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On Condition (A_2) of Bandlet and Petrich for Inverse Semirings

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Dedicated to the memory of Prof. A.B.Thaheem

Abstract

In this paper, we introduce the notion of commutators for a certain class of semirings satisfying the condition (A_2) of Bandlet and Petrich. We establish a few fundamental results of this class included Jacobian and other identities, that become special relevant cases of ring theory, and may be helpful to initiate Lie type theory of semirings (MA-semirings).

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1 Introduction and Preliminaries

The theory of commutators plays an important role in the study of Lie algebras [8], prime rings [14, 15] and C*-algebras [9]. It has tremendous applications in the theory of derivations of rings and modules as well.

The purpose of this paper is to initiate the study of commutators for semirings and to develop its first order theory, which is, indeed, the generalization of the commutators of rings. In this connection, we identify the class of additively commutative additively inverse semirings with 0 (see [7]), satisfying the condition (A_2) of Bandlet and Petrich [1]. We will call such semirings as MA-semirings. We introduce the notion of MA-semiring in section 2, which is useful to initiate the theory of commutators in semirings. Also some canonical constructions of MA-semirings from given MA-semirings R are presented in Proposition 2.6. A few identities of K.I Beidar [2] and M.A. Chebotar [4] are refined by using the inner derivations in MA-semirings. In section 3, we introduce the notion of commutators in MA-semirings and establish fundamental results for commutators including the Jacobian identity in the framework of MA-semirings. The notion of derivations of rings can be naturally extended in semirings [5, page 30]. The theory of derivations of semirings is not well developed as compared to the theory of derivations of rings (see[3, 10, 14, 15]) due to the absence of additive inverse and the lack of some important concepts included commutators. In section 4, we introduce the notion of inner derivations and study the derivations with the help of commutators and establish fundamental identities which generalize the relevant results [3, 5] of ring theory.

By semiring we mean a non empty set R with two binary operations '+' and '.' such that $(R,+)$ and $(R,.)$ are semigroups, where + is **' commutative** and $a.(b+c)=a.b+a.c$, $(b+c).a=b.a+c.a$ hold for all $a,b,c\in R$. If there exists $0\in R$ such that $a+0=0+a=a$ and $a0=0a=0$ for all $a\in R$, then R is said to be a **semiring with '0'** (see [5]). A non empty subset I of R is said to be ideal if $u,v\in I, r\in R$ imply $u+v\in I$ and $ur, ru\in I$. An ideal I is said to be k -ideal if $a+b\in I, b\in I$ imply that $a\in I$ (for more detail see [5]). An element $a\in R$ is said to be additively regular if there exists unique $b\in R$ such that $a+b+a=a$ (see [7]). In addition to this if $b+a+b=b$, then such semirings are called inverse semirings first introduced by P.H. Karvellas [7].

In 1982, H. J. Bandlet and M. Petrich [1] considered additively regular semirings which are also additively commutative, satisfying the conditions (A_1) $x(x+x^{'})=x+x^{'}\ \forall\ x\in S$; (A_2) $y(x+x^{'})=\left(x+x^{'}\right)y\ \forall x,y\in S$; (A_3) $x+(x+x^{'})y=1\ \forall x,y\in S$, where S is an inverse semiring. They proved some remarkable results in this class of semirings. The theory of inverse semirings (satisfying the conditions, (A_1), (A_2), (A_3) of [1]) is further expanded by several authors (see [11, 12]).

If R is an inverse semiring, then clearly $M_n(R)$, $R[x]$ is also an inverse semiring for every positive integer n. The centre of R is denoted by $Z(R)$. Now, we recall the following proposition of ([5, page 10]) which will be extensively used later.

Proposition 1.1.

Let R be an inverse semiring, let $a,b\in R$, then (i) $(a+b)^{'}=a^{'}+b^{'}$, (ii) $(ab)^{'}=a^{'}b=ab^{'}$, (iii) $a^{''}=a$, (iv) $a^{'}b^{'}=(a^{'}b)^{'}=(ab)^{''}=ab$, $a+a^{'}$ is additively idempotent. For undefined terms we refer to [5].

2 MA-semirings

We begin this section by introducing the notion of MA-semirings as a generalization of rings.

Definition 2.1. A non empty set R with binary operations $+$ and $.$ is called an MA-semiring if the following statements are satisfied:
(1) $(R, +, .)$ is an additively commutative inverse semiring with zero element.
(2) Satisfying the condition (A_2) of H.J. Bandlet and M. Petrich [1]. That is $a + a' \in Z(R)$ for all $a \in R$.
Now, we mention a few examples of MA-semirings.

