$ can be evaluated.

We have so far obtained some general expressions for moments of concomitants as applied to the Morgenstern family. We now specialize these results to some well-known members of the family.

2.5 Concomitants from Gumbel's Bivariate Exponential Distribution.

An important member of the Morgenstern family is the Gumbel's (1960) type II bivariate exponential distribution specified by

$$F_{X,Y}(x,y) = \left(1-\exp\left\{\frac{-x}{\theta_1}\right\}\right)\left(1-\exp\left\{\frac{-y}{\theta_2}\right\}\right)\left(1+\alpha\exp\left\{\frac{-x}{\theta_1}+\frac{-y}{\theta_2}\right\}\right),$$

$$x,y > 0;\ \theta_1, \theta_2 > 0. \qquad (2.5.1)$$

In this case the distribution function of $Y_{[r:n]}$ follows from (2.2.3) and is given by

$$F_{Y_{[r:n]}}(y) = [1-\exp(-y/\theta_2)][1+\frac{n-2r+1}{n+1}\alpha\exp(-y/\theta_2)]$$

$$y > 0,\ \theta_2 > 0 \qquad (2.5.2)$$

and the density function of $Y_{[r:n]}$ is

$$f_{Y_{[r:n]}}(y) = \frac{1}{\theta_2}\exp\left\{\frac{-y}{\theta_2}\right\}[1+\alpha\frac{n-2r+1}{n+1}\left(2\exp\left\{\frac{-y}{\theta_2}\right\}-1\right)], y > 0. \quad (2.5.3)$$

The k^{th} moment of $Y_{[r:n]}$ is directly calculated from (2.5.3) as

$$\mu_{[r:n]}^{(k)} = \Gamma(k+1)\theta_2{}^{k}[1 + \frac{n-2r+1}{n+1}\alpha(2^{-k}-1)]. \quad (2.5.4)$$

The following recurrence relations follow directly from (2.5.4).

Relation 2.5.1

$$\mu_{[r:n]}^{(k)} - \mu_{[r-1:n]}^{(k)} = 2(2^{-k}-1)\frac{\alpha}{n+1}\Gamma(k+1)\theta_2^{k}. \quad (2.5.5)$$

Relation 2.5.2

$$\mu_{[r:n]}^{(k)} - \mu_{[r:n-1]}^{(k)} = \alpha\theta_2^{k}\Gamma(k+1)\frac{2r}{n(n+1)}(2^{-k}-1). \quad (2.5.6)$$

Relation 2.5.3

$$\mu_{[r:n]}^{(k)} - \mu_{[r-1:n-1]}^{(k)} = \Gamma(k+1)\theta_2^{k}(2^{-k}-1)2\alpha\frac{(r-n-1)}{n(n+1)}. \quad (2.5.7)$$

We have

$$\mu_{[r:n]}^{(k)} - k\,\theta_2\mu_{[r:n]}^{(k-1)} = \Gamma(k+1)\theta_2^{k}\frac{(n-2r+1)}{n+1}\alpha[2^{-k}-2^{-k+1}]$$

$$= \Gamma(k+1)\theta_2{}^{k}\frac{(n-2r+1)}{n+1}(-\alpha)\,2^{-k}.$$

Hence we obtain the recurrence relation

$$\mu_{[r:n]}^{(k)} - \mu_{[r:n]}^{(k-1)}k\theta_2 + 2^{-k}\Gamma(k+1)\theta_2{}^{k}\alpha\frac{(n-2r+1)}{n+1} = 0. \quad (2.5.8)$$

The mean and variance of $Y_{[r:n]}$ are respectively

$$\mu_{[r:n]} = \theta_2[1 - \frac{n-2r+1}{2(n+1)}\alpha] \quad (2.5.9)$$

$$V(Y_{[r:n]}) = 2\theta_2^2[1-\frac{n-2r+1}{n+1}\frac{3\alpha}{4}] - \ [\theta_2\,(\ 1-\frac{n-2r+1}{2(n+1)}\alpha)]^2$$

$$=\theta_2^2[1-\frac{n-2r+1}{2(n+1)}\alpha-\frac{\alpha^2(n-2r+1)^2}{4(n+1)^2}]. \qquad (2.5.10)$$

It is worthwhile to examine how the orderings on the Y values in the present scheme contribute lower changes in the expectation when compared to the unordered Y values. The mean of the unordered Y values in the population is E(Y) $=\theta_2$, on the other hand the expected values of the concomitants are in arithmetic progression with difference of $-\frac{\alpha\theta_2}{n+1}$. Since θ_2 is always positive, the difference depends on the sign of α. When α is negative (positive) there is negative (positive) correlation between the (X,Y) values and the mean values increase (decrease) by $\frac{\alpha\theta_2}{n+1}$. The difference in the expected values of Y and $Y_{[r:n]}$ is $\frac{n-2r+1}{2(n+1)}\alpha\theta_2$ and will be zero only at r $=\frac{n+1}{2}$ in which case we have the middle value in the ordered sequence when n is an odd integer.

The difference between the variances of Y and $Y_{[r:n]}$ is

$$V(Y) - V(Y_{[r:n]}) = \frac{n-2r+1}{2(n+1)}\alpha\theta_2^2 + \frac{(n-2r+1)^2}{4(n+1)^2}\alpha^2\theta_2^2$$

$$= \frac{n-2r+1}{2(n+1)}(1+\frac{n-2r+1}{2(n+1)}\alpha)\alpha\theta_2^2.$$

For any value of r ranging from 1 to n, the term within the braces on the right side is positive irrespective of the value of α. Hence when α is negative (positive) $V(Y_{[r:n]})$ >(<) V(Y).

The joint cumulative distribution function of $(Y_{[r:n]}, Y_{[s:n]})$ is

$$F_{Y_{[r,s:n]}}(y_1, y_2) = [1\text{-exp}\{-\frac{y_1}{\theta_2}\}] \quad [1 - \exp\left\{\frac{-y_2}{\theta_2}\right\}][1 + \left[\frac{\alpha(n-2r+1)}{n+1}\left(\exp\left\{\frac{-y_1}{\theta_2}\right\}\right)\right]$$

$$+\frac{\alpha(n-2s+1)}{n+1}\left(\exp\left\{\frac{-y_2}{\theta_2}\right\}\right)+$$

$$\alpha^2\left[\frac{(n-2s+1)}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right]\left(\exp\left\{\frac{-y_1}{\theta_2}\right\}\right)\left(\exp\left\{\frac{-y_2}{\theta_2}\right\}\right)] \qquad (2.5.11)$$

and the joint probability density function of ($Y_{[r:n]}$, $Y_{[s:n]}$) is

$$f_{Y_{[r,s:n]}}(y_1, y_2) = \frac{1}{\theta_2^2}\exp\left\{\frac{-y_1 - y_2}{\theta_2}\right\}[1 + \left[\frac{\alpha(n-2r+1)}{n+1}\left(2\exp\left\{\frac{-y_1}{\theta_2}\right\} - 1\right)\right]$$

$$+\frac{\alpha(n-2s+1)}{n+1}\left(2\exp\left\{\frac{-y_2}{\theta_2}\right\} - 1\right) + \alpha^2\left[\frac{(n-2s+1)}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right]$$

$$\left(2\exp\left\{\frac{-y_1}{\theta_2}\right\} - 1\right)\left(2\exp\left\{\frac{-y_2}{\theta_2}\right\} - 1\right)];$$

$$y_1, y_2 > 0. \qquad (2.5.12)$$

The product moments are

for l, m > 0

$$\mu_{[r,s:n]}^{(l,m)} = \theta_2^{l+m}[\Gamma(l+1)\Gamma(m+1) + \frac{(n-2r+1)}{n+1}\alpha\{\frac{\Gamma(l+1)}{2^l} - \Gamma(l+1)\}\Gamma(m+1) +$$

$$\frac{(n-2s+1)}{n+1}\alpha\{\frac{\Gamma(m+1)}{2^m} - \Gamma(m+1)\}\Gamma(l+1) +$$

$$\alpha^2\{\frac{(n-2s+1)}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\}[\frac{\Gamma(l+1)}{2^l} - \Gamma(l+1)][\frac{\Gamma(m+1)}{2^m} - \Gamma(m+1)]$$

(2.5.13)

When l = 1, m = 1 (2.5.13) reduces to

$$E[Y_{[r:n]}Y_{[s:n]}] = \theta_2^2[1-\alpha\frac{(n-r-s+1)}{n+1}+\frac{\alpha^2}{4}\{\frac{(n-2s+1)}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\}] \quad (2.5.14)$$

and

$$\mathrm{Cov}\,(Y_{[r:n]},Y_{[s:n]})$$

$$= \theta_2^2[1-\alpha\frac{(n-r-s+1)}{n+1}+\alpha^2/4\{\frac{(n-2s+1)}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\}] -$$

$$\theta_2^2[1-\frac{\alpha}{2}\frac{(n-2r+1)}{(n+1)}][1-\frac{\alpha}{2}\frac{(n-2s+1)}{(n+1)}]$$

$$= \frac{\alpha^2}{4}\theta_2^2[\frac{(n-2s+1)}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}-\frac{(n-2r+1)(n-2s+1)}{(n+1)^2}]$$

$$=\frac{r(n-s+1)\alpha^2\theta_2^2}{(n+1)^2(n+2)} \quad (2.5.15)$$

It can be observed from (2.5.15) that the concomitants are positively correlated and the covariance decreases as r and s pull apart.

2.5.1 Estimation of the Parameters θ_2 and α

We point out a possible use of the theory of concomitants developed in the previous sections to inference problems relating to the parameters θ_2 and α. It may be noted that the parameter θ_1 is directly linked to the *X*-values and therefore is not an object of estimation using concomitants which are based on the *Y*-values. It is assumed that the ordering on *X* (not necessarily on the basis of the numerical values of *X*'s) is known to ascertain the concomitants $Y_{[r:n]}$.

The variance of $Y_{[r:n]}$ follows from (2.5.10) as

$$V(Y_{[r:n]}) = \theta_2^2[1-\frac{\alpha(n-2r+1)}{2(n+1)}-\frac{\alpha^2(n-2r+1)^2}{(n+1)^2 4}] \quad (2.5.16)$$

so that $E[T_r] = \theta_2$ (2.5.17)

$$V(T_r) = \theta_2^2\left(\frac{1}{2} - \frac{(n-2r+1)^2\alpha^2}{8(n+1)^2} + \frac{r^2\alpha^2}{2(n+1)^2(n+2)}\right),\ r \le \frac{n+1}{2}$$

$$= \theta_2^2\left(\frac{1}{2} - \frac{(n-2r+1)^2\alpha^2}{8(n+1)^2} + \frac{(n-r+1)^2\alpha^2}{2(n+1)^2(n+2)}\right),\ r > \frac{n+1}{2} \quad (2.5.18)$$

where T_r is the r[th] quasi-midrange defined by

$$T_r = \tfrac{1}{2}\,[(Y_{[r:n]} + Y_{[n-r+1:n]})]. \quad (2.5.19)$$

Thus all quasi-midranges are unbiased estimators of θ_2.

However

$$V(T_r) - V(T_{r-1}) = \frac{\theta_2^2\alpha^2}{(n+1)^2}\left\{\frac{n-2r+2}{2} + \frac{2r-1}{2(n+1)}\right\} > 0 \quad (2.5.20)$$

for every $r = 1, 2, \ldots$ and hence $V(T_r)$ is an increasing function of r for $r > \frac{n+1}{2}$. Thus among the unbiased estimators T_r, $r = 1, 2, \ldots$, minimum variance is attained by

$$T_1 = \frac{1}{2}\left(Y_{[1:n]} + Y_{[n:n]}\right) \quad (2.5.21)$$

and the smallest variance is

$$V(T_1) = \theta_2^2\left\{\frac{1}{2} - \frac{(n-1)^2\alpha^2}{8(n+1)^2} + \frac{\alpha^2}{2(n+1)^2(n+2)}\right\}. \quad (2.5.22)$$

The information required to unbiasedly estimate θ_2 is thus only the Y-values associated with the maximum and minimum of the X's. By comparison, the sample mean of Y is more efficient than T_1, but the former utilises the entire sample values.

Using $Y_{[1:n]} = T_1\left(1 - \frac{\alpha(n-1)}{2(n+1)}\right)$ and $Y_{[n:n]} = T_1\left(1 + \frac{\alpha(n-1)}{2(n+1)}\right)$ obtained by substituting the unbiased estimate T_1 of θ_2, we get

$$Y_{[n:n]} - Y_{[1:n]} = \alpha \frac{n-1}{n+1}\theta_2 . \tag{2.5.23}$$

Hence we obtain a quick estimate of α as

$$\hat{\alpha} = \begin{cases} 1 & T_2 \geq \frac{n-1}{2(n+1)} \\ \frac{2(n+1)}{n-1}T_2 & \frac{1-n}{2(n+1)} < T_2 < \frac{n-1}{2(n+1)} \\ -1 & T_2 \leq \frac{1-n}{2(n+1)} \end{cases}$$

where,

$$T_2 = \frac{Y_{[n:n]} - Y_{[1:n]}}{Y_{[n:n]} + Y_{[1:n]}} . \tag{2.5.24}$$

Another result concerning concomitants, that is not directly related to the estimation problem, but will be of interest is that the distribution of the ratio of concomitants $U_{r,s} = Y_{[r:n]}/ Y_{[s:n]}$. It is shown in the following Theorem that the distribution of $U_{r,s}$ is independent of θ_2.

Theorem 2.5.1

The density function of $U_{r,s}$ is

$$f(u) = C_1(1+u)^{-2} + 2C_2(1+2u)^{-2} + 2C_3(2+u)^{-2}, u > 0 \tag{2.5.25}$$

where

$$C_1 + C_2 + C_3 = 1,$$

$$C_1 = 1 - 2\alpha\frac{(n-r-s+1)}{(n+1)} + 2\alpha^2\left(\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right)$$

$$C_2 = \alpha\frac{n-2r+1}{n+1} - \alpha^2\left(\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right)$$

and

$$C_3 = \alpha\frac{n-2s+1}{n+1} - \alpha^2\left(\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right).$$

Proof

Introducing the transformation,

$$U = Y_{[r:n]}/Y_{[s:n]}$$

and

$V=Y_{[s:n]}$ in (2.5.12) we get

$$f(u,v) =$$

$$\frac{v}{\theta_2^2}\exp\{-(1+u)v/\theta_2\}[1+\frac{(n-2r+1)}{n+1}\alpha(2\exp-(\frac{uv}{\theta_2})-1)+$$
$$\frac{(n-2s+1)}{(n+1)}\alpha(2\exp(\frac{-v}{\theta_2})-1)$$
$$+\alpha^2\{\frac{(n-2s+1)}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\}\{2\exp-(\frac{uv}{\theta_2})-1\}\{2\exp-(\frac{v}{\theta_2})-1\}] \tag{2.5.26}$$

Integrating out v,

$$f(u) = \int_0^\infty f(u,v)dv$$

$$=\frac{1}{(1+u)^2}+\frac{(n-2r+1)\alpha}{n+1}\{\frac{2}{(1+2u)^2}-\frac{1}{(1+u)^2}\}+$$
$$\frac{(n-2s+1)\alpha}{n+1}\{\frac{2}{(2+u)^2}-\frac{1}{(1+u)^2}\}+$$
$$\{\frac{(n-2s+1)}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\}2\alpha^2\{\frac{1}{(1+u)^2}-\frac{1}{(1+2u)^2}-\frac{1}{(2+u)^2}\}]$$

$$= C_1(1+u)^{-2}+2C_2(1+2u)^{-2}+2C_3(2+u)^{-2},\ u > 0. \tag{2.5.27}$$

2.6 Distribution of Concomitants from Morgenstern type Uniform distribution

Another member of the family discussed in (2.1.1) is the bivariate uniform distribution (see Mardia (1970)) specified by the distribution function

$$F_{X,Y}(x, y) = \frac{x}{\theta_1}\frac{y}{\theta_2}\left[1+\alpha\left(1-\frac{x}{\theta_1}\right)\left(1-\frac{y}{\theta_2}\right)\right];\ -1 \le \alpha \le 1,\ 0 < x < \theta_1,$$

$$0 < y < \theta_2 . \tag{2.6.1}$$

The distribution function and density function of $Y_{[r:n]}$ follows from (2.2.3) and (2.2.4) respectively as

$$F_{Y[r:n]}(y) = \frac{y}{\theta_2}[1+\frac{(n-2r+1)}{n+1}\alpha(1-\frac{y}{\theta_2})];\quad 0< y <\theta_2 \tag{2.6.2}$$

and

$$f_{Y_{[r:n]}}(y) = \frac{1}{\theta_2}[1+\frac{(n-2r+1)}{n+1}\alpha(1-\frac{2y}{\theta_2})];\ 0 <y <\theta_2. \tag{2.6.3}$$

The moments of $Y_{[r:n]}$ are obtained as

$$\mu_{[r:n]}^{(k)} = \int_0^{\theta_2} y^k \frac{1}{\theta_2}[1+\frac{(n-2r+1)}{n+1}\alpha(1-\frac{2y}{\theta_2})]\,dy$$

$$= \frac{\theta_2^k}{k+1}+\alpha\frac{(n-2r+1)}{n+1}(\frac{\theta_2^k}{k+1}-\frac{2\theta_2^k}{k+2})$$

$$= \frac{\theta_2^k}{k+1}[1-\frac{\alpha(n-2r+1)k}{(n+1)(k+2)}],\ k = 1, 2,\ldots \tag{2.6.4}$$

see, Scaria and Nair (2003).

The means and variances of the concomitants can be evaluated from (2.6.4). The mean of $Y_{[r:n]}$,

$$E[Y_{[r:n]}] = \frac{\theta_2}{2}\{1-\frac{\alpha(n-2r+1)}{3(n+1)}\} \tag{2.6.5}$$

with variance

$$V[Y_{[r:n]}] = E[Y_{[r:n]}^2] - E[Y_{[r:n]}]^2$$

$$= \frac{\theta_2^2}{3}[1 - \frac{(n-2r+1)\alpha}{(n+1)2}] - [\frac{\theta_2}{2}(1 - \frac{(n-2r+1)\alpha}{3(n+1)})]^2$$

$$= \frac{\theta_2^2}{12}[1 - \frac{\alpha^2(n-2r+1)^2}{3(n+1)^2}]. \qquad (2.6.6)$$

The joint cumulative distribution function of $(Y_{[r:n]}, Y_{[s:n]})$ follows from (2.4.3) and is

$$F_{[r,s:n]}(y_1, y_2) = y_1 y_2 / \theta_2^2 \{1 + \frac{\alpha(n-2r+1)}{n+1}(1 - \frac{y_1}{\theta_2}) + \alpha \frac{(n-2s+1)}{n+1}(1 - \frac{y_2}{\theta_2}) + \alpha^2 \left[\frac{(n-2s+1)}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right](1 - \frac{y_1}{\theta_2})(1 - \frac{y_2}{\theta_2})\} \quad 0 < y_1, y_2 < \theta_2 . \qquad (2.6.7)$$

The joint probability density function corresponding to (2.6.7) is

$$f_{[r,s:n]}(y_1,y_2) = 1/\theta_2^2 \{1 + \frac{\alpha(n-2r+1)}{n+1}(1 - 2\frac{y_1}{\theta_2}) + \frac{\alpha(n-2s+1)}{n+1}(1 - 2\frac{y_2}{\theta_2}) + \alpha^2 \left[\frac{(n-2s+1)}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right] (1 - 2\frac{y_1}{\theta_2})(1 - 2\frac{y_2}{\theta_2})\} \; ; \; 0 < y_1, y_2 < \theta_2 . \qquad (2.6.8)$$

The product moments $E(Y_{[r:n]}^l Y_{[s:n]}^m)$, denoted by $\mu_{[r,s:n]}^{(l,m)}$, l, m > 0, are derived from (2.6.8)

$$\mu_{[r,s:n]}^{(l,m)} = \theta_2^{l+m}/(l+1)(m+1) + \alpha \frac{(n-2r+1)}{n+1}\{\theta_2^l / l+1 - 2\theta_2^l / l+2\}\theta_2^m/(m+1) + \alpha \frac{(n-2s+1)}{n+1}\{\theta_2^m/(m+1) - 2\theta_2^m/(m+2)\}\frac{\theta_2^l}{(l+1)} + \alpha^2\{\frac{(n-2s+1)}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\}[\frac{\theta_2^m}{(m+1)} - \frac{2\theta_2^m}{(m+2)}][\frac{\theta_2^l}{(l+1)} - \frac{2\theta_2^l}{(l+2)}]$$

$$= \frac{\theta_2^{l+m}}{(l+1)(m+1)}[1-\alpha\frac{(n-2r+1).l}{(n+1)(l+2)}-\alpha\frac{(n-2s+1).m}{(n+1)(m+2)}+$$

$$\alpha^2\frac{lm}{(l+2)(m+2)}\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\}] \tag{2.6.9}$$

In particular,

$$E[Y_{[r:n]}Y_{[s:n]}] = \frac{\theta_2^2}{4}[1-\frac{2\alpha(n-r-s+1)}{3}+\frac{\alpha^2}{9}\{\frac{(n-2s+1)}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\}]. \tag{2.6.10}$$

Covariance between $Y_{[r:n]}$ and $Y_{[s:n]}$ follows from (2.6.5) and (2.6.10)

$$Cov(Y_{[r:n]}Y_{[s:n]}) = \frac{\theta_2^2}{4}[1-\frac{2\alpha(n-r-s+1)}{3}+\frac{\alpha^2}{9}\{\frac{(n-2s+1)}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\}]-$$

$$\frac{\theta_2^2}{4}[1-\frac{\alpha(n-2r+1)}{3(n+1)}][1-\frac{\alpha(n-2s+1)}{3(n+1)}]$$

$$= \frac{\alpha^2\theta_2^2 r(n-s+1)}{9(n+1)^2(n+2)}. \tag{2.6.11}$$

It follows from (2.6.11) that the concomitants are positively correlated and the covariance between the concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$ decreases as r and s pull apart.

From (2.6.5) we get $E[T_r] = \theta_2$,

where

$$T_r = Y_{[r:n]} + Y_{[s:n]}, \text{the } r^{th} \text{ quasi range, } r = 1,2,\ldots,[\frac{n+1}{2}] \tag{2.6.12}$$

Thus all T_r are unbiased for θ_2 and variance of T_r,

$$V(T_r) = \frac{\theta_2^2}{6}[1-\frac{\alpha^2(n-2r+1)^2}{3(n+1)^2}]+\frac{2(\alpha\theta_2 r)^2}{9(n+1)^2(n+2)}$$

$$= \frac{\theta_2^2}{6}[1-\frac{\alpha^2}{3(n+1)^2}\{(n-2r+1)^2-\frac{4r^2}{(n+2)}\}]. \tag{2.6.13}$$

Among the unbiased estimators T_r, the minimum variance is attained by $T_1 = Y_{[1:n]} + Y_{[n:n]}$.

The smallest variance,

$$V(T_1) = \frac{\theta_2^2}{6}[1 - \frac{\alpha^2}{3(n+1)^2}\{(n-1)^2 - \frac{4}{(n+2)}\}] \qquad (2.6.14)$$

The information required to unbiasedly estimating θ_2 is thus only the Y-values paired with the maximum and minimum of the X-values. By comparison, the efficient estimator $\frac{n+1}{n}$ $Y_{n:n}$ is more efficient than T_1, but the former utilizes the entire sample values.

Using $Y_{[1:n]} = T_1[1 - \frac{\alpha(n-1)}{3(n+1)}]$ and

$Y_{[n:n]} = T_1[1 + \frac{\alpha(n-1)}{3(n+1)}]$ obtained by substituting the estimate of θ_2 we obtain a quick estimate of α as

$$\hat{\alpha} = \begin{cases} 1 & R_1 \geq \frac{n-1}{3(n+1)} \\ \frac{3(n+1)}{n-1}R_1 \quad \text{if} & \frac{1-n}{3(n+1)} < R_1 < \frac{n-1}{3(n+1)} \\ -1 & R_1 \leq \frac{1-n}{3(n+1)} \end{cases} \qquad (2.6.15)$$

where,

$$R_1 = \frac{Y_{[n:n]} - Y_{[1:n]}}{2(Y_{[n:n]} + Y_{[1:n]})}. \qquad (2.6.16)$$

The correlation coefficient of Morgenstern type uniform distribution is $\rho = \alpha/3$.

An efficient estimator of ρ based on concomitants is suggested by Tsukibayashi and is given in David and Nagaraja (1998). If X and Y have the same marginal distribution form then

$$\hat{\rho} = \frac{\bar{Y}'_{[k:n]} - \bar{Y}_{[k:n]}}{\bar{Y}'_{k:n} - \bar{Y}_{k:n}} \qquad (2.6.17)$$

where

$$\overline{Y}'_{[k:n]} = \frac{1}{k}\sum_{i=1}^{k} Y_{[n-i+1:n]}, \qquad \overline{Y}_{[k:n]} = \frac{1}{k}\sum_{i=1}^{k} Y_{[i:n]}$$

$$\overline{Y}'_{k:n} = \frac{1}{k}\sum_{i=n}^{k} Y_{n-i+1:n}, \qquad \overline{Y}_{k:n} = \frac{1}{k}\sum_{i=1}^{k} Y_{i:n}.$$

Hence an efficient estimate of α is

$$\hat{\alpha} = \begin{cases} 1 & \hat{\rho} \geq \frac{1}{3} \\ 3\hat{\rho} & -\frac{1}{3} < \hat{\rho} < \frac{1}{3} \\ -1 & \hat{\rho} \leq \frac{1}{3} \end{cases} \tag{2.6.18}$$

As in the exponential case all suggested estimators can be calculated even if only ranks of X's are available.

2.7 Concomitants from Morgenstern type Bivariate Logistic Distribution

Logistic distribution has many applications in life sciences and social sciences (see Balakrishnan (1992)). Durling (1969) has developed a 'Bivariate logit' method of analysis of bivariate quantal response data based on the Gumbel's (1961) type II logistic distribution which is specified by the distribution function

$$F_{X,Y}(x,y) = \frac{1}{\left[1+\exp-\left(\frac{x-\mu_1}{\sigma_1}\right)\right]}\frac{1}{\left[1+\exp-\left(\frac{y-\mu_2}{\sigma_2}\right)\right]}$$

$$\left[1+\alpha\frac{\exp-\left(\frac{x-\mu_1}{\sigma_1}\right)}{1+\exp-\left(\frac{x-\mu_1}{\sigma_1}\right)}\cdot\frac{\exp-\left(\frac{y-\mu_2}{\sigma_2}\right)}{\left(1+\exp-\left(\frac{y-\mu_2}{\sigma_2}\right)\right)}\right] \tag{2.7.1}$$

$$-\infty < x,\ y < \infty;\ -\infty < \mu_1,\ \mu_2 < \infty;\ \sigma_1, \sigma_2 > 0$$

The distribution function of $Y_{[r:n]}$ follows from (2.2.3) as

$$F_{Y[r:n]}(y) = \frac{1}{[1+\exp-(\frac{y-\mu_2}{\sigma_2})]}[1+\alpha\frac{(n-2r+1)}{(n+1)}\frac{\exp-(\frac{y-\mu_2}{\sigma_2})}{[1+\exp-(\frac{y-\mu_2}{\sigma_2})]}]$$

$$-\infty<y<\infty;\ -\infty<\mu_2<\infty; \sigma_2>0. \qquad (2.7.2)$$

For the Logistic distribution

$$f_Y(y) = \frac{1}{\sigma_2}\frac{\exp-(\frac{y-\mu_2}{\sigma_2})}{[1+\exp-(\frac{y-\mu_2}{\sigma_2})]^2}$$

and we note that

$$\begin{aligned}\sigma_2 f(y) &= F(y)[1-F(y)]\\ \sigma_2 f'(y) &= f(y)[1-2F(y)]\end{aligned} \qquad (2.7.3)$$

Hence (2.7.2) takes the form

$$F_{Y_{[r:n]}}(y) = F_Y(y)+\alpha\sigma_2\frac{(n-2r+1)}{n+1}f(y). \qquad (2.7.4)$$

It follows from (2.7.4) that the density function

$$f_{Y_{[r:n]}}(y) = f_Y(y)[1+\alpha\frac{(n-2r+1)}{n+1}(1-2F_Y(y))]$$

$$f_{Y[r:n]} = 1/\sigma_2\frac{\exp-\left(\frac{y-\mu_2}{\sigma_2}\right)}{\left[1+\exp-\left(\frac{y-\mu_2}{\sigma_2}\right)\right]^2}\left[1-\alpha\frac{n-2r+1}{n+1}\left\{\frac{1-\exp-\left(\frac{y-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y-\mu_2}{\sigma_2}\right)}\right\}\right]$$

$$-\infty<y<\infty,\ -\infty<\mu_2<\infty,\ \sigma_2>0 \qquad (2.7.5)$$

For the logistic distribution, the moment generating function $M_Y(t)$ satisfies the relation

$$t\sigma_2 M_Y(t) = M_{2:2}(t) - M_{1:2}(t).$$

Hence from (2.3.5) we get

$$M_{[r:n]}(t) = M_Y(t)[1-\alpha t\sigma_2 \frac{(n-2r+1)}{n+1}]$$

$$= \Gamma(1-t)\Gamma(1+t)[1-\alpha t\sigma_2 \frac{(n-2r+1)}{n+1}]. \qquad (2.7.6)$$

It follows directly from (2.7.6) that the k^{th} moment of $Y_{[r:n]}$ as

$$\mu_{[r:n]}^{(k)} = \mu^{(k)} - k\alpha\sigma_2 \frac{(n-2r+1)}{n+1}\mu^{(k-1)} \quad k = 1, 2, \ldots \qquad (2.7.7)$$

It also from (2.3.1) that

$$\mu_{[r:n[}^{(k)} = \mu^{(k)} + \frac{\alpha}{2}\frac{n-2r+1}{n+1}[\mu^{(k)}{}_{1:2} - \mu^{(k)}{}_{2:2}]. \qquad (2.7.8)$$

Using the recurrence relation, Shaw (1970),

$$\mu_{1:2}^{(k)} - \mu_{2:2}^{(k)} = -2k\ \sigma_2\mu^{(k-1)}$$

we get

$$\mu_{[r:n[}^{(k)} = \mu^{(k)} - k\alpha\sigma_2 \frac{(n-2r+1)}{n+1}\mu^{(k-1)}. \qquad (2.7.9)$$

Thus (2.7.9) agrees with (2.7.7).

The mean of $Y_{[r:n]}$ is

$$E[Y_{[r:n]}] = \mu_2 - \alpha\sigma_2 \frac{(n-2r+1)}{(n+1)} \qquad (2.7.10)$$

and

$$\mu_{[r:n]}^2 = \mu^{(2)} - \alpha\sigma_2\mu^{(1)} = \mu_2^2 + \frac{\pi^2}{3}\sigma_2^2 - 2\alpha\mu_2 \frac{(n-2r+1)}{(n+1)}. \qquad (2.7.11)$$

Variance of $Y_{[r:n]}$,

$$V(Y_{[r:n]}) = \sigma_2^2[\frac{\pi^2}{3} - \alpha^2 \frac{(n-2r+1)^2}{(n+1)^2}]. \tag{2.7.12}$$

The following recurrence relations immediately follows from (2.7.9).

Relation 2.7.1

For the bivariate population with pdf as in (2.7.2) we have the relation

$$\mu_{[r:n]}^{(k+1)} = \mu^{(k+1)} - \frac{k}{(k+1)}\beta_1^2 \mu^{(k-1)} - \beta_1 \mu_{[r:n]}^{(k)}$$

where

$$\beta_1 = \alpha\sigma_2(k+1)\frac{(n-2r+1)}{(n+1)}. \tag{2.7.13}$$

Proof

From (2.7.9)

$$\mu_{[r:n]}^{(k+1)} = \mu^{(k+1)} - \alpha(k+1)\sigma_2 \frac{(n-2r+1)}{(n+1)}\mu^{(k)}$$

$$= \mu^{(k+1)} - \alpha(k+1)\sigma_2 \frac{(n-2r+1)}{(n+1)}[\mu_{[r:n]}^{(k)} + k\alpha\sigma_2 \frac{(n-2r+1)}{(n+1)}\mu^{(k-1)}]$$

$$= \mu^{(k+1)} - \frac{k}{(k+1)}\beta_1^2 \mu^{(k-1)} - \beta_1 \mu_{[r:n]}^{(k)}.$$

Relation 2.7.2

$$(n+1)[\mu_{[r:n]}^{(k)} - \mu_{[1:n]}^{(k)}] = 2k(r-1)\alpha\sigma_2\mu^{(k-1)}, \quad 1 < r \le n; k = 1,2,\ldots \tag{2.7.14}$$

Proof

From (2.7.9)

$$\mu_{[r:n]}^{(k)} - \mu_{[1:n]}^{(k)} = (-k)\alpha\sigma_2\mu^{(k-1)}[\frac{(n-2r+1)}{n+1} - \frac{(n-1)}{n+1}]$$

$$= 2k\alpha\sigma_2(\frac{r-1}{n+1})\mu^{(k-1)}$$

Hence $(n+1)[\mu^{(k)}_{[r:n]} - \mu^{(k)}_{[1:n]}] = 2k(r-1)\alpha\sigma_2\mu^{(k-1)}$

Relation 2.7.3

$$n\,(n+1)[\mu^{(k)}_{[r:n]} - \mu^{(k)}_{[r:n-1]}] + 2rk\alpha\sigma_2\mu^{(k-1)} = 0. \qquad (2.7.15)$$

From (2.7.9)

$$\mu^{(k)}_{[r:n]} - \mu^{(k)}_{[r:n-1]} = k\alpha\sigma_2\mu^{(k-1)}[\{(n-2r)/n\} - \{(n-2r+1)/(n+1)\}]$$

$$= \alpha\sigma_2 k\mu^{(k-1)}\frac{(-2r)}{n(n+1)}$$

Hence $n(n+1)[\mu^{(k)}_{[r:n]} - \mu^{(k)}_{[r:n-1]}] + 2rk\alpha\sigma_2\mu^{(k-1)} = 0.$

In general we replace $E[Y^k_{[r:n]}]$ by $E[h(Y_{[r:n]})]$ then the above recurrence relations are all satisfied , provided $E[h(Y_{[r:n]})]$ exists .

The joint distribution function of $Y_{[r:n]}$ and $Y_{[s:n]}$ $(r < s)$ follows from (2.4.3) and is

$$F_{[r,s:n]}(y_1,y_2) = \frac{1}{\left[1+\exp-\left(\frac{x-\mu_1}{\sigma_1}\right)\right]}\ \frac{1}{\left[1+\exp-\left(\frac{y-\mu_2}{\sigma_2}\right)\right]}$$

$$[1+\alpha\frac{n-2r+1}{n+1}\left\{\frac{\exp-\left(\frac{y_1-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y_1-\mu_2}{\sigma_2}\right)}\right\} + \alpha\frac{n-2s+1}{n+1}\left\{\frac{\exp-\left(\frac{y_2-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y_2-\mu_2}{\sigma_2}\right)}\right\}$$

$$\alpha^2\left\{\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right\}\left\{\frac{\exp-\left(\frac{y_1-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y_1-\mu_2}{\sigma_2}\right)}\right\}\left\{\frac{\exp-\left(\frac{y_2-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y_2-\mu_2}{\sigma_2}\right)}\right\}]$$

$$-\infty < y_1,\ y_2 < \infty;\ -\infty < \mu_2 < \infty\ ;\ \sigma_2 > 0. \qquad (2.7.16)$$

Using $\sigma_2 f(y) = F(y)[1-F(y)]$ in (2.7.16) it reduces to

$$F_{[r,s:n]}(y_1,y_2) = F_Y(y_1)F_Y(y_2) +$$

$$\alpha\sigma_2\frac{(n-2r+1)}{n+1}f(y_1)F(y_2)+\alpha\sigma_2\frac{(n-2s+1)}{n+1}f(y_2)F(y_1)$$

$$+(\alpha\sigma_2)^2\{\frac{(n-2s+1)}{(n+1)}-\frac{2r(n-2s)}{(n+1)(n+2)}\}f(y_1)f(y_2) \tag{2.7.17}$$

The joint probability function follows from (2.4.5)

$$f_{Y[r,s:n]}(y_1,y_2) = \frac{1}{\sigma_2^2}\;\frac{\exp-\left(\dfrac{y_1+y_2-2\mu_2}{\sigma_2}\right)}{\left[1+\exp-\left(\dfrac{y_1-\mu_2}{\sigma_2}\right)\right]^2\left[1+\exp-\left(\dfrac{y_2-\mu_2}{\sigma_2}\right)\right]^2}$$

$$[1-\alpha\frac{n-2r+1}{n+1}\left\{\frac{1-\exp-\left(\dfrac{y_1-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\dfrac{y_1-\mu_2}{\sigma_2}\right)}\right\}-\alpha\frac{n-2s+1}{n+1}\left\{\frac{1-\exp-\left(\dfrac{y_2-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\dfrac{y_2-\mu_2}{\sigma_2}\right)}\right\}$$

$$+\alpha^2\left\{\frac{n-2s+1}{n+1}-\frac{2r(n-2s)}{(n+1)(n+2)}\right\}$$

$$\left\{\frac{1-\exp-\left(\dfrac{y_1-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\dfrac{y_1-\mu_2}{\sigma_2}\right)}\right\}\left\{\frac{1-\exp-\left(\dfrac{y_2-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\dfrac{y_2-\mu_2}{\sigma_2}\right)}\right\}]$$

$$-\infty < y_1,y_2 < \infty\ ;\quad -\infty < \mu_2 < \infty\ ;\quad \sigma_2 > 0. \tag{2.7.18}$$

Using $\mu_{1:2}^{(k)} - \mu_{2:2}^{(k)} = -2k\sigma_2\mu^{(k-1)}$ in the formula (2.4.6) for the product moments, we get $\mu^{(l,m)}_{[r,s:n]} = \mu^{(l)}\mu^{(m)} -$

$$\alpha\sigma_2 l\frac{(n-2r+1)}{n+1}\mu^{(l-1)}\mu^{(m)} - \alpha\sigma_2 m\frac{(n-2s+1)}{n+1}\mu^{(m-1)}\mu^{(l)}$$

$$+(\alpha\sigma_2)^2 lm\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\}\mu^{(l-1)}\mu^{(m-1)} \qquad (2.7.19)$$

In particular for l = 1 and m = 1

$$E[Y_{[r:n]}Y_{[s:n]}] = \mu_2^2 - 2\alpha\sigma_2\mu_2\frac{(n-r-s+1)}{(n+1)} + \alpha^2\sigma_2^2\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\}. \qquad (2.7.20)$$

Using (2.7.10) and (2.7.20)

Cov $[Y_{[r:n]}Y_{[s:n]}]$ =

$$\mu_2^2 - 2\alpha\sigma_2\mu_2\frac{(n-r-s+1)}{(n+1)} + \alpha^2\sigma_2^2\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\} -$$

$$[\mu_2-\alpha\sigma_2\frac{(n-2r+1)}{(n+1)}]\,[\mu_2-\alpha\sigma_2\frac{(n-2s+1)}{(n+1)}]$$

$$= \alpha^2\sigma_2^2[\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)} - \frac{(n-2r+1)(n-2s+1)}{(n+1)^2}]$$

$$= \alpha^2\sigma_2^2[\frac{(n-2s+1)}{n+1}\{1-\frac{(n-2r+1)}{(n+1)}\} - \frac{2r(n-2s)}{(n+1)(n+2)}\}]$$

$$= \frac{4r(n-s+1)}{(n+1)^2(n+2)}\alpha^2\sigma_2^2. \qquad (2.7.21)$$

As in the bivariate exponential case, the quasi mid ranges,

$$T_r = \frac{Y_{[r:n]} + Y_{[n-r+1]}}{2}$$

are all unbiased for the location parameter μ_2. Among all the unbiased estimators T_r, $T_1 = \{Y_{[1:n]} + Y_{[n:n]}\}/2$ has the least variance.