Example 2.2. Let R be a commutative ring and S be a subsemiring of the semiring of all ideals of R with respect to ordinary addition and product of ideals. Set $R_1 = \{(a, I) : a \in R, I \in S\}$ and define operations $\oplus$ and $\odot$ on R_1 by setting $(a, I) \oplus (b, J) = (a + b, I + J)$ and $(a, I) \odot (b, J) = (ab, IJ)$ for all $I, J \in S$. If $(a, I) \in R_1$, we define pseudo inverse of the element of R_1 as $(a, I)' = (-a, I)$. Then R_1 is an MA-semiring.

Lemma 2.3. Every additively idempotent and multiplicative semiring is trivially an MA-semiring; so as any commutative ring and hence their cartesian product is also such.

The following example reflects that every inverse semiring may not be MA-semiring.

Example 2.4. Let $R = N_\circ = \{0, 1, 2, ...\}$ in which addition $\oplus$ and multiplication $\odot$ are defined as, $x \oplus y = sup(x, y)$ and $x \odot y = inf(x, y)$ for all $x, y \in R$. It can be observed that $M_2(R)$, the set of all 2×2 matrices over R, is an inverse semiring with $A' = A$ for all $A \in M_2(R)$ under usual addition and multiplication of matrices but not an MA-semiring. If we take $A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 0 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$ from $M_2(R)$. Then it can be easily seen that $(A + A')C \neq C(A + A')$ and hence axiom (A_2) of H.J. Bandlet and M. Petrich [1] is not satisfied. This shows that $M_2(R)$ is not an MA-semiring while R is an MA-semiring. Thus in general, if R is an MA-semiring $M_n(R)$ may not be an MA-semiring for $n > 1$.

Example 2.5. Let R be a non commutative ring and S be an MA-semiring and $M_2(R \times S)$ be the collection of 2×2 matrices. Take the subset $R_2 \subseteq M_2(R \times S)$ defined as

$R_1 = \left\{ \begin{bmatrix} 0 & 0 \\ (r,a) & (s,a) \end{bmatrix} : r,s \in R, a \in S \right\}$ and for $A = \begin{bmatrix} 0 & 0 \\ (r,a) & (s,a) \end{bmatrix}$
taking additive pseudo inverse as $A' = \begin{bmatrix} 0 & 0 \\ (r,a) & (s,a) \end{bmatrix}' = \begin{bmatrix} 0 & 0 \\ (-r,a') & (-s,a') \end{bmatrix}$
then under the matrix addition and multiplication R_2 is an MA-semiring, which indeed is non commutative since R is non commutative.

Proposition 2.6. Let R, S be MA-semirings, $R[x]$ the set of all polynomials over R, then $R \times S$ and $R[x]$ are MA-semirings.

Definition 2.7. Let R be MA-semiring. A subsemiring S is said to be MA-subsemiring, if $a \in S$ implies that $a' \in S$.

Proposition 2.8. Let R be an MA-semiring, $M_n(R)$ be the set of all $n \times n$ matrices over $R, Dig(n,R)$ be the set of all diagonal matrices over R, then the following statements hold.

(i) $(M_n(R), +)$ is an inverse semiring.

(ii) $Dig(n,R)$ is an MA-semiring.

From Example 2.4, it follows that $M_n(R)$ need not be always an MA-semiring. The part of Proposition 2.6, by Definition 2.7, can be refined in the following way.

Proposition 2.9. If S is an MA-subsemiring of an MA-semiring R, then $S[x]$ is an MA-subsemiring of an MA-semiring $R[x]$.

Definition 2.10. Let R be an MA-semiring. An ideal I of R is said to be MA-ideal if it is MA-subsemiring.

Proposition 2.11. Every k-ideal of an MA-semiring R is an MA-subsemiring and hence is an MA-ideal of R.

Proof. Let I be a k-ideal. Let $a \in I$, as R is MA-semiring, $a' \in R$ such that $a + a' + a = a \in I$. This implies that $a' + 2a \in I, 2a \in I$. As I is k-ideal, therefore, $a' \in I$. Hence $a + a' \in I, a + a' \in Z(R)$. Since $Z(I) = I \cap Z(R)$, therefore, $a + a' \in Z(I)$ and so I is an MA-semiring.

Proposition 2.12. Let R be an MA-semiring and I be an MA-ideal of R. Then $I[x]$ is an MA-ideal of $R[x]$.

Proof. By Proposition 2.9, it follows that $I[x]$ is MA-subsemiring of $R[x]$. As I is an ideal of R, therefore $I[x]$ is an ideal of $R[x]$, that is $I[x]$ is an MA-ideal of $R[x]$.