The smallest variance is

$$V(T_1) = \frac{\sigma_2^2}{2}\left[\frac{\pi^2}{3} - \frac{\alpha^2(n-1)^2}{(n+1)^2} + \frac{4\alpha^2}{(n+1)^2(n+2)}\right]. \qquad (2.7.22)$$

Thus the information required to estimate the location parameter μ_2 is only the Y- values associated with the maximum and minimum of the X's.

2.8 Concomitants from Morgenstern type Bivariate Gamma distribution

A Morgenstern type bivariate gamma distribution is constructed by D'Este (1981) and is specified by the cumulative distribution function

$$F_{X,Y}(x,y) = P(\alpha,x)P(\beta,y)[1+\lambda(1-P(\alpha,x))(1-P(\beta,y))];\ 0 < x,\ y < \infty \qquad (2.8.1)$$

where $P(\alpha, x)$ is the incomplete gamma function.

The joint probability density function of (X,Y) is

$$f_{X,Y}(x,y) = [\Gamma(\alpha)\Gamma(\beta)]^{-1} x^{\alpha-1} y^{\beta-1} \exp -(x+y)$$
$$[1+\lambda(1-2P(\alpha,x))(1-2P(\beta,y))]$$
$$0 < x,\ y < \infty. \qquad (2.8.2)$$

The distribution function and probability density function of $Y_{[r:n]}$ follows from (2.2.3) and (2.2.4) as

$$F_{Y_{[r:n]}}(y) = P(\beta,y)[1+\frac{(n-2r+1)}{n+1}\lambda(1-P(\beta,y))];\ 0<y<\infty \qquad (2.8.3)$$

and

$$f_{Y_{[r:n]}}(y) = \frac{1}{\Gamma(\beta)}\exp(-y)y^{\beta-1}[1+\frac{(n-2r+1)}{n+1}\lambda(1-2P(\beta,y))];\ 0 < y < \infty. \qquad (2.8.4)$$

The moments follows from (2.3.1)

For k = 1, 2,...

$$\mu_{[r:n]}^{(k)} = \mu^{(k)} + \frac{\lambda}{2}\frac{n-2r+1}{n+1}[\mu^{(k)}{}_{1:2} - \mu^{(k)}{}_{2:2}]$$

where ,

$$\mu^{(k)} = \Gamma(k+\beta)/\Gamma(\beta). \tag{2.8.5}$$

Since 1-2F = 2(1-F)-1, for any distribution

$$\frac{\mu_{1:2}^{(k)} - \mu_{2:2}^{(k)}}{2} = \mu_{1:2}^{(k)} - \mu^{(k)}. \tag{2.8.6}$$

Then (2.8.5) becomes

$$\begin{aligned}\mu_{[r:n]}^{(k)} &= \frac{\Gamma(k+\beta)}{\Gamma(\beta)} + \frac{(n-2r+1)}{n+1}\lambda(\mu_{1:2}^{(k)} - \frac{\Gamma(k+\beta)}{\Gamma(\beta)}) \\ &= \frac{\Gamma(k+\beta)}{\Gamma(\beta)}[1-\lambda\frac{(n-2r+1)}{(n+1)}] + \frac{(n-2r+1)}{(n+1)}\lambda\mu_{1:2}^{(k)}.\end{aligned} \tag{2.8.7}$$

Gupta ((1960), (1962)) has tabulated the values of $\mu_{1:2}^{(k)}$ for k =1, 2,3 and 4 for sample sizes up to 15 when β = 1(1)5. Breiter and Krishnaiah (1968) have provided the values of $\mu_{1:2}^{(k)}$ for k=1,2,3 and 4 for sample sizes up to 9 when β = 0.5(1) 10.5.

If β is an integer $\mu_{1:2}^{(k)} = \frac{2}{\Gamma(\beta)}\sum_{s=0}^{\beta-1} a_s(\beta,1)\Gamma(k+\beta+s)2^{-(k+\beta+s)}$ (2.8.8)

where,

$a_s(\beta,1) = \frac{1}{\Gamma(s+1)}$ s = 0,1,2,... β-1, see Balakrishnan and Cochen (1991).

Thus if β is an integer

$$\mu_{[r:n]}^{(k)} = \frac{\Gamma(k+\beta)}{\Gamma(\beta)}[1-\frac{(n-2r+1)}{n+1}\lambda] + \frac{(n-2r+1)}{n+1}\lambda\frac{2}{\Gamma(\beta)}\sum_{s=0}^{\beta-1}\frac{\Gamma(k+\beta+s)}{\Gamma(s+1)2^{(k+\beta+s)}}. \tag{2.8.9}$$

Using (2.8.9) we can tabulate the values of the means and variances of $Y_{[r:n]}$, r = 1,2,...n for various values of n, λ and β(integer).

The joint cumulative distribution function of $(Y_{[r:n]}, Y_{[s:n]})$ (r <s) follows from, (2.4.3) and is given by

$$F_{[r,s:n]}(y_1, y_2) = P(\beta, y_1)P(\beta, y_2)[1 + \lambda(1 - P(\beta, y_1))\frac{(n-2r+1)}{(n+1)} +$$

$$\lambda\frac{(n-2s+1)}{(n+1)}(1 - P(\beta, y_2)) +$$

$$\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\}\lambda^2(1 - P(\beta, y_1))(1 - P(\beta, y_2))]$$

$$0 < y_1, y_2 < \infty. \qquad (2.8.10)$$

The joint probability density function is

$$f_{[r,s:n]}(y_1, y_2) = \Gamma(\beta)^{-2}(y_1 y_2)^{\beta-1}\exp-(y_1 + y_2)$$

$$[1 + \lambda(1 - 2P(\beta, y_1))\frac{(n-2r+1)}{(n+1)} + \lambda\frac{(n-2s+1)}{(n+1)}(1 - 2P(\beta, y_2)) +$$

$$\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\}\lambda^2(1 - 2P(\beta, y_1))(1 - 2P(\beta, y_2))]$$

$$0 < y_1, y_2 < \infty. \qquad (2.8.11)$$

The product moments $\mu_{[r,s:n]}^{(l,m)}$ are obtained directly from (2.8.11) as $\mu_{[r,s:n]}^{(l,m)}$

$$= \mu^{(l)}\mu^{(m)} + \frac{(n-2r+1)}{(n+1)}\lambda\mu^{(m)}[\mu_{1:2}^{(l)} - \mu^{(l)}] +$$

$$\frac{(n-2s+1)}{(n+1)}\lambda\mu^{(l)}[\mu_{1:2}^{(m)} - \mu^{(m)}] + \lambda^2\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\}$$

$$[\mu_{1:2}^{(l)} - \mu^{(l)}][\mu_{1:2}^{(m)} - \mu^{(m)}]$$

$$(2.8.12)$$

In particular

$E[Y_{[r:n]}Y_{[s:n]}]$

$$= \beta^2 + \frac{(n-2r+1)}{(n+1)}\lambda\beta[\mu_{1:2} - \beta] + \frac{(n-2s+1)}{(n+1)}\lambda\beta[\mu_{1:2} - \beta] +$$

$$\lambda^2\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\}[\mu_{1:2} - \beta\ ]^2. \tag{2.8.13}$$

From (2.8.7) and (2.8.13),

Cov $(Y_{[r:n]}Y_{[s:n]})$

$$= \beta^2 + \frac{(n-2r+1)}{(n+1)}\lambda\beta[\mu_{1:2} - \beta] + \frac{(n-2s+1)}{(n+1)}\lambda\beta[\mu_{1:2} - \beta]$$

$$+ \lambda^2\{\frac{(n-2s+1)}{(n+1)} - \frac{2r(n-2s)}{(n+1)(n+2)}\}[\mu_{1:2} - \beta\ ]^2 -$$

$$[\beta(1-\lambda\frac{(n-2r+1)}{(n+1)}) + \frac{(n-2r+1)}{(n+1)}\lambda\mu_{1:2}]$$

$$[\beta(1-\lambda\frac{(n-2s+1)}{(n+1)}) + \frac{(n-2s+1)}{(n+1)}\lambda\mu_{1:2}]$$

$$= \lambda^2(\mu_{1:2} - \beta\ )^2\{\frac{(n-2s+1)}{(n+1)}(1 - \frac{(n-2r+1)}{(n+1)}) - \frac{2r(n-2s)}{(n+1)(n+2)}\}$$

$$= 4\lambda^2(\mu_{1:2} - \beta)^2\frac{r(n-s+1)}{(n+1)^2(n+2)}. \tag{2.8.14}$$

Thus covariance between the r[th] and s[th] concomitants is positively correlated and it decreases as r and s pull apart.

Chapter 3

DISTRIBUTION OF THE MAXIMUM OF CONCOMITANTS OF SELECTED ORDER STATISTICS FROM THE MORGENSTERN FAMILY OF DISTRIBUTIONS

3.1 Introduction

As described earlier, the concomitants have found a wide variety of applications in many areas such as selection problems, prediction analysis, double sampling plans, ocean engineering and inference problems. The most important use of concomitants arise in selection problems as described in Yeo and David (1984) when k (<n) individuals are chosen on the basis of their X-values where the corresponding Y- values are of primary interest. For example X could be the score in a screening test and Y the score obtained after a further training and a second test. Then $V_{k,n} = \max(Y_{[n-k+1:n]} \ldots, Y_{[n:n]})$, $k = 1, 2, \ldots, n$ represents the score of the best performer in the second test. The ratio $e_{k,n} = E[V_{k,n}]/E[Y_{n:n}]$ which clearly increases to 1 with k, is a measure of effectiveness of the screening procedure. One may wish to choose k to make this ratio sufficiently close to 1.

Feinberg (1991) and Feinberg and Huber (1996) have investigated some properties of $v_{k,n}$ in a study of cutoff rules under imperfect information. Feinberg (1991) used simulation to examine the behavior of E $(V_{k,n})$ for selected values of n assuming the sample is drawn from a bivariate normal distribution.

Motivated by Feinberg's work, Nagaraja and David (1994) investigated the finite sample and asymptotic properties of $V_{k,n}$ for an arbitrary absolutely continuous bivariate c.d.f F. They have established the following important results concerning the finite and asymptotic distribution of $V_{k,n}$.

The finite sample cumulative distribution function of $V_{k,n}$ is derived using symmetry arguments . They have shown that

$$F_{k,n}(y) = P[V_{k,n} \leq y] = P[Y_{[n-k+1:n]} \leq y, \ldots Y_{[n:n]} \leq y]$$

$$= \int_{-\infty}^{\infty} [F^{*}_{Y|X}(y \mid x)]^{k} f_{X_{n-k:n}}(x)dx \tag{3.1.1}$$

where

$$F^{*}_{Y|X}(y|x) = P[Y \leq y | X > x]$$

and

$f_{n-k:n}(x)$ is the pdf of $X_{n-k:n}$.

The asymptotic distribution of $V_{k,n}$ in the extreme case can be given in the following result.

Lemma 3.1.1

If Von Mises conditions are satisfied, there exists constants $a_n, b_n > 0$ such that the p.d.f of $(X_{n-i+1:n}-a_n)/b_n$ converges to g_i, the p.d.f of W_i, the i^{th} lower record value from the c.d.f G , for any fixed i. Further, the joint p.d.f of $W_1, W_2 \ldots W_{k+1}$ and the marginal p.d.f of W_{k+1} are given respectively by

$$g_{(k+1)}(w_1, \ldots w_{k+1}) = g(w_{k+1}) \prod_{r=1}^{k} \frac{g(w_r)}{G(w_r)} \qquad w_1 > w_2 \ldots \quad > w_{k+1}$$

and

$$g_{k+1}(w) = \frac{[-\log G(w)]^{k}}{k!} g(w) \tag{3.1.2}$$

where

$G = G_1, G_2$ or G_3 , see page 10.

Result 3.1.1

Suppose the conditions of Lemma 3.1.1 hold and assume there exist constants A_n and $B_n > 0$ such that

$$F_2^*(A_n + B_n y \mid a_n + b_n x) \to H(y \mid x) \text{ as } n \to \infty \text{, for all x and y .}$$

Then as as $n \to \infty$

$$F_{k,n}(A_n + B_n y) \to \int_{-\infty}^{\infty} [H(y \mid x)]^k g_{k+1}(x) dx \tag{3.1.3}$$

Now suppose the joint distribution of (X,Y) is such that as $x \to \xi_1(1), F_2(y \mid x) \to H(y)$. Where $\xi_1(p)$ the p^{th} quantile of X.

Then (3.1.1) holds with $H(y|x) = H(y)$ and $A_n = 0$, $B_n = 1$ then the lemma follows.

Lemma 3.1.2

If, for some y,

$$\lim_{x \to \xi_1(1)} F_2(y \mid x) = H(y)$$

then

$$\lim_{x \to \xi_1(1)} F^*{}_2(y \mid x) = H(y) \tag{3.1.4}$$

If (3.1.4) holds and the appropriate Von Mises condition hold for F_1, then as $n \to \infty$

$$P[V_{k,n} \le y] = [H(y)]^k \tag{3.1.5}$$

Thus if H is a c.d.f then $V_{k,n} \to V$, where V behaves like the maximum of a random sample of size k from the c.d.f H .

The asymptotic distribution of $V_{k,n}$ in the quantile case is given in the following result.

Result 3.1.2

Assume that $f_1(x') > 0$, where $x' = \xi(q)$, with $q = 1\text{-}p$. For constants A_k and $B_k > 0$ free of x, and for fixed y define

$$H_n(y|x) = [F_2^*(A_n + B_n y \mid a_n + b_n x)]^k. \tag{3.1.6}$$

Assume that, as $n \to \infty$, for all u

$$H_n(y \mid x' + u/n^{1/2}) \to H(y \mid x') \tag{3.1.7}$$

then $$F_{k,n}(A_k + B_k y) \to H(y \mid x').$$

Using these basic results the present chapter deals with the distribution theory of $V_{k,n}$ for the Morgenstern family of bivariate distributions and obtain certain characteristics that could be useful in selection problems. The importance of this family discussed in Chapter 2 arises from the fact that it enables construction of bivariate distributions with specified marginals. When prior information is available in the form of marginal distributions, we get a class of distributions indexed by the parameter, from which a suitable member that is appropriate to the data could be found.

3.2 Distribution of $V_{k,n}$

The cumulative distribution function (cdf) of $V_{k,n}$ follows from (3.1.1) and is

$$F_{k,n}(y) = P[V_{k,n} \le y] = \int_{-\infty}^{\infty} \left[F_{Y|X}^*(y|x)\right]^k f_{X_{n-k:n}}(x)\,dx$$

where

$$F_{Y|X}^*(y|x) = P[Y \le y \mid X > x]$$

and

$$f_{X_{n-k:n}}(x) = \frac{1}{B(n-k, k+1)} [F_X(x)]^{n-k-1} [1-F_X(x)]^k f_X(x). \tag{3.2.1}$$

For the Morgenstern family we have

$$F^{*}_{Y|X}(y|x) = \frac{\int_x^{\infty} F_{Y|X}(y \mid x) f_X(x)dx}{\int_x^{\infty} f_X(x)dx}$$

$$= [\int_x^{\infty} F_Y(y)\{1+\alpha(1-F_Y(y))(1-2F_X(x))\}f_X(x)dx]/(1-F_X(x))$$

$$= \frac{F_Y\ (y)\{(1-F_X(x))-\alpha(1-F_Y(y))(1-F_X(x))F_X(x)\}}{1-F_X(x)}$$

$$= F_Y(y)\ [1-\alpha F_X(x)(1-F_Y(Y))] \tag{3.2.2}$$

and hence

$$F_{k,n}(y) = \int_{-\infty}^{\infty} [F_Y(y)\{1-\alpha F_X(x)(1-F_Y(Y))\}]^k\ f_{X_{n-k:n}}(x)\ dx$$

$$= \int_{-\infty}^{\infty} [F_Y(y)\{1-\alpha F_X(x)(1-F_Y(Y))\}]^k \frac{1}{B(n-k,k+1)}\ [F_X(x)]^{n-k-1}\ [1-F_X(x)]^k\ f_X(x)\ dx$$

$$= \frac{[F_Y(y)]^k}{B(n-k,k+1)} \int_{-\infty}^{\infty} [1-\alpha(1-F_Y(y))F_X(x)]^k [1-F_X(x)]^k F_X(x)^{n-k-1} f_X(x)dx\,.$$

Using binomial expansion and simplifying we get

$$F_{k,n}(y) = \frac{[F_Y(y)]^k}{B(n-k,k+1)} \sum_{t=0}^{k} (-\alpha)^t \binom{k}{t} B(n-k+t,\ k+1)\ [1-F_Y(y)]^t\,. \tag{3.2.3}$$

It follows from (3.2.3) that the probability density function of $V_{k,n}$ is

$$f_{k,n}(y) = [F_Y(y)]^{k-1} f_y(y)\ [k+\sum_{t=1}^{k} (-\alpha)^t \binom{k}{t} \frac{B(n-k+t,k+1)}{B(n-k,k+1)}$$

$$[1-F_Y(y)]^{t-1}\ \{k-(k+t)F_Y(y)\}]$$

$$= f_{k:k}(y) + \frac{1}{B(n-k,k+1)} \sum_{t=1}^{k} (-\alpha)^t \frac{B(n-k+t,k+1)}{\binom{k+t-1}{k}} [f_{k:k+t-1}(y) - f_{k+1:k+t}(y)] \tag{3.2.4}$$

where $f_{r:s}(y)$ is the probability density function of $Y_{r:s}$. The moments of $V_{k,n}$ are derived from (3.2.4) as