Let X be a non empty set, R be a semiring, $Map(X,R)$ be the set of all mappings from X in to R, define $+$, . pointwise addition and multiplication respectively. Then $(Map(X,R),+,.)$ form a semiring. However, if R is an MA-semiring, then one can establish the following proposition.

Proposition 2.13. Let X be a non empty set, $x \in X$ and R be an MA-semiring. Then the following statements hold.

(i) $(Map(X,R),+,.)$ is an MA-semiring.

(ii) If $I_x = \{f \in Map(X,R) : f(x) = 0\}$, then I_x is a k-ideal and hence MA-ideal of $Map(X,R)$.

(iii) If $S \subseteq X, I_s = \{f \in Map(X,R) : f(x) = 0 \ \ \forall x \in S\}$, then $I_s = \bigcap_{x \in S}$ and I_s is an MA-ideal of $Map(X,R)$.

Proposition 2.14. If R is an MA-semiring, then $Z(R)$ is an MA-semiring.

Proof. It is well known that $Z(R)$ is subsemiring of R. Let $a \in Z(R)$, then $ax = xa$ implies that $(ax)^{'} = (xa)^{'}$ that is $a^{'}x = xa^{'}$ implying that $a^{'} \in Z(R)$ and hence $Z(R)$ is an MA-semiring.

If I is a k-ideal of an MA-semiring R, then by Proposition 2.11 I is an MA-ideal of R. As $Z(I) = Z(R) \cap I$. Therefore by Proposition 2.13, $Z(I)$ is MA-semiring. This leads to the following.

Corollary 2.15. If I is a k-ideal of MA-semiring R, then $Z(I)$ is an MA-semiring.

3 Commutators

In this section, we introduce the notion of commutators for MA-semirings, which is, indeed, a generalization of commutators of rings.

Definition 3.1. If R is an MA-semiring, define commutator as a mapping $[.,.] : R \times R \to R$ by $[x,y] = xy + (yx)^{'} = xy + y^{'}x = xy + yx^{'}$ for all $x,y \in R$. Then $[x,y]$ is called a commutator of x,y.

Theorem 3.2. If R is an MA-semiring, then for all $x,y,z \in R$, the following identities are valid.

$(i)[x,yz] = [x,y]z + y[x,z]$,(Jacobian Identity)

$(ii)[xy,z] = x[y,z] + [x,z]y$,(Jacobian Identity)

$(iii)[x+y,z] = [x,z] + [y,z]$

$(iv)[x,0] = [0,x] = 0$

$(v)([x,y])' = [y,x] = [x,y'] = [x',y]$
$(vi)[[x,y],z] = [x,y]z + z[y,x]$
$(vii)[nx,y] = n[x,y]$, for any positive integer n.

Proof. (i) Take the left hand side of (i), then by Definition 3.1.

$$\begin{aligned}[x,yz] &= x(yz) + (yz)x' = x(yz) + yz(x' + x + x') \\ &= x(yz) + yz(x') + yz(x + x') = x(yz) + yz(x') + y(x + x')z \\ &= (xy + yx')z + y(xz + zx') = [x,y]\,z + y\,[x,z].\end{aligned}$$

The proof of (ii) is similar to the proof of (i). The identities (iii) to (vii) can be proved by using more or less similar technique.

Corollary 3.3. If R is a ring, then the following statements are valid for all $x,y,z \in R$.
(1) $[x,x] = 0$; (2) $[x+y,z] = [x,z] + [y,z]$; (3) $[x,y+z] = [x,y] + [x,z]$;
(4) $[xy,z] = x[y,z] + [x,z]y$; (5) $[x,yz] = [x,y]z + y[x,z]$

Lemma 3.4. Let R be an MA-semiring. For each $x \in R$, denote $x + x'$ by x_0. Then (i) $x_0 + x_0 = x_0 = x_0'$, (ii) $x + x_0 = x$, (iii) $(xy)_0 = x_0y = xy_0 = x_0y_0 = y_0x_0 = (yx)_0$ for all $x,y \in R$.
Proof. We only establish $(xy)_0 = x_0y = xy_0$. The proofs of other identities are quite obvious. Indeed, $x_0y_0 = (x+x')(y+y') = xy + xy' + x'y + x'y' = xy + xy' + x'y + xy = xy + xy' + xy + x'y = (x + x' + x)y + x'y = xy + x'y = (x+x')y = x_0y$. Similarly we can show that $(xy)_0 = xy_0$, as $x_0, y_0 \in Z(R)$. Thus $(xy)_0 = x_0y = xy_0$.