For s = 1, 2...

$$\mu_{k,n}^{(s)} = E[V_{k,n}^{s}] = \mu_{k:k}^{(s)} + \frac{1}{B(n-k,k+1)}$$

$$\sum_{t=1}^{k} (-\alpha)^{t} \frac{B(n-k+t,k+1)}{\binom{k+t-1}{k}} \left(\mu_{k:k+t-1}^{(s)} - \mu_{k+1:k+t}^{(s)}\right) \qquad (3.2.5)$$

where

$$\mu_{r:j}^{(s)} = E[Y_{r:j}^{s}];\ s = 1, 2, \ldots,\ k = 1, 2, \ldots, n.$$

The distribution function (3.2.2) can be written as

$$F_{k,n}(y) = F_Y^k(y) \sum_{t=0}^{k} (-1)^t \binom{k}{t} \frac{(n-k)(n-k+1)\ldots(n-k+t-1)}{(n+1)(n+2)\ldots(n+t)} \alpha^t [1-F_Y(y)]^t$$

After using the reduction formula for gamma functions,

$$F_{k,n}(y) = F_Y^k(y) \sum_{t=0}^{k} (-1)^t \binom{k}{t} \frac{\left(1-\frac{k}{n}\right)\ldots\left(1-\frac{k-t+1}{n}\right)}{\left(1+\frac{1}{n}\right)\ldots\left(1+\frac{t}{n}\right)} \alpha^t [1-F_Y(y)]^t.$$

As $n \to \infty$, keeping k fixed, we find that the asymptotic distribution of $V_{k,n}$ has distribution function

$$F_k(y) = F_Y^k(y) \sum_{t=0}^{k} (-1)^t \binom{k}{t} \alpha^t [1-F_Y(y)]^t.$$

$$= F_Y^k(y) [1- \alpha \{1-F_Y(y)\}]^k. \qquad (3.2.6)$$

The density function corresponding to (3.2.6) becomes

$$f_k(y) = k F_Y^{k-1}(y) f_y(y) [1- \alpha \{1-F_Y(y)\}]^{k-1} [1- \alpha \{1-2F_Y(y)\}]. \qquad (3.2.7)$$

(3.2.7) may be represented as

$$f_k(y) = k F_Y^{k-1}(y)\, f_Y(y)\, [1-\alpha\{1-F_Y(y)\}]^k + k F_Y^k(y)\, f_Y(y)\, \alpha[1-\alpha\{1-F_Y(y)\}]^{k-1}.$$

$$= (k!)^2 \left[\frac{(-\alpha)^k}{(2k)!} f_{k:2k}(y) + \sum_{t=0}^{k-1} \frac{(-\alpha)^t}{(k-t)!(k+t)!} [f_{k:k+t}(y) + \frac{\alpha(k-t)}{(k+t+1)} f_{k+1:k+t+1}(y)] \right].$$

(3.2.8)

Moments of (3.2.8) are calculated as

$$\mu_k^{(s)} = E[V_k^s] = (k!)^2 \left[\frac{(-\alpha)^k}{(2k)!} \mu_{k:2k}^{(s)} + \sum_{t=0}^{k-1} \frac{(-\alpha)^t}{(k-t)!(k+t)!} [\mu_{k:k+t}^{(s)} + \frac{\alpha(k-t)}{(k+t+1)} \mu_{k+1:k+t+1}^{(s)}] \right] \quad (3.2.9)$$

where

$$\mu_{r:j}^{(s)} = E[Y_{r:j}^s]; \; s = 1, 2, \; \ldots .$$

3.3 Particular Cases

In this section we specialise the results in the previous section for some well-known members of the family.

3.3.1 Bivariate Exponential Distribution

For the bivariate exponential distribution specified by (2.5.1) it follows from (3.2.3) that,

$$F_{k,n}(y) = \frac{\left[1-\exp\left\{-\frac{y}{\theta_2}\right\}\right]^k}{B(n-k,k+1)} \left[\sum_{t=0}^{k} (-\alpha)^t \binom{k}{t} B(n-k+t,k+1) \exp\left\{-\frac{ty}{\theta_2}\right\} \right];$$

$$y > 0, \; \theta_2 > 0 \quad (3.3.1.1)$$

$$f_{k,n}(y) = \frac{1}{\theta_2} \left[1-\exp\left\{-\frac{y}{\theta_2}\right\}\right]^{k-1} \exp\left\{-\frac{y}{\theta_2}\right\} \left[k + \sum_{t=1}^{k} (-\alpha)^t \binom{k}{t} \frac{B(n-k+t,k+1)}{B(n-k,k+1)} \exp\left\{-\frac{(t-1)y}{\theta_2}\right\} \left(k - (k+t)\left(1-\exp\left\{-\frac{y}{\theta_2}\right\}\right)\right) \right]. \quad (3.3.1.2)$$

Using $\mu_{r:s} = \theta_2 \sum_{t=1}^{r} \frac{1}{s-t+1}$ in (3.2.5) we get

$$E[V_{k,n}] = \theta_2 \left[\sum_{t=1}^{k} \frac{1}{t} - \frac{1}{k+1} \sum_{t=1}^{k} \frac{B(n-k+t,k+1)}{B(n-k,k+1)} \frac{B(k+1,t)}{B(t+1,k-t+1)} (-\alpha)^t \right]$$

$$= \theta_2 \sum_{t=1}^{k} \frac{1}{t} \left[1 - \frac{(n-k)_t (k+1-t)_t}{(n+1)_t (k+1)_t} (-\alpha)^t \right] \qquad (3.3.1.3)$$

where $(n)_x = n(n+1) \ldots (n+x-1)$.

It may be noted that $E[V_{k,n}]$ is an increasing function of α and depends only on θ_2. Specializing for $k = n$

$$E[V_{n,n}] = \theta_2 \sum_{t=1}^{n} \frac{1}{t}.$$

Also

$$e_{k,n} = \frac{E[V_{k,n}]}{E[Y_{n:n}]} = \frac{\sum_{t=1}^{k} \frac{1}{t} \left[1 - \frac{(n-k)_t (k+1-t)_t}{(k+1)_t (n+1)_t} (-\alpha)^t \right]}{\sum_{t=1}^{n} \frac{1}{t}}. \qquad (3.3.1.4)$$

The gain in effectiveness due to an additional observation is decreasing function of k. $e_{k,n}$ is independent of the parameters θ_1 and θ_2. More over it is an increasing function of α and $e_{k,n} = 1$ when $k = n$. Minimum and maximum values of $e_{k,n}$ occurs when $\alpha = -1$ and $\alpha = 1$ and these values for n =10,k =1 are .2017 and .4811 respectively.

The asymptotic distribution of $V_{k,n}$ for fixed k has distribution function

$$F_k(y) = \left[1 - \exp\left\{ -\frac{y}{\theta_2} \right\} \right]^k \left[1 - \alpha \exp\left\{ -\frac{y}{\theta_2} \right\} \right]^k \qquad (3.3.1.5)$$

and with density function

$$f_k(y) = \frac{k}{\theta_2}\left[1-\exp\left\{-\frac{y}{\theta_2}\right\}\right]^{k-1}\exp\left\{-\frac{y}{\theta_2}\right\}\left[1-\alpha\exp\left\{-\frac{y}{\theta_2}\right\}\right]^{k-1}$$

$$\left[1-\alpha\left(2\exp\left\{-\frac{y}{\theta_2}\right\}-1\right)\right]. \qquad (3.3.1.6)$$

Also

$$E[V_k] = (k!)^2\,\theta_2\left[\frac{(-\alpha)^k}{(2k+1)!}\sum_{l=1}^{k}\frac{1}{(2k-l+1)}+\sum_{t=0}^{k}\frac{(-\alpha)^t}{(k-t)!(k+t)!}\right.$$

$$\left.\left\{\sum_{l=1}^{k}\frac{1}{(k+t-l+1)}+\frac{\alpha(k-t)}{(k+t+1)}\sum_{l=1}^{k+1}\frac{1}{(k+t-l+2)}\right\}\right]. \qquad (3.3.1.7)$$

3.3.2 Bivariate Uniform Distribution

The distribution function of the bivariate uniform distribution is specified by (2.6.1). The distribution function of $V_{k,n}$ follows from (3.2.3) as,

$$F_{k,n}(y) = \left(\frac{y}{\theta_2}\right)^k \frac{1}{B(n-k,k+1)}\left[\sum_{t=0}^{k}(-\alpha)^t\binom{k}{t}B(n-k+t,k+1)\left(1-\frac{y}{\theta_2}\right)^t\right];\ 0<y<\theta_2.$$

(3.3.2.1)

and

$$f_{k,n}(y)=\frac{1}{\theta_2^k}y^k\left[k+\sum_{t=1}^{k}(-\alpha)^t\binom{k}{t}\frac{B(n-k+t,k+1)}{B(n-k,k+1)}\left(1-\frac{y}{\theta_2}\right)^{t-1}\left\{k-(k+t)\frac{y}{\theta_2}\right\}\right] \qquad (3.3.2.2)$$

Using $\mu_{r:s} = \theta_2\,\dfrac{r}{s+1}$ in (3.2.5), we get

$$E[V_{k,n}] = \left[\frac{k}{k+1}-\frac{\Gamma(k+1)\Gamma(n+1)}{\Gamma(n-k)}\sum_{t=1}^{k}(-\alpha)^t\,\frac{\Gamma(n-k+t)\Gamma(t+1)}{\Gamma(k+t+2)\Gamma(n+t+1)}\right]\theta_2$$

$$=\left[\frac{k}{k+1}-\sum_{t=1}^{k}(-\alpha)^t\frac{(n-k)_t\,t!}{(k+1)_{t+1}(n+1)_t}\right]\theta_2. \qquad (3.3.2.3)$$

Since $E[V_{n,n}] = E[Y_{n:n}] = \frac{n}{n+1}\,\theta_2$, we get

$$e_{k,n} = \frac{n+1}{n}\left[\frac{k}{k+1} - \sum_{t=1}^{k}(-\alpha)^t \frac{(n-k)_t\, t!}{(k+1)_{t+1}(n+1)_t}\right] \tag{3.3.2.4}$$

$e_{k,n}$ is an increasing function of α and is independent of the other model parameters. The asymptotic distribution of $V_{k,n}$ for fixed k is obtained from (3.2.6) given by

$$F_k(y) = \left(\frac{y}{\theta_2}\right)^k \left[1-\alpha\left(1-\frac{y}{\theta_2}\right)\right]^k ;\ 0 < y < \theta_2 \tag{3.3.2.5}$$

or

$$f_k(y) = \frac{k}{\theta_2^k} y^{k-1}\left[1-\alpha\left(1-\frac{y}{\theta_2}\right)\right]^{k-1}\left[1-\alpha\left(1-\frac{2y}{\theta_2}\right)\right];\ 0 < y < \theta_2. \tag{3.3.2.6}$$

Also

$$E[V_k] = (k!)^2\,\theta_2\left[\frac{(-\alpha)^k k}{(2k+1)!} + \sum_{t=0}^{k-1}(-\alpha)^t \frac{k+\alpha\dfrac{(k-t)(k+1)}{k+t+2}}{(k+t+1)!(k-t)!}\right] \tag{3.3.2.7}$$

and

$$e_k = (k!)^2\left[\frac{(-\alpha)^k k}{(2k+1)!} + \sum_{t=0}^{k-1}(-\alpha)^t \frac{k+\alpha\dfrac{(k-t)(k+1)}{k+t+2}}{(k+t+1)!(k-t)!}\right]. \tag{3.3.2.8}$$

3.3.3 Bivariate Logistic Distribution

For the Gumbel's (1961) type II logistic distribution specified by (2. 7.1) $V_{k,n}$ has distribution function

$$F_{k,n}(y) = \frac{1}{\left[1+\exp\left(-\frac{y}{\theta_2}\right)\right]^k B(n-k,k+1)}$$

$$\left\{\sum_{t=0}^{k}(-\alpha)^t\binom{k}{t}B(n-k+t,k+1)\left[\frac{\exp\left\{-\frac{y}{\theta_2}\right\}}{1+\exp\left\{-\frac{y}{\theta_2}\right\}}\right]^t\right\}; \qquad (3.3.3.1)$$

and probability density function

$$f_{k,n}(y) = \frac{1}{\theta_2}\frac{\exp\left(-\frac{y}{\theta_2}\right)}{\left[1+\exp\left(-\frac{y}{\theta_2}\right)\right]^{k+1}}$$

$$\left\{k+\sum_{t=1}^{k}(-\alpha)^t\binom{k}{t}\frac{B(n-k+t,k+1)}{B(n-k,k+1)}\left[\frac{\exp\left\{-\frac{y}{\theta_2}\right\}}{1+\exp\left\{-\frac{y}{\theta_2}\right\}}\right]^{t-1}\left[k-\frac{(k+t)}{1+\exp\left\{-\frac{y}{\theta_2}\right\}}\right]\right\};$$

$$-\infty < y < \infty. \qquad (3.3.3.2)$$

Using $\mu_{k:k} = \theta_2 \sum_{t=1}^{k-1}\frac{1}{t}$ and $\mu_{k:k+t-1} - \mu_{k+1:k+t} = \frac{-\theta_2}{k}$ (3.3.3.3)

we get

$$E[V_{k,n}] = \theta_2\left[\sum_{t=1}^{k-1}\frac{1}{t} - \frac{\Gamma(n+1)\Gamma(k)\Gamma(k+1)}{\Gamma(n-k)}\sum_{t=1}^{k}\frac{(-\alpha)^t}{t}\frac{\Gamma(n-k+t)}{\Gamma(n+t+1)\Gamma(k+t)\Gamma(k-t+1)}\right]$$

$$= \theta_2\left[\sum_{t=1}^{k-1}\frac{1}{t}\left(1-(-\alpha)^t\frac{(n-k)_t(k-t+1)_t}{(k)_t(n+1)_t}\right) - \frac{\Gamma(k)(n-k)_k(-\alpha)^k}{(k)_k(n+1)_k}\right].$$

(3.3.3.4)

As in the above cases E[$V_{k,n}$] is an increasing function of α and depends only on θ_2. Specializing for $k = n$

$$E[V_{n,n}] = \theta_2 \sum_{t=1}^{k-1} \frac{1}{t}$$

Hence

$$e_{k,n} = \frac{\left[\sum_{t=1}^{k-1} \frac{1}{t}\left(1-(-\alpha)^t \frac{(n-k)_t (k-t+1)_t}{(k)_t (n+1)_t}\right) - \frac{\Gamma(k)(n-k)_k (-\alpha)^k}{(k)_k (n+1)_k}\right]}{\sum_{t=1}^{n-1} \frac{1}{t}}. \quad (3.3.3.5)$$

The asymptotic distribution of $V_{k,n}$ for fixed k is

$$F_k(y) = \frac{1}{\left[1+\exp\left(-\frac{y}{\theta_2}\right)\right]^k} \left\{1-\alpha \frac{\exp\left\{-\frac{y}{\theta_2}\right\}}{1+\exp\left\{-\frac{y}{\theta_2}\right\}}\right\}^k \quad (3.3.3.6)$$

from which

$$E[V_k] = (k!)^2 \theta_2 \left[\frac{(-\alpha)^{k+1}}{(2k)!}\frac{1}{k} + \sum_{t=0}^{k-1} \frac{(-\alpha)^t}{(k-t)!(k+t)!} \cdot \left\{\sum_{j=t+1}^{k-1} \frac{1}{j} + \alpha \frac{(k-t)}{(k+t+1)}\left(\sum_{j=t+1}^{k} \frac{1}{j}\right)\right\}\right]$$

(3.3.3.7)

3.4 Asymptotic Approximation

When asymptotic values are discussed, it is pertinent to ask the question: How large n should be, in order that the asymptotic expressions could be practically useful. To answer this question we undertake a comparison of $e_{k,n}$ and e_k, the asymptotic values give us a fair indication of the sample size for which the asymptotic results hold good. Since the theoretical expressions are too complex for an analytical solution, we have resorted to simulation for an approximate answer.

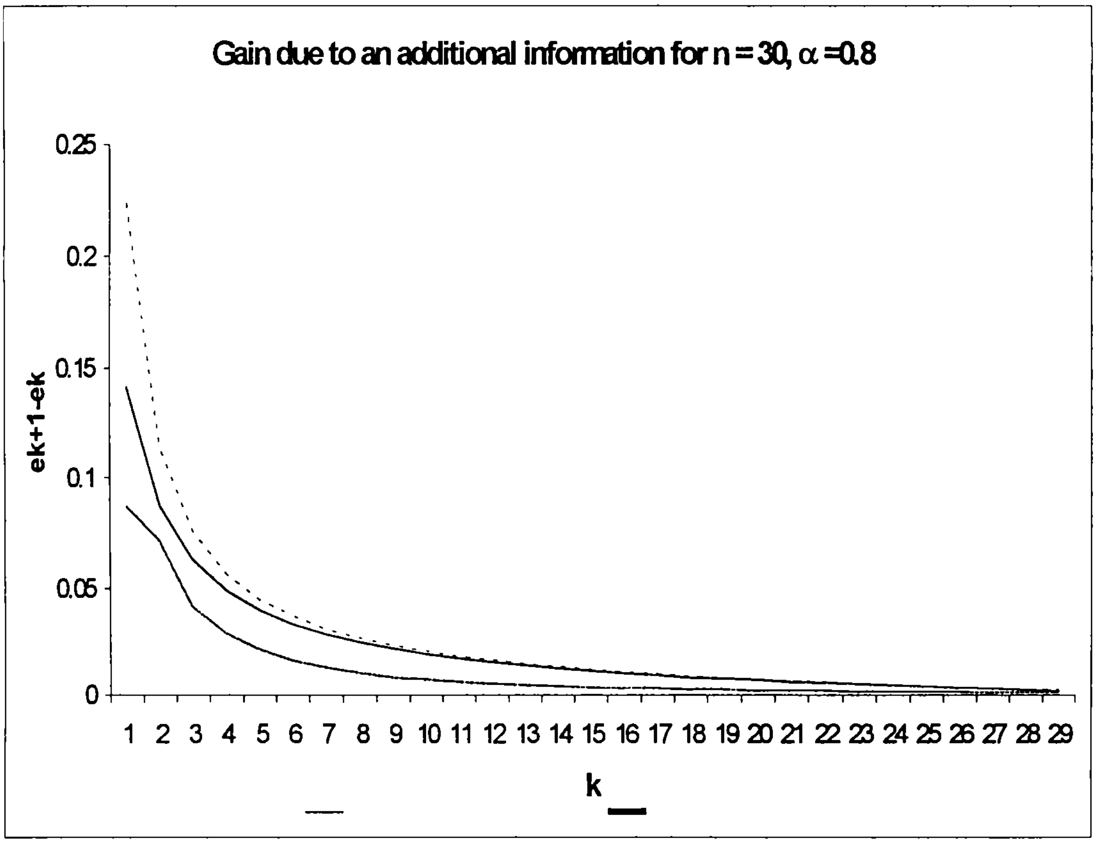
The results in Table 3.3.1 indicates that the values are sufficiently close enough with the results from the sample size for n = 80 onwards, with the value becoming still nearer for increasing values of n. Thus when the test involves more than 80 contestants, the asymptotic distribution provides a reasonable approximation in the case of the uniform distribution, for various values of α and k.