Now, we prove the Jacobian theorem which may be useful in the development of Lie type theory of semirings.

Theorem 3.5. Let R be an MA-semiring, then $[x,[y,z]]+[y,[z,x]] = [[x,y],z]$ holds for all $x,y,z \in R$.

Proof. By Theorem 3.2 (v), we have $[x,[y,z]] = [[y,z],x]' = [y,z]'x + x[z,y]' = [z,y]x + x[y,z]$ for all $x,y,z \in R$. Thus $[x,[y,z]] + [y,[z,x]] = [z,y]x + x[y,z] + [x,z]y + y[z,x]$
$= (zy + y'z)x + x(yz + z'y) + (xz + z'x)y + y(zx + x'z) = (xyz + yx'z + zyx + z'xy) + (y'zx + yzx) + (xz'y + xzy) = (xyz + yx'z + zyx + z'xy) + y_0zx + xz_0y = [x,y]z + z[y,x] = [[x,y],z]$, as by Lemma 2.3, $zyx + y_0zx = zyx + (zyx)_0 = zyx$ and $xyz + xz_0y = xyz + (xyz)_0 = xyz$.

Remark 3.6. As we have already shown in Example 2.4 that if R is an MA-semiring, then $M_2(R)$ may not be an MA-semiring. Also we note that the validity of (i) and (ii) of theorem 3.2 in the context of Example 2.4, depending upon the particular selection of the matrices. It may hold or may not. However, in general, the Jacobian identity is not hold.

The parts (i) and (ii) of theorem 3.2 may valid for some particular case. But if we take $A = \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$, then $[A, BC] \neq [A, B]C + B[A, C]$ and also $(A + A') \notin Z(M_2(R))$.

Proposition 3.7. Let R be an MA-semiring, then for all $x, y, z \in R$ the following identities hold:

(i) $[xy, z] + [yz, x] + [zx, y] = [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$

(ii) $[xyz, u] = xy\,[z, u] + x\,[y, u]\,z + [x, u]\,yz$.

Proof.(i) In view of Theorem 3.2, (i) becomes $x\,[y, z] + [y, z]'\,x + y\,[z, x] + [z, x]'\,y + z\,[x, y] + [x, y]'\,z = [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$.

(ii) Using the Definition 3.1 and Lemma 3.4, the right hand side becomes: $xyzu + xyu_0z + xu'yz + xuyz + ux'yz = xyzu + xyzu_0 + xu_0yz + u'xyz$ $= xyzu + xyzu_0 + u_0xyz + u'xyz = xyz(u_0 + u) + (u_0 + u')xyz = xyzu + u'xyz = [xyz, u]$.

Proposition 3.8. Let R be an MA-semiring and $x, y, z \in R$. Then

(i) $[xy, z] + [yz, x] = [y, zx]$

(ii) $[xyz, u] + [yzu, x] + [zux, y] = [z, uxy]$

Proof. (i): By Definition 3.1 and Lemma 3.4, (i) reduces to: $yzx + z'xy + xyz_0 = yzx + z'xy + z_0xy = yzx + (z_0 + z')xy = yzx + z'xy = yzx + zxy' = [y, zx]$

(ii): By Definition 3.1 and Lemma 3.4, (ii) becomes. $xyzu + u'xyz + yzux + x'yzu + zuxy + y'zux$

$= u'xyz + xyzu_0 + y_0zux + zuxy = u'xyz + u_0xyz + zuxy_0 + zuxy = (u' + u_0)xyz + zux(y_0 + y)$

$= zuxy + uxyz' = [z, uxy]$.

We define Jordan product as $(x \circ y) = xy + yx$. Observe that $x \circ y = y \circ x$. and $(x + y) \circ z = (x \circ z) + (y \circ z)$.

Proposition 3.9. Let R be an MA-semiring and $x, y, z \in R$. Then the following Jordan identities hold:

(i) $x \circ y = y \circ x$.

(ii) $(x + y) \circ z = (x \circ z) + (y \circ z)$.

(iii) $[x \circ y, z] + [y \circ z, x] = [y, z \circ x]$.

Proof. The proofs of (i), (ii) are obvious. Now by using the Definition 3.1 and Lemma 3.4 the left hand side of the (iii) becomes:

$(x \circ y)z + z'(x \circ y) + (y \circ z)x + x'(y \circ z) = yzx + yxz + z'xy + x'zy + xyz_0 + z_0yx$ $= yzx + yxz + z'xy + x'zy + z_0xy + yxz_0 = yzx + x'zy + (z_0 + z')xy + yx(z_0 + z)$

$= y(zx + xz) + (zx + xz)y^{'} = y(z \circ x) + (z \circ x)y^{'} = [y, z \circ x]$.