Some typical values of $e_{k,n}$ for sample of size n = 10,20,30 and 40 for various distributions are exhibited in Tables 3.3.2 to 3.3.4 for selected values of k and selected values α from 0.2(0.2)1.0.

Another point of interest is the gain in efficiency due to an additional observation. This is measured through the difference $e_{k+1,n}$ - $e_{k,n}$ for different values of k. As an illustration Figure 3.3 provides a graphical representation of the gain for the uniform, exponential and logistic models. From the graph it could be seen that for values of k larger than 10 in a sample of size 30, the gain is not substantial for α = 0.8.

Table 3.3.5 illustrates that the selection procedure performs well in the bivariate uniform case. The main advantage is that in this case top 17 concomitants include the best individual with efficiency 0.95 for all values of α>0 and n > 80. In the bivariate exponential case a substantial reduction of the value of k (expensive measurements) is possible for α > 0.8. For example if n=1000, k = 100 gives an efficiency $e_{k,n}$ = 0.95 for α = 1. But in the bivariate logistic case the selection procedure has no significant use. It needs almost 50% of the top concomitants to get the best individual in the selection procedure. Thus we conclude that the data is approximately closet to the bivariate independent case and the marginals are uniform the selection procedure discussed in this chapter is very effective and useful.

Figure 3.3



Logistic **Exponential** **Uniform**

Table 3.3.1

Values of e_k and $e_{k,80}$ for uniform distribution

	Values of e_k for uniform dis.					Values of $e_{k,80}$ for uniform dis.				
	α					α				
k	0.2	0.4	0.6	0.8	1	0.2	0.4	0.6	0.8	1
1	.5333	.5667	.6000	.6333	.6667	.5392	.5721	.6050	.6379	.6708
11	.9290	.9384	.9458	.9517	.9565	.9292	.9308	.9312	.9322	.9331
21	.9616	.9669	.9710	.9742	.9767	.9668	.9670	.9673	.9676	.9677
31	.9737	.9774	.9802	.9824	.9841	.9810	.9811	.9812	.9813	.9814
41	.9800	.9828	.9850	.9866	.9880	.9884	.9885	.9886	.9886	.9887
51	.9839	.9862	.9879	.9892	.9903	.9931	.9931	.9931	.9931	.9932
61	.9865	.9884	.9898	.9910	.9919	.9962	.9962	.9962	.9962	.9962
71	.9884	.9900	.9913	.9922	.9930	.9984	.9984	.9985	.9985	.9985

Table 3.3.2

Values of $e_{k,n}$ for Bivariate exponential distribution

		α				
n	k	0.2	0.4	0.6	0.8	1
10	1	0.3694	0.3973	0.4252	0.4531	0.4811
	2	0.5466	0.5759	0.6059	0.6346	0.6621
	3	0.6577	0.6877	0.7162	0.7432	0.7688
	4	0.7402	0.7676	0.7934	0.8178	0.8409
	5	0.8047	0.8285	0.8510	0.8724	0.8927
	6	0.8572	0.8769	0.8957	0.9136	0.9307
	7	0.9012	0.9165	0.9311	0.9452	0.9587
	8	0.9388	0.9493	0.9594	0.9692	0.9788
	9	0.9714	0.9768	0.9820	0.9872	0.9922

20	1	0.3031	0.3282	0.3534	0.3785	0.4037
	2	0.4481	0.4777	0.5060	0.5330	0.5586
	3	0.5422	0.5728	0.6014	0.6282	0.6532
	4	0.6117	0.6419	0.6699	0.6960	0.7203
	5	0.6644	0.6987	0.7228	0.7479	0.7713
	6	0.7115	0.7395	0.7655	0.7895	0.8119
	7	0.7496	0.7762	0.8008	0.8238	0.8451
	8	0.7826	0.8076	0.8309	0.8525	0.8728
	9	0.8115	0.8349	0.8567	0.8770	0.8961
	10	0.8373	0.8589	0.8791	0.8980	0.9159
	11	0.8605	0.8802	0.8987	0.9162	0.9327
	12	0.8814	0.8993	0.9161	0.9320	0.9471
	13	0.9006	0.9165	0.9315	0.9457	0.9543
	14	0.9182	0.9320	0.9451	0.9576	0.9783
	15	0.9344	0.9461	0.9572	0.9679	0.9853
	16	0.9494	0.9589	0.9680	0.9768	0.9909
	17	0.9634	0.9705	0.9775	0.9843	0.9952
	18	0.9764	0.9813	0.9860	0.9907	0.9933
	19	0.9886	0.9910	0.9935	0.9958	0.9982
30	1	0.2737	0.2971	0.3206	0.3440	0.3674
	3	0.4928	0.5291	0.5679	0.6095	0.6538
	5	0.6068	0.6456	0.6884	0.7357	0.7880
	7	0.6832	0.7212	0.7637	0.8113	0.8649
	9	0.7403	0.7760	0.8160	0.8612	0.9124
	11	0.7855	0.8182	0.8548	0.8960	0.9426
	13	0.8227	0.8521	0.8848	0.9212	0.9623
	15	0.8543	0.8802	0.9086	0.9401	0.9752
	17	0.8816	0.9038	0.9281	0.9546	0.9838

	19	0.9053	0.9242	0.9442	0.9659	0.9895
	21	0.9267	0.9418	0.9579	0.9750	0.9933
	23	0.9458	0.9574	0.9696	0.9824	0.9959
	25	0.9631	0..9713	0.9797	0.9885	0.9977
	27	0.9788	0.9836	0.9886	0.9936	0.9989
	29	0.9932	0.9948	0.9964	0.9980	0.9997
40	1	0.2560	0.2782	0.3004	0..3227	0.3449
	3	0.4613	0.4965	0.5343	0.5748	0.6182
	5	0.5686	0.6073	0.6501	0.6978	0.7508
	7	0.6408	0.6798	0.7238	0.7735	0.8303
	9	0.6949	0.7328	0.7759	0.8251	0.8819
	11	0.7379	0.7740	0.8150	0.8622	0.9168
	13	0.7735	0.8074	0.8458	0.8899	0.9409
	15	0.8037	0.8352	0.8707	0.9111	0.9578
	17	0.8299	0.8588	0.8912	0.9278	0.9697
	19	0.8530	0.8793	0.9085	0.9412	0.9782
	21	0.8736	0.8972	0.9232	0.9520	0.9843
	23	0.8921	0.9130	0.9359	0.9610	0.9887
	25	0.9089	0.9272	0.9470	0.9685	0.9919
	27	0.9243	0.9400	0.9568	0.9748	0.9943
	29	0.9384	0.9515	0.9654	0.9802	0.9960
	31	0.9514	0.9620	0.9732	0.9849	0.9972
	33	0.9636	0.9717	0.9802	0.9890	0.9982
	35	0.9748	0.9805	0.9865	0.9926	0.9989
	37	0.9854	0.9888	0.9923	0.9958	0.9994
	39	0.9953	0.9964	0.9975	0.9987	0.9998

Table 3.3.3

Values of $e_{k,n}$ for bivariate uniform distribution

		α				
n	k	0.2	0.4	0.6	0.8	1
10	1	0.5860	0.6100	0.6400	0.6700	0.7000
	2	0.7459	0.7568	0.7661	0.7739	0.7800
	3	0.8317	0.8379	0.8477	0.8492	0.8545
	4	0.8839	0.8875	0.8909	0.8942	0.8972
	5	0.9189	0.9212	0.9233	0.9254	0.9273
	6	0.9443	0.9456	0.9469	0.9482	0.9494
	7	0.9633	0.9641	0.9644	0.9657	0.9664
	8	0.9782	0.9787	0.9791	0.9795	0.9799
	9	0.9908	0.9904	0.9905	0.9907	0.9909
20	1	0.5567	0.5883	0.6200	0.6157	0.6883
	2	0.7140	0.7259	0.7357	0.7434	0.7491
	3	0.7954	0.8020	0.8097	0.8162	0.8225
	4	0.8451	0.8498	0.8542	0.8582	0.8619
	5	0.8784	0.8870	0.8847	0.8875	0.8903
	6	0.9024	0.9047	0.9069	0.9090	0.9109
	7	0.9205	0.9222	0.9238	0.9253	0.9268
	8	0.9346	0.9359	0.9371	0.9383	0.9399
	9	0.9460	0.9469	0.9478	0.9487	0.9496
	10	0.9553	0.9560	0.9567	0.9574	0.9581
	11	0.9631	0.9636	0.9642	0.9647	0.9652
	12	0.9697	0.9701	0.9705	0.9709	0.9713

	13	0.9753	0.9757	0.9760	0.9763	0.9766
	14	0.9802	0.9805	0.9807	0.9810	0.9812
	15	0.9846	0.9847	0.9849	0.9851	0.9853
	16	0.9884	0.9884	0.9886	0.9887	0.9889
	17	0.9918	0.9918	0.9918	0.9920	0.9921
	18	0.9948	0.9948	0.9949	0.9949	0.9950
	19	0.9975	0.9975	0.9976	0.9976	0.9976
30	1	0.5489	0.5411	0.6133	0.6456	0.6778
	3	0.7835	0.7912	0.7983	0.8051	0.8118
	5	0.8649	0.8685	0.8718	0.8749	0.8778
	7	0.9062	0.9082	0.9100	0.9118	0.9135
	9	0.9312	0.9324	0.9336	0.9347	0.99357
	11	0.9480	0.9488	0.9495	0.9503	0.9504
	13	0.9600	0.9606	0.9611	0.9616	0.9620
	15	0.9691	0.9695	0.9698	0.9702	0.9705
	17	0.9762	0.9764	0.9767	0.9769	0.9771
	19	0.9818	0.9820	0.9822	0.9823	0.9825
	21	0.9865	0.9866	0.9867	0.9868	0.9869
	23	0.9903	0.9904	0.9905	0.9905	0.9907
	25	0.9936	0.9937	0.9937	0.9937	0.9938
	27	0.9964	0.9965	0.9965	0.9965	0.9965
	29	0.9989	0.9989	0.9989	0.9989	0.9989

40	1	0.5450	0.5775	0.6100`	0.6425	0.6750
	3	0.7775	0.7854	0.7926	0.7996	0.8065
	5	0.8582	0.8619	0.8653	0.8685	0.8715
	7	0.8991	0.9012	0.9032	0.9050	0.9068
	9	0.9239	0.9253	0.9264	0.9276	0.9288
	11	0.9405	0.9414	0.9422	0.9430	0.9438
	13	0.9524	0.9530	0.9536	0.9542	0.9548
	15	0.9614	0.9618	0.9623	0.9627	0.9631
	17	0.9684	0.9687	0.9690	0.9693	0.9696
	19	0.9740	0.9742	0.9745	0.9747	0.9749
	21	0.9785	0.9788	0.9790	0.9791	0.9793
	23	0.9824	0.9826	0.9827	0.9828	0.9830
	25	0.9857	0.9858	0.9859	0.9860	0.9861
	27	0.9885	0.9886	0.9886	0.9887	0.9888
	29	0.9909	0.9909	0.9910	0.9911	0.9911
	31	0.9930	0.9930	0.9931	0.9931	0.9932
	33	0.9949	0.9949	0.9949	0.9950	0.9950
	35	0.9965	0.9966	0.9966	0.9966	0.9966
	37	0.9980	0.9980	0.9981	0.9981	0.9981
	39	0.9994	0.9994	0.9994	0.9994	0.9994

Table 3.3.4

Values of $e_{k,n}$ for Bivariate logistic distribution

		α				
n	k	0.2	0.4	0.6	0.8	1
10	1	0.0578	0.1157	0.1735	0.2314	0.2892
	2	0.4036	0.4512	0.4962	0.5386	0.5784
	3	0.5737	0.6144	0.6524	0.6880	0.7211
	4	0.6853	0.7201	0.7526	0.7829	0.8114
	5	0.7675	0.7967	0.8240	0.8498	0.8740
	6	0.8321	0.8557	0.8779	0.8991	0.9191
	7	0.8848	0.9028	0.9199	0.9362	0.9519
	8	0.9291	0.9413	0.9530	0.9643	0.9752
	9	0.9671	0.9732	0.9793	0.9852	0.9909
20	1	0.0510	0.1020	0.1530	0.2040	0.2550
	2	0.3288	0.3729	0.4143	0.4529	0.4887
	3	0.4666	0.5069	0.5440	0.5781	0.6094
	4	0.5578	0.5952	0.6295	0.6608	0.6897
	5	0.6256	0.6606	0.6925	0.7218	0.7487
	6	0.6794	0.7121	0.7419	0.7694	0.7947
	7	0.7239	0.7543	0.7822	0.8079	0.8317
	8	0.7617	0.7899	0.8159	0.8399	0.8623
	9	0.7944	0.8204	0.8445	0.8669	0.8878
	10	0.8232	0.8471	0.8693	0.8899	0.9094
	11	0.8489	0.8705	0.8908	0.9098	0.9277
	12	0.8720	0.8914	0.9097	0.9269	0.9431

	13	0.8929	0.9101	0.9263	0.9417	0.9563
	14	0.9121	0.9269	0.9411	0.9547	0.9674
	15	0.9296	0.9422	0.9541	0.9656	0.9767
	16	0.9458	0.9560	0.9657	0.9751	0.9842
	17	0.9609	0.9685	0.9760	0.9832	0.9903
	18	0.9748	0.9790	0.9851	0.9900	0.9948
	19	0.9878	0.9904	0.9950	0.9956	0.9980
30	1	0.0472	0.0945	0.1417	0.1889	0.2361
	3	0.4207	0.4592	0.4944	0.5266	0.5560
	5	0.5645	0.5993	0.6307	0.6593	0.6853
	7	0.6539	0.6858	0.7146	0.7409	0.7649
	9	0.7184	0.7477	0.7743	0.7986	0.8209
	11	0.7687	0.7954	0.8198	0.8423	0.8629
	13	0.8097	0.8338	0.8561	0.8766	0.8957
	15	0.8442	0.8657	0.8857	0.9043	0.9217
	17	0.8737	0.8927	0.9104	0.9270	0.9426
	19	0.8996	0.9158	0.9311	0.9456	0.9593
	21	0.9224	0.9359	0.9487	0.9609	0.9726
	23	0.9428	0.9534	0.9637	0.9735	0.9830
	25	0.9611	0.9689	0.9764	0.9836	0.9907
	27	0.9778	0.9825	0.9871	0.9916	0.9961
	29	0.9929	0.9945	0.9961	0.9977	0.9992

40	1	0.0447	0.0894	0.1342	0.1789	0.2236
	3	0.3932	0.4302	0.4639	0.4946	0.5226
	5	0.5277	0.5617	0.5923	0.6200	0.6451
	7	0.6117	0.6435	0.6721	0.6980	0.7215
	9	0.6725	0.7024	0.7293	0.7538	0.7761
	11	0.7200	0.7481	0.7735	0.7967	0.8179
	13	0.7589	0.7852	0.8092	0.8310	0.8511
	15	0.7916	0.8163	0.8387	0.8593	0.8784
	17	0.8199	0.8428	0.8637	0.8830	0.9010
	19	0.8447	0.8658	0.8852	0.9032	0.9200
	21	0.8667	0.8860	0.9038	0.9205	0.9361
	23	0.8864	0.9039	0.9202	0.9354	0.9498
	25	0.9043	0.9199	0.9345	0.9483	0.9613
	27	0.9206	0.9343	0.9472	0.9594	0.9702
	29	0.9356	0.9472	0.9584	0.9690	0.9792
	31	0.9493	0.9590	0.9683	0.9772	0.9858
	33	0.9620	0.9697	0.9770	0.9842	0.9911
	35	0.9738	0.9794	0.9847	0.9900	0.9951
	37	0.9848	0.9882	0.9915	0.9947	0.9979
	39	0.9951	0.9962	0.9974	0.9985	0.9996

Table 3.3.5

Values of k (number of expensive measurements) for $e_{k,n} = 0.95$

n		0.2	0.4	0.6	0.8	1
40	uni	7	7	7	7	7
	exp	30	28	25	20	14
	log	31	29	25	23	21
100	uni	15	15	15	14	14
	exp	73	67	59	43	25
	log	73	71	64	56	51
500	uni	17	17	17	17	17
	exp	325	290	240	170	65
	log	325	300	275	250	225
1000	uni	17	17	17	17	17
	exp	620	550	450	300	100
	log	620	570	520	470	425
2000	uni	17	17	17	17	17
	exp	1200	1050	850	550	150
	log	1200	1100	1000	900	820

Chapter 4

CONCOMITANTS OF RECORDS FROM MORGENSTERN FAMILY

4.1 Introduction

Let X_1, X_2... be an infinite sequence of independent and identically distributed random variables having the same absolutely continuous distribution function F(x). An observation X_j will be called an upper record (or simply a record) if its value exceeds that of all previous observations. Thus X_j is a record if $X_j > X_i$ for every $i < j$. An analogous definition deals with lower record values. The time at which records appear are of interest. For convenience, let us assume that X_j is observed at time j. Then the record time sequence $\{T_n, n \geq 0\}$ is defined in the following manner.

$$T_0 = 1 \text{ with probability one}$$

and for $n \geq 1$

$$T_n = \min\{j: X_j > X_{T_{n-1}}\}.$$

The record value sequence $\{R_n\}$ is then defined by $R_n = X_{T_n}$; n = 0, 1, 2...

Here R_0 is referred to as the reference value or the trivial record. An excellent work on record value theory is available on Nevzorov and Balakrishnan (1998). A comprehensive study on record values is presented in Arnold Balakrishnan and Nagaraja (1998).

Even the definition of record times and record values for multivariate observations are open to discussion. Several competing definitions are introduced. Very limited progress is documented in the development of appropriate multivariate record theory. In this chapter our discussion is focus on the bivariate case. A generalization of concomitants of order statistics in the record value theory was initiated by Houchens (1984). Some applications of record concomitants we refer to Nevzorov and Ahsanullah (2000).

Let (X_i, Y_i), i = 1, 2... be a sequence of i.i.d random variable from an absolutely continuous distribution with distribution function F(x,y) and density function f(x,y). Let R_n denote the n^{th} record value in the sequence of the X's. The corresponding random variable Y, ie the Y-value paired with the X-value R_n is called the n^{th} record concomitant and will be denoted by $R_{[n]}$.

The present chapter deals with the distribution theory of record concomitants from the Morgenstern family of bivariate distributions. We also discuss the distribution theory of record concomitants some important members of the family namely, bivariate exponential, bivariate uniform and bivariate logistic distributions.

4.2 Distribution of n^{th} Record Concomitant, $R_{[n]}$

For the Morgenstern system we have

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y)\, [1+\alpha(1-F_X(x))\,(1-F_Y(y))], \quad -1 \leq \alpha \leq 1 \qquad (4.2.1)$$

and

$$F_{Y|X}(y|x) = F_Y(y)\, [1+\alpha(1-2\, F_X(x))(1-F_Y(y))]. \qquad (4.2.2)$$

Since (X_i, Y_i), i = 1,2...are independent and identically distributed random variables, it follows that

$$F_{R_{[n]}|R_n}(y \mid x) = F_{Y|X}(y \mid x) \qquad (4.2.3)$$

see, Arnold Balakrishnan and Nagaraja (1998) p.272. We can immediately write the distribution function of the n[th] record concomitant in the form

$$F_{R_{[n]}}(y) = \int F_{Y|X}(y \mid x) f_{R_n}(x) dx, \tag{4.2.4}$$

where

$$f_{R_n}(x) = f_X(x) \frac{1}{n!}[-\log(1 - F_X(x))]^n \text{, see Arnold et al. (1998).} \tag{4.2.5}$$

In the Morgenstern family

$$F_{R_{[n]}}(y) = \int \quad F_Y(y)[1+\alpha(1-2F_X(x))(1-F_Y(y))] \; f_X(x)\frac{1}{n!}[-\log(1-F_X(x))]^n \, dx$$

$$= F_Y(y)\int f_{R_n}(x)dx + \alpha F_Y(y)[1-F_Y(y)]\int [1-2F_X(x)]$$

$$f_X(x)\frac{1}{n!}[-\log(1-F_X(x))]^n dx$$

$$= F_Y(y) + \alpha F_Y(y)[1-F_Y(y)]\int_0^\infty \frac{(2e^{-2u} - e^{-u})u^n}{n!} du$$

$$= F_Y(y)\left[1-(1-\frac{1}{2^n})\alpha \;[1-F_Y(y)]\right]. \tag{4.2.6}$$

The corresponding density function is

$$f_{R_{[n]}}(y) = f_Y(y)\left[1-(1-\frac{1}{2^n})\alpha \;[1-2F_Y(y)]\right] \tag{4.2.7}$$

Writing 1-2F as 1-F-F in (4.2.7) and using the formula for the density of order statistics, we find

$$f_{R_{[n]}}(y) = f_Y(y) - \alpha(1-\frac{1}{2^n})\{[1-F_Y(y)]f_Y(y) - F_Y(y)f_Y(y)\}$$

$$= f_Y(y) - \alpha/2(1-\frac{1}{2^n})\{2[1-F_Y(y)]f_Y(y) - 2F_Y(y)f_Y(y)\}$$

$$= f_Y(y) - \frac{\alpha}{2}(1-\frac{1}{2^n}) \; [f_{1:2}(y) - f_{2:2}(y)]. \tag{4.2.8}$$

Equation (4.2.8) reveals that the density function of the n^{th} record concomitant depends only on the marginal distribution of Y and the distribution of the order statistics $Y_{1:2}$ and $Y_{2:2}$. Getting from (4.2.8) the k^{th} moment of $R_{[n]}$,

$$\mu_{[n]}^{(k)} = \mu^{(k)} + \frac{\alpha}{2}(\frac{1}{2^n} - 1)[\mu^{(k)}{}_{1:2} - \mu^{(k)}{}_{2:2}] \tag{4.2.9}$$

where,

$$\mu^{(k)} = E(Y^k)$$

and

$$\mu_{1:2}^{(k)} = E[Y_{r:2}^k] \quad r = 1,2.$$

It follows from (4.2.8) that

$$f_{R_{[n]}}(y) = (2 - \frac{1}{2^{n-1}}) f_{R_{[1]}}(y) + f_Y(y)(\frac{1}{2^{n-1}} - 1). \tag{4.2.10}$$

Two interesting recurrence relations connecting the moments of record concomitants immediately follows from (4.2.10) as

$$\begin{aligned} & 2^{(n+1)}[\mu_{[n]}^{(k)} - \mu_{[n-1]}^{(k)}] + \alpha[\mu_{1:2}^{(k)} - \mu_{2:2}^{(k)}] = 0 \\ & \textit{and} \\ & \mu_{[n]}^{(k)} = (2 - \frac{1}{2^{n-1}})\mu_{[1]}^{(k)} + \mu^{(k)}(\frac{1}{2^{n-1}} - 1) \end{aligned} \tag{4.2.11}$$

4.3 Joint Distribution of $R_{[m]}$ and $R_{[n]}$ (m<n)

In this section, we derive the joint distribution of m^{th} and n^{th} record concomitants. Since $R_{[m]}$ and $R_{[n]}$ are conditionally independent given R_m and R_n, the conditional distribution function of

$(R_{[m]}, R_{[n]})$ given $(R_m = x_1, R_n = x_2)$ is

$$F_{R_{[m]},R_{[n]}|R_m,R_n}(y_1, y_2 \mid x_1, x_2) = F_{Y|X}(y_1 \mid x_1) F_{Y|X}(y_2 \mid x_2)$$

see, (Bhattacharya (1984), Lemma 3.1). Hence the joint distribution function of $(R_{[m]}, R_{[n]})$ is

$$F_{R_{[m]},R_{[n]}}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} F_{Y|X}(y_1|x_1)\, F_{Y|X}(y_2|x_2)\, f_{R_m,R_n}(x_1, x_2)dx_1dx_2 \quad (4.3.1)$$

where,

$$f_{R_m,R_n}(x_1, x_2) = \frac{[-\log(1-F_X(x_1))]^m}{m!} \frac{\{-\log[\frac{1-F_X(x_2)}{1-F_X(x_1)}]\}^{n-m-1}}{(n-m-1)!} \frac{f_X(x_1)}{[1-F_X(x_1)]} f_X(x_2)$$

(4.3.2)

see, Arnold et al. (1998).

$$F_{R_{[m]},R_{[n]}}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} F_Y(y_1)F_Y(y_2)[1+\alpha(1-F_Y(y_2))(1-2F_X(x_2))]$$

$$[1+\alpha(1-F_Y(y_1))(1-2F_X(x_1))]\ f_{R_m,R_n}(x_1, x_2)dx_1dx_2$$

$$= F_Y(y_1)\ F_Y(y_2)[1+\alpha\,(1-F_Y(y_2))\left\{2\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}(1-F_x(x_2))f_{R_m,R_n}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 - 1\right\}$$

$$+ \alpha\,[1-F_Y(y_1)]\left\{1-2\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}F_x(x_1)f_{R_m,R_n}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2\right\}$$

$$+ \alpha^2\,[1-F_Y(y_1)]\,[1-F_Y(y_2)]\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}(1-2F_x(x_1))[2(1-F_x(x_2))-1]f_{R_m,R_n}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2\right\}]$$

$$= F_Y(y_1)\ F_Y(y_2)[1+\alpha\,(1-F_Y(y_1))(\frac{1}{2^m}-1) + \alpha\,[1-F_Y(y_2)](\frac{1}{2^n}-1) +$$

$$\alpha^2\,[1-F_Y(y_1)][1-F_Y(y_2)]\int_{-\infty}^{\infty}[1-2F_X(x_1)]\int_{x_1}^{\infty}[1-2F_X(x_2)]f_{R_m,R_n}(x_1, x_2)dx_1dx_2.$$

(4.3.3)

We have

$$\int_{x_1}^{\infty}[1-2F_X(x_2)]f_{R_m,R_n}(x_1,x_2)dx_1dx_2=\int_{x_1}^{\infty}[1-2F_X(x_2)]$$

$$\frac{[-\log(1-F_X(x_1))]^m}{m!}\frac{\{-\log[\frac{1-F_X(x_2)}{1-F_X(x_1)}]\}^{n-m-1}}{(n-m-1)!}\frac{f_X(x_1)}{[1-F_X(x_1)]}f_X(x_2)dx_2$$

$$=\frac{[-\log(1-F_X(x_1))]^m}{m!}\int_{x_1}^{\infty}[1-2F_X(x_2)]\frac{\{-\log[\frac{1-F_X(x_2)}{1-F_X(x_1)}]\}^{n-m-1}}{(n-m-1)!}\frac{f_X(x_1)}{[1-F_X(x_1)]}f_X(x_2)dx_2.$$

(4.3.4)

Using the transformation U = $-\log\left[\frac{1-F_X(x_2)}{1-F_X(x_1)}\right]$

we have $e^{-u}=\frac{1-F_X(x_2)}{1-F_X(x_1)}$ and hence the integral on the RHS of (4.3.4) reduces to

$$\int_{x_1}^{\infty}[1-2F_X(x_2)]f_{R_m,R_n}(x_1,x_2)dx_1dx_2=\int_{0}^{\infty}[2e^{-u}(1-F_X(x_1))-1]\frac{e^{-u}u^{n-m-1}}{(n-m-1)!}du$$

$$=\left[\frac{(1-F_X(x_1))}{2^{n-m-1}}-1\right]. \qquad (4.3.5)$$

Using (4.3.5) in (4.3.3) we get

$$\int_{-\infty}^{\infty}[1-2F_X(x_1)]\int_{x_1}^{\infty}[1-2F_X(x_2)]f_{R_m,R_n}(x_1,x_2)dx_1dx_2=$$

$$\int_{-\infty}^{\infty}[1-2F_X(x_1)]\left[\frac{([1-F_X(x_1))}{2^{n-m-1}}-1\right]$$

$$f_X(x)\frac{1}{n!}[-\log(1-F_X(x))]^n dx_1.$$

Using the transformation U = –log [1–$F_X(x_1)$] we have

$$\int_{-\infty}^{\infty}[1-2F_X(x_1)]\int_{x_1}^{\infty}[1-2F_X(x_2)]f_{R_m,R_n}(x_1,x_2)dx_1dx_2 =$$

$$\frac{1}{2^{n-m-2}}\int_0^{\infty}\frac{e^{-3u}u^m}{m!}du-\frac{1}{2^{n-m-1}}\int_0^{\infty}\frac{e^{-2u}u^m}{m!}du-(\frac{1}{2^m}-1)$$

$$=1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}. \quad (4.3.6)$$

Using (4.3.6) in (4.3.3) we have

$$F_{R_{[m]},R_{[n]}}(y_1,y_2)=F_Y(y_1)F_Y(y_2)[1+\alpha(1-F_Y(y_1))(\frac{1}{2^m}-1)+\alpha[1-F_Y(y_2)]\,(\frac{1}{2^n}-1)$$

$$+\alpha^2\,[1-F_Y(y_1)]\,[1-F_Y(y_2)]\,\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}]. \quad (4.3.7)$$

The joint density function corresponding to (4.3.7) is

$$f_{R_{[m]}R_{[n]}}(y_1,y_2) = f_Y(y_1)f_Y(y_2)\,[1+\alpha\,[1-2F_Y(y_1)]\,(\frac{1}{2^m}-1)$$

$$+\alpha\,[1-2F_Y(y_2)]\,(\frac{1}{2^n}-1)$$

$$+\alpha^2\,[1-2F_Y(y_1)]\,[1-2F_Y(y_2)]\,\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}]. \quad (4.3.8)$$

The asymptotic joint distribution of ($R_{[m]}$, $R_{[n]}$) is evident from (4.3.8). Without any normalizing we have

$$\lim_{m,n\to\infty} f_{R_{[m]}R_{[n]}}(y_1,y_2) = f_Y(y_1)f_Y(y_2)\,[1-\alpha(1-2F_Y(y_1))$$

$$-\alpha\,[1-2F_Y(y_2)]\;+\alpha^2[1-2F_Y(y_1)]\,[1-2F_Y(y_2)]]. \quad (4.3.9)$$

It follows from (4.3.9) that the record concomitants are asymptotically independent for all values of m and n ($m \neq n$).

The product moments can be directly follows from (4.3.8).

For $1 \le m < n$; l = 1, 2... and k = 1, 2...

$$E\,[R^{l}_{[m]}R^{k}_{[n]}] = \iint y_1^l y_2^k \; f_{R_{[m]}R_{[n]}}(y_1, y_2)\,dy_1dy_2$$

$$= \iint y_1^l y_2^k \, f_Y(y_1) f_Y(y_2)[1+\alpha(1-2F_Y(y_1))\,(\frac{1}{2^m}-1)$$

$$+\,\alpha\,[1-2F_Y(y_2)]\,(\frac{1}{2^n}-1)$$

$$+\,\alpha^2\,[1-2F_Y(y_1)]\,[1-2F_Y(y_2)]\,\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}]\,dy_1dy_2$$

$$=\mu^{(l)}\mu^{(k)} + (\frac{1}{2^m}-1)\frac{\alpha}{2}[\mu_{1:2}^{(l)} - \mu_{2:2}^{(l)}]\mu^{(k)} + (\frac{1}{2^n}-1)\frac{\alpha}{2}[\mu_{1:2}^{(k)} - \mu_{2:2}^{(k)}]\mu^{(l)} +$$

$$\frac{\alpha^2}{4}[\mu_{1:2}^{(l)} - \mu_{2:2}^{(l)}][\mu_{1:2}^{(k)} - \mu_{2:2}^{(k)}]\;\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}. \qquad (4.3.10)$$

4.4 Record Concomitants from Gumbel's Bivariate Exponential distribution

The distribution theory of concomitants from Gumbel's bivariate exponential distribution is discussed in chapter 2. In this section we deal with the distribution theory of record concomitants from this distribution.

The Gumbel's bivariate type1 1 exponential distribution is specified by (2.5.1). In this case the distribution function of $R_{[n]}$ follows from (4.2.6) as

$$F_{R_{[n]}}(y) = [1-\exp(-y/\theta_2)][1+(\frac{1}{2^n}-1)\alpha\exp(-y/\theta_2)];\ y > 0,\ \theta_2 > 0. \qquad (4.4.1)$$

The density function of $R_{[n]}$ follows from (4.2.7) as

$$f_{R_{[n]}}(y) = \frac{1}{\theta_2}\exp(-y/\theta_2)[1+(\frac{1}{2^n}-1)\alpha(2\exp(-y/\theta_2)-1)];\ y > 0,\ \theta_2 > 0. \qquad (4.4.2)$$

The joint probability density function of ($R_{[m]}$, $R_{[n]}$) follows from (4.3.8) as

$$f_{R_m,R_n}(y_1,y_2) = \frac{1}{\theta_2^2}\exp\left\{\frac{-y_1-y_2}{\theta_2}\right\}\left[1+\alpha(\frac{1}{2^m}-1)\left(2\exp\left\{\frac{-y_1}{\theta_2}\right\}-1\right)\right]$$

$$+\alpha(\frac{1}{2^n}-1)\left(2\exp\left\{\frac{-y_2}{\theta_2}\right\}-1\right)+\alpha^2\,\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}$$

$$\left(2\exp\left\{\frac{-y_1}{\theta_2}\right\}-1\right)\left(2\exp\left\{\frac{-y_2}{\theta_2}\right\}-1\right)],\ y_1, y_2 > 0. \qquad (4.4.3)$$

The moments of $R_{[n]}$ is readily obtained from (4.4.2) as

$$\mu_{R_{[n]}}^{(k)} = \Gamma(k+1)\theta_2^{\,k}[1+\alpha(\frac{1}{2^k}-1)(\frac{1}{2^n}-1)]\quad k = 1, 2\ldots \qquad (4.4.4)$$

In particular

$$\mu_{R_{[n]}} = \theta_2[1-\frac{\alpha}{2}(\frac{1}{2^n}-1)]. \qquad (4.4.5)$$

The asymptotic mean of $R_{[n]}$ is

$$\mu_{R_{[n]}} = \theta_2[1+\frac{\alpha}{2}]. \qquad (4.4.6)$$

The variance of $R_{[n]}$ follows from (4.4.4) and (4.4.5) as

$$V(R_{[n]}) = \theta_2^2[1-\frac{\alpha}{2}(\frac{1}{2^n}-1)-\frac{\alpha^2}{4}(\frac{1}{2^n}-1)^2]. \qquad (4.4.7)$$

From (4.3.10) we get

$$\text{for } 1\le m<n\ \ E[R_{[m]}R_{[n]}] = \theta_2^2[1-\frac{\alpha}{2}(\frac{1}{2^n}+\frac{1}{2^m}-2)+\frac{\alpha^2}{4}$$

$$\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}]. \qquad (4.4.8)$$

$$\text{Hence}\quad Cov(R_{[m]}R_{[n]}) = \frac{\alpha^2\theta_2^2}{4}[\frac{1}{2^{n-m-2}3^{m+1}}-\frac{1}{2^n2^m}]$$

$$= \frac{\alpha^2\theta_2^2}{2^{n-m}}[\frac{1}{3^{m+1}}-\frac{1}{4^{m+1}}]. \qquad (4.4.9)$$

As in the case of concomitants record concomitants are also positively correlated and Cov ($R_{[m]}$ $R_{[n]}$) decreases as m and n pull apart.

4.5 Record concomitants from Morgenstern type Bivariate Uniform Distribution

The Morgenstern family is the bivariate uniform distribution specified by (2.6.1). The cumulative distribution function and density function of $R_{[n]}$ follow from (4.2.6) and (4.2.7) respectively as

$$F_{R_{[n]}}(y) = \frac{y}{\theta_2}[1+\alpha(\frac{1}{2^n}-1)(1-\frac{y}{\theta_2})] \tag{4.5.1}$$

and

$$f_{R_{[n]}}(y) = \frac{1}{\theta_2}[1+\alpha(\frac{1}{2^n}-1)(1-\frac{2y}{\theta_2})]. \tag{4.5.2}$$

The moments of $R_{[n]}$ follows from (4.5.2) as

$$\mu_{R_{[n]}}^{(k)} = \frac{\theta_2^k}{k+1}[1-(\frac{1}{2^n}-1)\frac{\alpha k}{(k+2)}] \qquad k = 1, 2\ldots \tag{4.5.3}$$

The mean of $R_{[n]}$

$$E[R_{[n]}] = \frac{\theta_2}{2}[1+\frac{\alpha}{3}(1-\frac{1}{2^n})]. \tag{4.5.4}$$

Using (4.5.3) and (4.5.4) the variance of $R_{[n]}$,

$$V[R_{[n]}] = \frac{\theta_2^2}{6}[\frac{1}{2}-\frac{\alpha^2}{6}(\frac{1}{2^n}-1)^2]. \tag{4.5.5}$$

The asymptotic mean and variance of $R_{[n]}$ are respectively

$$E[R_{[.]}] = \frac{\theta_2}{2}[1+(\alpha/3)] \tag{4.5.6}$$

$$V(R_{[.]}) = \frac{\theta_2^2}{6}[\frac{1}{2}-\frac{\alpha^2}{6}]. \tag{4.5.7}$$

The joint density function of ($R_{[m]}$, $R_{[n]}$) is

$$f_{R_{[m]}R_{[n]}}(y_1,y_2) = \theta_2^2[1+\alpha(1-2\frac{y_1}{\theta_2})(\frac{1}{2^m}-1) + \alpha(1-2\frac{y_2}{\theta_2})(\frac{1}{2^n}-1)$$

$$+\alpha^2\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}(1-2\frac{y_1}{\theta_2})(1-2\frac{y_2}{\theta_2})]. \quad (4.5.8)$$

From (4.3.10) for $1 \le m < n$

$$E[R_{[m]}, R_{[n]}] = \frac{1}{\theta_2^2}[1-(\frac{\alpha}{6})(\frac{1}{2^m}-1) -(\frac{\alpha}{6})((\frac{1}{2^n}-1)$$

$$+\alpha^2/36\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}]. \quad (4.5.9)$$

$$Cov(R_{[m]}, R_{[n]}) = \alpha^2\theta_2^2/36[\{1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\}-(1-\frac{1}{2^m})(1-\frac{1}{2^n})]$$

$$=\frac{\alpha^2\theta_2^2}{9(2^{n-m})}[\frac{1}{3^{m+1}}-\frac{1}{4^{m+1}}]. \quad (4.5.10)$$

Thus the record concomitants are positively correlated and Cov($R_{[m]}$, $R_{[n]}$) decreases as m and n pull apart.

4.6 Record Concomitants from Gumbel's type 11 Logistic Distribution

The joint cumulative distribution of Gumbel's type 11 logistic distribution is given in (2.7.1).

The distribution function of $R_{[n]}$ follows from (4.2.6)

$$F_{R_{[n]}}(y) = \frac{1}{[1+\exp-(\frac{y-\mu_2}{\sigma_2})]}[1+\alpha(\frac{1}{2^n}-1)\frac{\exp-(\frac{y-\mu_2}{\sigma_2})}{[1+\exp-(\frac{y-\mu_2}{\sigma_2})]}]$$

$$-\infty<y<\infty;\ --\infty<\mu_2<\infty; \sigma_2>0. \quad (4.6.1)$$

For the logistic distribution we have

$$\begin{aligned}&\sigma_2 f(y) = F(y)[1-F(y)]\\ &\text{and}\\ &\sigma_2 f'(y) = f(y)[1-2F(y)]\ .\end{aligned} \tag{4.6.2}$$

Hence (4.6.1) takes the form

$$F_{R_{[n]}}(y) = F_Y(y) + \alpha\sigma_2(\frac{1}{2^n}-1)f_Y(y)\,. \tag{4.6.3}$$

Thus the probability function of $R_{[n]}$ follows from (4.6.3) as

$$f_{R_{[n]}}(y) = f_y(y) + \alpha\sigma_2(\frac{1}{2^n}-1)f'_Y(y)$$

$$= f_y(y) + \alpha(\frac{1}{2^n}-1)f_Y(y)[1-2F_Y(y)]$$

$$= f_y(y)[1+\alpha(\frac{1}{2^n}-1)(1-2F_Y(y))]$$

$$= 1/\sigma_2 \frac{\exp-\left(\dfrac{y-\mu_2}{\sigma_2}\right)}{\left[1+\exp-\left(\dfrac{y-\mu_2}{\sigma_2}\right)\right]^2}\left[1-\alpha(\frac{1}{2^n}-1)\left\{\frac{1-\exp-\left(\dfrac{y-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\dfrac{y-\mu_2}{\sigma_2}\right)}\right\}\right]$$

$$-\infty < y < \infty,\ -\infty < \mu_2 < \infty,\ \sigma_2 > 0. \tag{4.6.4}$$

Using the recurrence relation, Shaw (1970), $\mu_{1:2}^{(k)} - \mu_{2:2}^{(k)} = -2k\ \sigma_2 \mu^{(k-1)}$ the moments of $R_{[n]}$ are given by

$$\mu_{R_{[n]}}^{(k)} = \mu^{(k)} - \alpha\sigma_2 k(\frac{1}{2^n}-1)\mu^{(k-1)}\,. \tag{4.6.5}$$

In particular

$$\mu_{R_{[n]}} = \mu_2 - \alpha\sigma_2(\frac{1}{2^n}-1)\,. \tag{4.6.6}$$

Also for k = 2

$$\mu_{R_{[n]}}^{(2)} = \mu_2^2 + \frac{\pi^2}{3}\sigma_2^2 - 2\alpha\mu_2\sigma_2(\frac{1}{2^n}-1)\,. \tag{4.6.7}$$

Hence

$$V(R_{[n]}) = \sigma_2^2[\frac{\pi^2}{3} - \alpha^2(\frac{1}{2^n} - 1)]. \qquad (4.6.8)$$

The joint probability density function of ($R_{[m]}$, $R_{[n]}$)

$$f_{R_{[m]},R_{[n]}}(y_1, y_2) = \frac{1}{\sigma_2^2} \frac{\exp-\left(\frac{y_1 + y_2 - 2\mu_2}{\sigma_2}\right)}{\left[1+\exp-\left(\frac{y_1 - \mu_2}{\sigma_2}\right)\right]^2\left[1+\exp-\left(\frac{y_2 - \mu_2}{\sigma_2}\right)\right]^2}$$

$$[1-\alpha(\frac{1}{2^m} - 1)\left\{\frac{1-\exp-\left(\frac{y_1-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y_1-\mu_2}{\sigma_2}\right)}\right\} - \alpha(\frac{1}{2^n} - 1)\left\{\frac{1-\exp-\left(\frac{y_2-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y_2-\mu_2}{\sigma_2}\right)}\right\}$$

$$+\alpha^2(1 - \frac{1}{2^m} - \frac{1}{2^n} - \frac{1}{2^{n-m-2}3^{m+1}})\left\{\frac{1-\exp-\left(\frac{y_1-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y_1-\mu_2}{\sigma_2}\right)}\right\}\left\{\frac{1-\exp-\left(\frac{y_2-\mu_2}{\sigma_2}\right)}{1+\exp-\left(\frac{y_2-\mu_2}{\sigma_2}\right)}\right\}]$$

$$-\infty < y_1, y_2 < \infty \ ; \ -\infty < \mu_2 < \infty \ ; \ \sigma_2 > 0. \qquad (4.6.9)$$

Using the relation $\mu_{1:2}^{(k)} - \mu_{2:2}^{(k)} = -2k\ \sigma_2 \mu^{(k-1)}$ we get the product moment of $R_{[m]}$ and $R_{[n]}$ as

$$\mu_{R_{[m]}R_{[n]}}^{(l,k)} = \mu^{(l)}\mu^{(k)} +$$

$$\alpha\sigma_2 l(1 - \frac{1}{2^m})\mu^{(l-1)}\mu^{(k)} + \alpha\sigma_2 k(1 - \frac{1}{2^n})\mu^{(k-1)}\mu^{(l)}$$
$$+ (\alpha\sigma_2)^2 lk\{1 - \frac{1}{2^m} - \frac{1}{2^n} - \frac{1}{2^{n-m-2}3^{m+1}}\}\mu^{(l-1)}\mu^{(k-1)}.$$

(4.6.10)

In particular l = 1, k = 1

$$\mu_{R_{[m]},R_{[n]}} = E[R_{[m]}R_{[n]}] = \mu_2^2 - \alpha\sigma_2\ \mu_2\ [\frac{1}{2^m} + \frac{1}{2^n} - 2]$$

$$+ \alpha^2\ \sigma_2^2\ \{1 - \frac{1}{2^m} - \frac{1}{2^n} + \frac{1}{2^{n-m-2}3^{m+1}}\}. \qquad (4.6.11)$$

Using (4.6.11) and (4.6.6) we have

$$\mathrm{Cov}(R_{[m]}, R_{[n]}) = \alpha^2\ \sigma_2^2[\frac{1}{2^{n-m-2}3^{m+1}} - \frac{1}{2^n 2^m}]$$

$$= \frac{4\alpha^2\theta_2^2}{2^{n-m}}[\frac{1}{3^{m+1}} - \frac{1}{4^{m+1}}]. \qquad (4.6.12)$$

As in the previous cases, the record concomitants $R_{[m]}$ and $R_{[n]}$ are positively correlated and the $\mathrm{Cov}(R_{[m]},R_{[n]})$ decreases as m and n pull apart .

Chapter 5

DISTRIBUTION OF THE EXTREMES OF THE r^{th} CONCOMITANT FROM THE MORGENSTERN FAMILY

5.1 Introduction

In many real life sampling situations the variable of interest from the experimental units can be more easily ranked than quantified. In agricultural and environmental studies, it is indeed possible to rank the experimental or sampling units without actually measuring them. In such cases ranked set sampling is highly beneficial for the estimation of the population mean. The procedure of ranked set sampling was suggested by McIntyre (1952) for improving the precision of $\bar{Y}$ as an estimator of the population mean. This method is applicable for situations where the primary variable of interest, Y, is difficult or expensive to measure, but where ranking in small sets is easy. The process involves selecting m samples, each of size m, and ordering each of the samples by eye or some relatively inexpensive means, without actual measurement of the individual, see David and Levine (1972), Dell and Clutter (1972), Stokes (1977). The smallest observation from the first sample is chosen for measurement, as is the second smallest observation from the second sample. The process continues in this way until the largest observation from the nth sample is measured, producing a total of n measured observations one from each order class. When the ranking is subject to error the ranked set sampling have been effectively extended by the use of concomitants to the estimation of mean, Stokes (1977), variance Stokes (1980a), correlation coefficient Stokes (1980b) and to the situation with size biased selection of X's Muttlak and Mc Donald (1990).

Motivated from the ranked set sampling we use the following sampling method for selection of primary variable. Suppose there are two correlated variables Y and X, where Y is difficult to measure or to rank, see Kaiser (1983). For example X may represent an inexpensive rough measurement and Y corresponding refined more expensive measurement. By using the method of concomitants we may reduce the number of expensive measurements on the basis of inexpensive measurements. Consider a bivariate sample of size n = mk, where k is an integer. Randomly subdivide the sample in to k sub samples (groups) each of size m. In each sub sample we measured only the Y-value corresponding to the r^{th} order statistic $X_{r:m}$. Then the Y- value measured in the i^{th} sample is the r^{th} concomitant will be denoted by $Y_{[r:m],i}$ i=1,2,...k. The $Y_{[r:m],i}$ are independent random variables having the same marginal distribution as $Y_{[r:m]}$.

$$M_{k,[r:m]} = \max[Y_{[r:m],1}, Y_{[r:m],2}, \ldots Y_{[r:m],k}]$$

Let the above, and

$$m_{k,[r:m]} = \min[Y_{[r:m],1}, Y_{[r:m],2}, \ldots Y_{[r:m],k}]$$

denote the largest and smallest among the selected concomitants. Thus $M_{k,[r:m]}$ and $m_{k,[r:m]}$ are the extremes of the selected expensive measurements in the samples . In particular $M_{k,[m:m]}$ is the largest observation of the concomitants of maximum of order statistics in the sub samples. It is an extremely useful statistic for the inference on the parameter of the expensive marginal variable Y in bivariate uniform model. It is a very useful statistic in selection problems where the selection is based on the marginal inexpensive variable. For example X be the score of a preliminary test and Y the score in a final test. Suppose we divide the contestants in to k groups and select the top performer in the screening test. Then the ratio $\frac{E[M_{k,[m:m]}]}{E[Y_{n:n}]}$ which clearly increases to 1 with k, is a measure of effectiveness of the selection procedure. One may wish to choose the value of the number of subdivisions (populations), k, to make this ratio sufficiently close to 1. See also Nagaraja and David (1994) Yeo and David (1984) for different approaches of selection.

The present chapter deals with the general distribution theory of $M_{k,[r:m]}$ and $m_{k,[r:m]}$ from the Morgenstern family of distributions and discuss some applications in inference, estimation of the parameter of the marginal variable Y in the Morgenstern type uniform distributions. We also apply the results to the selection problem discussed earlier. Section 5.5 is fully devoted to the distribution of the rank of the r^{th} concomitant from the Morgenstern family.

5.2 Distribution of $M_{k,[r:m]}$

We have discussed the distribution theory of concomitants from the Morgernstern family in Chapter 2.

The cdf of [X,Y] is specified in (1.2.1).

The cumulative distribution function of $Y_{[r:m],i}$ i=1,2,...k is given in (2.2.3) and is

$$F_{Y_{[r:m],i}}(y) = F_Y(y)\left[1+\frac{m-2r+1}{m+1}\alpha\ [1-F_Y(y)]\right] \tag{5.2.1}$$

and the corresponding density function

$$f_{Y_{[r:m],i}}(y) = f_Y(y)[1+\frac{m-2r+1}{m+1}\alpha\ [1-2F_Y(y)]]. \tag{5.2.2}$$

The distribution function of $M_{k,[r:m]}$, denoted by $F_{k:k,[r:m]}$ is

$$F_{k:k;[r:m]}(y) = [F_{Y_{[r:m],i}}(y)]^k = \{ F_Y(y)\left[1+\frac{m-2r+1}{m+1}\alpha\ [1-F_Y(y)]\right] \}^k. \tag{5.2.3}$$

The density function corresponding to (5.2.3) is

$$f_{k:k;[r:m]}(y) = \mathrm{k}$$

$$[F_Y(y)]^{k-1}[1-\frac{(2r-m-1)}{(m+1)}\alpha(1-F_Y(y))]^{k-1} f_Y(y)\{1-\frac{(2r-m-1)}{(m+1)}\alpha+2\alpha(\frac{(2r-m-1)}{m+1})F_Y(y)\}. \quad (5.2.4)$$

Using binomial expansion in (5.2.4) we have

$$f_{k:k;[r:m]}(y) =$$

$$\mathrm{k}[F_Y(y)]^{k-1} f_Y(y)\{1-\frac{(2r-m-1)}{(m+1)}\alpha+2\alpha(\frac{(2r-m-1)}{m+1})F_Y(y)\}$$

$$\sum_{t=o}^{k-1}\binom{k-1}{t}(-\alpha)^t[(\frac{(2r-m-1)}{m+1})(1-F_Y(y))]^t$$

$$=\sum_{t=0}^{k-1}\frac{(k-t)_{t+1}}{(k)_t}[-\alpha\frac{(2r-m-1)}{m+1}]^t\{f_{k:k+t}(y)[1-\frac{(2r-m-1)}{m+1}\alpha]+f_{k+1:k+t+1}(y)2\alpha\frac{k(2r-m-1)}{(k+t+1)(m+1)}\} \quad (5.2.5)$$

where $(n)_x = n(n+1)(n+2)\dots(n+x-1)$ and $f_{r:s}(y)$ is the probability density function of $Y_{r:s}$.

The moments of $M_{k,[r:m]}$ are derived from (5.2.5).

For s = 1, 2...

$$\mu^{(s)}_{k:k;[r:m]} = E[M^s_{k,[r:m]}]$$

$$=\sum_{t=0}^{k-1}\frac{(k-t)_{t+1}}{(k)_t}[-\alpha\frac{(2r-m-1)}{m+1}]^t\{\mu^{(s)}{}_{k:k+t}[1-\frac{(2r-m-1)}{m+1}\alpha]+\mu^{(s)}{}_{k+1:k+t+1}2\alpha\frac{k(2r-m-1)}{(k+t+1)(m+1)}\} \quad (5.2.6)$$

where

$$\mu^{(s)}_{r:j} = E[Y^s_{r:j}].$$

Particularly for r = m

$$\sum_{t=0}^{k-1}\frac{(k-t)_{t+1}}{(k)_t}[-\alpha\frac{(m-1)}{m+1}]^t\{f_{k:k+t}(y)[1-\frac{(m-1)}{m+1}\alpha]+f_{k+1:k+t+1}(y)2\alpha\frac{k(m-1)}{(k+t+1)(m+1)}\}. \quad (5.2.7)$$

and

$$\mu^{(s)}_{k:k;[m:m]} =$$

$$\sum_{t=0}^{k-1}\frac{(k-t)_{t+1}}{(k)_t}[-\alpha\frac{(m-1)}{m+1}]^t\{\mu^{(s)}{}_{k:k+t}[1-\frac{(m-1)}{m+1}\alpha]+\mu^{(s)}{}_{k+1:k+t+1}2\alpha\frac{k(m-1)}{(m+1)(k+t+1)}\}. \tag{5.2.8}$$

For large values of m (5.2.7) and (5.2.8) may be approximated as

$$f_{k:k;[:.]}(y) = \sum_{t=0}^{k-1}\frac{(k-t)_{t+1}}{(k)_t}(-\alpha)^t\{f_{k:k+t}(y)(1-\alpha)+f_{k+1:k+t+1}(y)2\alpha\frac{k}{(k+t+1)}\} \tag{5.2.9}$$

and

$$\mu^{(s)}_{k:k;[:.]} = \sum_{t=0}^{k-1}\frac{(k-t)_{t+1}}{(k)_t}(-\alpha)^t\{\mu^{(s)}{}_{k:k+t}(1-\alpha)+\mu^{(s)}{}_{k+1:k+t+1}2\alpha\frac{k}{(k+t+1)}\} \tag{5.2.10}$$

respectively.

5.3 Distribution of $m_{k,[r:m]}$

The distribution function of $m_{k,[r:m]}$, denoted by $F_{1:k;[r:m]}(y)$, is given by

$$F_{1:k;[r:m]}(y) = 1-[1-F_Y(y)]^k[1-\frac{(2r-m-1)}{m+1}\alpha F_Y(y)]^k. \tag{5.3.1}$$

The probability density function corresponds to (5.3.1) is

$$f_{1:k;[r:m]}(y) =$$

$$\sum_{t=0}^{k-1}\frac{(k-t)_{t+1}}{(k)_t}[-\alpha\frac{(2r-m-1)}{m+1}]^t\{f_{t+1:k+t}(y)(1+\frac{(2r-m-1)}{m+1}\alpha)-f_{t+2:k+t+1}(y)2\alpha\frac{(t+1)(2r-m-1)}{(k+t+1)(m+1)}\}. \tag{5.3.2}$$

The moments of $m_{k,[r:m]}$ is denoted by $\mu^{(s)}_{1:k;[r:m]}$ s = 1,2.. and is derived from (5.3.2) as

$$\mu^{(s)}_{1:k;[r:m]} =$$

$$\sum_{t=0}^{k-1}\frac{(k-t)_{t+1}}{(k)_t}[-\alpha\frac{(2r-m-1)}{m+1}]^t\{\mu^{(s)}{}_{t+1:k+t}(1+\frac{(2r-m-1)}{m+1}\alpha)-\mu^{(s)}{}_{t+2:k+t+1}2\alpha\frac{(t+1)(2r-m-1)}{(k+t+1)(m+1)}\}. \tag{5.3.3}$$

In particular r = 1

$$m_{k;[1:m]} = \min(Y_{[1:m],1} \ldots Y_{[1:m],k}).$$

The probability density function of $m_{k,[1:m]}$ is

$$f_{1:k;[1:m]}(y) =$$

$$\sum_{t=0}^{k-1} \frac{(k-t)_{t+1}}{(k)_t} [-\alpha \frac{(1-m)}{m+1}]^t \{ f_{t+1:k+t}(y)(1+\frac{(1-m)}{m+1}\alpha) - f_{t+2:k+t+1}(y) 2\alpha \frac{(t+1)(1-m)}{(k+t+1)(m+1)} \}.$$

(5.3.4)

The moments of $m_{k,[1:m]}$ is obtained from (5.3.4) as

$$\mu^{(s)}_{1:k;[1:m]} =$$

$$\sum_{t=0}^{k-1} \frac{(k-t)_{t+1}}{(k)_t} [-\alpha \frac{(1-m)}{m+1}]^t \{ \mu^{(s)}{}_{t+1:k+t} (1+\frac{(1-m)}{m+1}\alpha) - \mu^{(s)}{}_{t+2:k+t+1} 2\alpha \frac{(t+1)(1-m)}{(k+t+1)(m+1)} \}.$$

(5.3.5)

If m is sufficiently large (5.3.4) and (5.3.5) may be approximated to

$$f_{1:k;[1:.]} = \sum_{t=0}^{k-1} \frac{(k-t)_{t+1}}{(k)_t} [\alpha]^t \{ f_{t+1:k+t}(y)(1-\alpha) + f_{t+2:k+t+1}(y) 2\alpha \frac{(t+1)}{(k+t+1)} \} \quad (5.3.6)$$

and

$$\mu^{(s)}{}_{1:k;[1:.]} = \sum_{t=0}^{k-1} \frac{(k-t)_{t+1}}{(k)_t} [\alpha]^t \{ \mu^{(s)}{}_{t+1:k+t}(y)(1-\alpha) + \mu^{(s)}{}_{t+2:k+t+1}(y) 2\alpha \frac{(t+1)}{(k+t+1)} \}. \quad (5.3.7)$$

5.4 (i) Application to Inference

In this section we use the above sampling method to illustrate the application of the above result to estimation problem. The bivariate Morgenstern type uniform distribution is specified by the distribution function

$$F_{X,Y}(x, y) = \frac{x}{\theta_1} \frac{y}{\theta_2} \left[1 + \alpha \left(1 - \frac{x}{\theta_1} \right) \left(1 - \frac{y}{\theta_2} \right) \right]; -1 \le \alpha \le 1; 0 < x < \theta_1; 0 < y < \theta_2.$$

(5.4.1)

From (5.2.7) the density function of $M_{k,[m:m]}$ under model (5.4.1) is

$$f_{k:k;[m:m]}(y)$$
$$=\frac{k}{\theta_2^k}y^{k-1}\sum_{t=0}^{k-1}\binom{k-1}{t}\{-\alpha\frac{(m-1)}{m+1}(1-\frac{y}{\theta_2})\}^t[(1-\frac{m-1}{m+1}\alpha)+2\alpha\frac{(m-1)k}{(m+1)(k+t+1)}\frac{y}{\theta_2}] \tag{5.4.2}$$

and

$$\mu_{k:k;[m:m]}=\theta_2\sum_{t=0}^{k-1}\{\alpha\frac{(1-m)}{(1+m)}\}^t\frac{(k-t)_{t+1}}{(k+1)_{t+2}}[(1-\frac{(m-1)}{(m+1)}\alpha)+2\alpha\frac{(m-1)(k+1)}{(m+1)(k+t+2)}]$$
$$=\theta_2 a_{k,m}. \tag{5.4.3}$$

Also

$$\mu^2_{k:k;[m:m]}=\theta^2{}_2(k+1)\sum_{t=0}^{k-1}\{\alpha\frac{(1-m)}{(1+m)}\}^t\frac{(k-t)_{t+1}}{(k+1)_{t+2}}[(1-\frac{(m-1)}{(m+1)}\alpha)+2\alpha\frac{(m-1)(k+2)}{(m+1)(k+t+3)}]$$
$$=\theta_2^2 b_{k,m}. \tag{5.4.4}$$

An unbiased estimator of θ_2 follows from (5.4.3) and is

$$\hat{\theta}_{2[k,m]}=\frac{Mk,[m:m]}{a_{k,m}} \tag{5.4.5}$$

and

$$V\ (\hat{\theta}_{2[k,m]})=\theta_2^2(\frac{b_{k,m}}{a^2_{k,m}}-1). \tag{5.4.6}$$

The UMVUE for θ_2 based on k independent observations is

$$\hat{\theta}_2=\frac{(k+1)}{k}Y_{k:k},\text{ see Rohatgi (1976).} \tag{5.4.7}$$

The relative efficiency of $\hat{\theta}_{2[k,m]}$ over $\hat{\theta}_2$ is

$$R_{[k,m]}=\frac{a^2_{k,m}}{k(k+2)[b_{k,m}-a^2_{k,m}]}. \tag{5.4.8}$$

The values of $R_{[k,m]}$ for α = 0.2(0.2)1, m = 10,20 and1000 are tabulated in Table 5.4.1.

Table 5.4.1

Values of $R_{[k,m]}$

m = 10

k	0.2	0.4	0.6	0.8	1
1	1.1221	1.2756	1.4723	1.7320	2.0851
2	1.1653	1.3734	1.6364	1.9706	2.3956
3	1.1959	1.4407	1.7454	2.1217	2.5810
4	1.2182	1.4890	1.8214	2.2239	2.7036
5	1.2352	1.5250	1.8769	2.2972	2.7907
6	1.2484	1.5528	1.9192	2.3523	2.8538
7	1.2591	1.5748	1.9523	2.3952	2.9062
8	1.2751	1.5927	1.9723	2.4259	2.9465
9	1.2812	1.6075	2.0008	2.4576	2.9794
10	1.2852	1.6199	2.0191	2.4809	3.0067

m = 20

1	1.1367	1.3132	1.5466	1.8668	2.3299
2	1.1852	1.4240	1.7339	2.1381	2.6645
3	1.2193	1.4999	1.8566	2.3054	2.8610
4	1.2442	1.5539	1.9412	2.4172	2.9906
5	1.2632	1.5941	2.0027	2.4970	3.0826
6	1.2779	1.6250	2.0492	2.5567	3.1513
7	1.2897	1.6494	2.0856	2.6031	3.2046
8	1.2994	1.6692	2.1149	2.6409	3.2472
9	1.3075	1.6855	2.1388	2.6704	3.2820
10	1.3143	1.6992	2.1588	2.6957	3.3110

m = 1000

2	1.208	1.483	1.850	2.343	3.000
3	1.246	1.568	1.988	2.526	3.200
4	1.274	1.629	2.081	2.646	3.330
5	1.295	1.673	2.149	2.732	3.429
6	1.311	1.708	2.200	2.796	3.500
7	1.324	1.735	2.24.	2.846	3.556
8	1.335	1.757	2.272	2.886	3.600
9	1.344	1.775	2.298	2.918	3.636
10	1.351	1.790	2.320	2.945	3.667

From the above table it can be observed that the relative efficiency of the estimator $\hat{\theta}_{2[k,m]}$ is an increasing function of α and k. Moreover for high values of α, $\hat{\theta}_{2[k,m]}$ is highly efficient than the UMVU estimator $\hat{\theta}_2$.

(ii) Application to the Selection Problem

It follows from (5.4.3) that the effectiveness of the selection problem mentioned in the introduction is denoted by $e_{k,n}$ and is given by

$$e_{k,n} = \frac{E[M_{k,[m:m]}]}{E[Y_{n:n}]} = \frac{n+1}{n} a_{k,n} \tag{5.4.8}$$

It is noted that $e_{k,n}$ is free of the parameter θ_2 and is equal to 1 when k = n. The values of $e_{k,n}$ for $\alpha = 0.2(0.2)1, n = 10,20,50,100$ are tabulated in the following table.

Table 5.4.2

Values of $e_{k,n}$

n	m	k	0.2	0.4	0.6	0.8	1.0
10	10	1	.5800	.6100	.6400	.6700	.7000
	2	5	.7570	.7796	.8008	.8207	.839
	5	2	.9251	.9330	.9404	.9473	.9538
20	20	1	.5567	.5883	.6200	.6517	.6833
	10	2	.7227	.7535	.7775	.7996	.8198
	5	4	.8576	.8733	.8873	.8997	.9108
	4	5	.8891	.9017	.9128	.9227	.9315
	2	10	.9596	.9642	.9685	.9724	.9759
50	50	1	.5427	.5753	.6050	.6407	.673
	25	2	.7107	.7381	.7687	.7870	.8080
	10	5	.8683	.8839	.8971	.9084	.9181
	5	10	.9367	.9446	.9513	.9569	.9618
	2	25	.9831	.9851	.9870	.9886	.9902
100	100	1	.5380	.5710	.6040	.6370	.6700
	50	2	.7015	.7305	.7969	.7807	.8020
	25	4	.8310	.8506	.8672	.8813	.8933
	20	5	.8616	.8782	.8921	.9038	.9136
	10	10	.9294	.9385	.9459	.9520	.9570
	5	20	.9672	.9714	.9750	.9779	.9805
	4	25	.9751	.9783	.9810	.9833	.9853
	2	50	.9914	.9925	.9934	.9943	.9950

Table 5.4.2 illustrates that $e_{k,n}$ increases with k and α. It tends to 1 as k approaches to n. This method of selection will substantially reduce the number of expensive measurements. For example, if n = 100 twenty-five percent of the expensive measurements give an efficiency greater than .975 for all $\alpha > 0$.

5.5 Distribution and Expected values of the Rank of the r^{th} Concomitant from the Morgenstern family

In this section we are concerned with the distribution and expected value of the rank $R_{[r:n]}$ of $Y_{[r:n]}$ among n Y_i . Suppose we have independent measurements (X_i,Y_i), i = 1,2...n, with cumulative distribution function specified in (2.2.1), on individuals $A_1,A_2,\ldots A_n$ and that A_i rank r^{th} on the X measurements. Here we study the following questions: (a) What is the probability that A_i will rank s^{th} on the Y-measurement? (b) What is A_i's expected rank on the Y-measurement?.

Let $R_{[r:n]}$denote the rank of $Y_{[r:n]}$. Then

$$R_{[r:n]}=\sum_{i=1}^{n} I(Y_{[r:n]}-Y_i),$$

where

$$\begin{aligned} I(x) &= 1 \quad \text{if } x \geq 0 \\ &= 0 \quad \text{if } x<0 \end{aligned} \tag{5.5.1}$$

The small sample distribution of $R_{[r:n]}$ is given in O'Connell David and Yang (1977). They have derived the general formula for the finite distribution of $R_{[r:n]}$ and is given in (1.2.18).

If (X, Y) have the distribution specified by (2.1.1) and X and Y have marginal distribution of the same form then it is quick to obtain the following three relations.

Relation 5.5.1

$$\pi_{rs} = \pi_{sr} \qquad \text{for } r, s = 1,2\ldots n\,. \qquad (5.5.2)$$

Since for the Morgenstern family $F_{X,Y}(x,y) = F_{Y,X}(y,x)$

$$P(r(X_i) = r, r(Y_i) = s) = P(r(Y_i) = r, r(X_i) = s)$$

$$\pi_{rs} = \pi_{sr} \qquad \text{for } r, s = 1,2,\ldots n.$$

Relation 5.5.2

$$\pi_{rs} = \pi_{n+1-r,n+1-s}. \qquad (5.5.3)$$

It follows directly from (1.2.18).

Relation 5.5.3

$$\pi_{rs}(\alpha) = \pi_{r,n+1-s}(-\alpha)\,. \qquad (5.5.4)$$

Since for the Morgenstern family α is the correlation parameter.

The values of π_{rs} for the Morgenstern type uniform distribution is tabulated for $\alpha = 0.25\ (.25)\ 1$, n = 5,10 in Table 5.5.1.

Table 5.5.1

$\pi_{rs} = P(R_{[r:n]} = s)$ **as function of α for n = 5 and n = 10**

		α			
n=5; r	s	0.25	0.5	0.75	1
5	5	.2282	.2573	.2873	.3184
	4	.2137	.2268	.2394	.2512
	3	.1996	.1983	.1962	.1933
	2	.1859	.1715	.1568	.1418
	1	.1726	.1461	.1203	.0952

4	4	.2071	.2145	.2226	.2314
	3	.2002	.2008	.2018	.2014
	2	.1931	.1863	.1795	.1724
	1	.1859	.1715	.1568	.1418
3	3	.2004	.2017	.2040	.2073
	2	.2002	.2008	.2018	.2030
	1	.1996	.1983	.1962	.1933
n=10; r 10	10	.1189	.183	.1583	.1789
	9	.1145	.1292	.1441	.1591
	8	.1103	.1204	.1305	.1403
	7	.1061	.1118	.1173	.1225
	6	.1019	.1034	.1046	.1055
	5	.0975	.0952	.0923	.0891
	4	.0937	.0871	.0803	.0734
	3	.0886	.0792	.0687	.0582
	2	.0876	.0714	.0574	.0436
	1	.0867	.0638	.0464	.0294
9	9	.1113	.1226	.1340	.1457
	8	.1080	.1160	.1240	.1320
	7	.1048	.1095	.1141	.1187
	6	.1015	.1030	.1043	.1057
	5	.0983	.0966	.0946	.0926
	4	.0951	.0902	.0851	.0793
	3	.0919	.0839	.0757	.0696
	2	.0886.	.0776	.0665	.0555
	1	.0856	.0704	.0574	.0436

8	8	.1058	.1115	.1174	.1223
	7	.1035	.1070	.1106	.1143
	6	.1012	.1024	.1037	.1051
	5	.0989	.0978	.0968	.0958
	4	.0966	.0932	.0898	.0864
	3	.0943	.0885	.0828	.0763
	2	.0916	.0839	.0757	.0676
	1	.0896	.0792	.0687	.0582
7	7	.1021	.1044	.1069	.1085
	6	.1008	.1017	.1029	.1042
	5	.0994	.0989	.0987	.0986
	4	.0984	.0961	.0943	.0926
	3	.0911	.0932	.0898	.0864
	2	.0951	.0902	.0851	.0799
	1	.0937	.0871	.0803	.0734
6	6	.1003	.1009	.1017	.1028
	5	.0999	.1000	.1033	.1009
	4	.0994	.0990	.0987	.0986
	3	.0989	.0978	.0968	.0958
	2	.0983	.0966	.0946	.0926
	1	.0918	.0952	.0923	.0891
5	5	.1003	.1009	.1017	.1028
	4	.1008	.1017	.1029	.1042
	3	.1012	.1024	.1037	.1051
	2	.1015	.1030	.1043	.1053
	1	.1019	.1034	.1046	.1055

The, expected rank of $Y_{[r:n]}$, $E[R_{[r:n]}]$, r = 1,2,...n may be obtained from the formula (1.2.19) see David and Nagaraja (1998).

We have

$$\int_{-\infty}^{\infty}\theta_1 f(y\mid x)dy = \int_{-\infty}^{\infty}F_X(x)F_Y(y)[1+\alpha(1-F_x(x))(1-F_Y(y))]f_Y(y)[1+\alpha(1-2F_x(x))(1-F_Y(y))]dy$$

$$= F_X(x)[\int_0^1 udu + \alpha(1-F_X(x))\int_0^1 u(1-u)du + \alpha(1-2F_X(x))\int_0^1 u(1-2u)du + \alpha^2(1-F_X(x))(1-2F_X(x))$$

$$\int_{-\infty}^{\infty} u(1-u)(1-2u)du], \qquad \text{by the transformation } u = F_Y(y)$$

$$= F_X(x)\frac{[1+\frac{\alpha}{3}F_X(x)]}{2}. \tag{5.5.5}$$

$$\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\theta_1(x,y)f(y\mid x)dy]f_{r-1:n-1}(x)dx = \frac{1}{B(r-1,n-r+1)}\int_{-\infty}^{\infty} F_X(x)\frac{[1+\frac{\alpha}{3}F_X(x)]}{2}$$

$$[F_x(x)]^{r-2}[1-F_x(x)]^{n-r}f_X(x)dx$$

$$= \frac{B(r,n-r+1)+(\alpha/3)B(r+1,n-r+1)}{2B(r-1,n-r+1)}$$

$$= \frac{1}{2}[\frac{r-1}{n}+\frac{r(r-1)\alpha}{3n(n+1)}]$$

$$= \frac{(r-1)}{2n}[1+\frac{\alpha.r}{3(n+1)}]. \tag{5.5.6}$$

$$\int_{-\infty}^{\infty}\theta_3(x,y)f(y\mid x)dy = \int_{-\infty}^{\infty}[F_Y(y)-F_{X,Y}(x,y)]f(y\mid x)dy$$

$$= \int_{-\infty}^{\infty}F_Y(y)f_Y(y)[1+\alpha(1-2F_X(x))(1-F_Y(y))]dy - F_X(x)\frac{\{1+\frac{\alpha}{3}F_X(x)\}}{2}$$

$$= \quad \frac{1}{2} - \frac{\alpha[1-2F_X(x)]}{6} - F_X(x)\frac{\{1+\frac{\alpha}{3}F_X(x)\}}{2}$$

$$= \quad \frac{[1-F_X(x)]}{2}\{1-\alpha/3(1-F_X(x))\}. \qquad (5.5.7)$$

$$\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\theta_3(x,y)f(y\mid x)dy]f_{r:n-1}(x)dx = \frac{1}{B(r,n-r)}\int_{-\infty}^{\infty}\frac{[1-F_X(x)]}{2}\{1-\alpha/3(1-F_X(x)\}$$

$$[F_X(x)]^{r-1}[1-F_X(x)]^{n-r+1}f_X(x)dx$$

$$= \quad \frac{1}{2B(r,n-r)}[B(r,n-r+1)-(\alpha/3)B(r,n-r+2)]$$

$$= \quad \frac{(n-r)}{2n}[1-(\alpha/3)\frac{n-r+1}{n+1}]. \qquad (5.5.8)$$

Hence $$E[R_{[r:n]}] = 1+n\{\frac{(r-1)}{2n}[1+\frac{\alpha r}{3(n+1)}]+\frac{(n-r)}{2n}[1-(\alpha/3)\frac{n-r+1}{n+1}]\}$$

$$= \frac{1}{2}[1+n\{1-\frac{(n-2r+1)\alpha}{3(n+1)}\}]. \qquad (5.5.9)$$

It follows from (5.5.9) that $E[R_{[r:n]}]$ is a linear function of α and it increases(decreases) with r for $\alpha > 0$ (<0).

It readily follows that

$$E[R_{[r:n]}] = n+1 - E[R_{[n+1-r:n]}] \qquad (5.5.10)$$

and for negative α, we can use

$$E[R_{[r:n]}(-\alpha)] = n+1 - E[R_{[r:n]}(\alpha)]. \qquad (5.5.11)$$

The values of $E[R_{[r:n]}]$ are tabulated for specific values of α and n in the following table .

Table 5.5.2

$E[R_{[r:n]}]$ as a function of α for n=5, 10 and 20

		α			
n	**r**	**.25**	**.5**	**.75**	**1**
5	1	2.861	2.722	2.583	2.440
	2	2.931	2.861	2.792	2.722
	3	3	3	3	3
	4	3.069	3.139	3.208	3.278
	5	3.139	3.278	3.412	3.556
10	1	5.159	4.818	4.477	4.136
	2	5.235	4.970	4.705	4.439
	3	5.311	5.121	4.932	4.742
	4	5.386	5.273	5.159	5.045
	5	5.462	5.424	5.386	5.348
	6	5.538	5.576	5.614	5.652
	7	5.614	5.727	5.814	5.955
	8	5.689	5.879	6.018	6.258
	9	5.765	6.030	6.295	6.561
	10	5.841	6.182	6.523	6.864

20	1	9.746	8.992	8.238	7.484
	2	9.825	9.151	8.476	7.802
	3	9.905	9.310	8.714	8.119
	4	9.984	9.468	8.952	8.437
	5	10.043	9.627	9.190	8.754
	6	10.143	9.786	9.426	9.071
	7	10.222	9.994	9.667	9.389
	8	10.302	10.103	9.905	9.701
	9	10.381	10.262	10.143	10.024
	10	10.460	10.421	10.381	10.341
	11	10.540	10.579	10.619	10.659
	12	10.619	10.738	10.857	10.976
	13	10.698	10.897	11.095	11.294
	14	10.788	11.056	11.373	11.611
	15	10.857	11.214	11.571	11.929
	16	10.937	11.373	11.810	12.246
	17	11.016	11.532	12.048	12.563
	18	11.095	11.690	12.286	12.881
	19	11.175	11.849	12.524	13.198
	20	11.254	12.008	12.762	13.516

LIST OF TABLES

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