Proposition 3.10. Let R be an MA-semiring and $x, y, z \in R$. Then the following Jordan identity holds $x \circ [y, z] + [x, z] \circ y = [(x \circ y), z]$.
Proof. By using the Definition 3.1 and Lemma 3.4, we have:
$x \circ [y, z] + [x, z] \circ y = x \circ (yz + zy^{'}) + (xz + zx^{'}) \circ y = xyz + yxz + zx^{'}y + zy^{'}x + xzy_0 + yzx_0 = xyz + yxz + zx^{'}y + zy^{'}x + y_0xz + x_0yz = (x_0 + x)yz + (y_0 + y)xz + zx^{'}y + zy^{'}x = (xy + yx)z + z^{'}(xy + yx) = (x \circ y)z + z^{'}(x \circ y) = [x \circ y, z]$.

Remark 3.11. If R is commutative ring, then the commutators become zero. But it is not the case with the commutative MA-semiring. Proposition 3.7, 3.8, 3.9, 3.10 are refinements of the relevant commutator identities of ring theory (see [2, 4]).

4 Derivations of semirings.

In this section, we develop a relationship between commutators and derivations. In this connection, we are able to generalize some results of ring theory. Also we introduce here the notion of inner derivation in canonical fashion.

Definition 4.1 ([5]). A derivation on a semiring R is a function $d : R \to R$ satisfying: $(i) d(r_1 + r_2) = d(r_1) + d(r_2)$, $(ii) d(r_1r_2) = d(r_1)r_2 + r_1d(r_2)$ for all $r_1, r_2 \in R$.

If R is an additively idempotent semiring, that is, $r+r = r$, then the identity mapping $I(r) = r$ is a derivation on R. In particular, identity mapping is a derivation in distributive lattice.

Definition 4.2. Let R be an MA-semiring, Let $a \in R$ be a fixed element of R. Define $d : R \to R$ by $d(x) = [a, x]$, for all $x \in R$. Then d is a derivation and said to be an inner derivation, indeed, for all $x, y \in R$, $d(xy) = [a, xy] = x[a, y] + [a, x]y = xd(y) + d(x)y$ (cf. Theorem 3.2 (i)).

For convenience we prefer to write $d([x, y]) = [a, [x, y]]$ by $[x, y]^d$.

Proposition 4.3. Let R be an MA-semiring, d be a derivation for all $x, y \in R$, then $d(xy) = d(x^{'}y^{'})$

Proof. By Proposition 1.1, we have, $d(x^{'}y^{'}) = d(x^{'})y^{'} + x^{'}d(y^{'}) = (d(x))^{'}y^{'} + x^{'}(d(y))^{'} = (d(x)y^{'})^{'} + (x^{'}d(y))^{'} = (d(x)y)^{''} + (xd(y))^{''} = d(x)y + xd(y) = d(xy)$.

Proposition 4.4. Let R be an MA-semiring, $a \in R$ and d be an inner derivation $d(x) = [a, x]$ for all $x \in R$, then for all $x, y, z \in R$ $[xy, z]^d + [yz, x]^d =$

$[y, zx]^d$.

Proof. By using Definition 4.2, Definition 3.1 and Lemma 3.4, the left hand side of the given expression becomes: $az^{'}xy + z^{'}xya' + ayzx + yzxa^{'} + ax_0yz + x_0yza^{'} = az^{'}xy + z^{'}xya' + ayzx + yzxa^{'} + ayzx_0 + yzx_0a' = az^{'}xy + z^{'}xya^{'} + ayz(x_0 + x) + yz(x_0 + x)a^{'} = az^{'}xy + z^{'}xya^{'} + ayzx + yzxa^{'} = a(yzx + zxy^{'}) + (yzx + zxy^{'})a^{'} = a\,[y, zx] + [y, zx]\,a^{'} = [a, [y, zx]] = [y, zx]^d$.

Proposition 4.5. Let R be an MA-semiring, $a \in R$ and d be an inner derivation determined by a, that is, $d(x) = [a, x]$ for all $x \in R$, then for all $x, y, z, u \in R$, the following identities hold:

(i) $[xyz, u]^d = (xy\,[z, u])^d + (x\,[y, u]\,z)^d + ([x, u]\,yz)^d$.

(ii) $[xyz, u]^d + [yzu, x]^d + [zux, y]^d = [z, uxy]^d$.

Proof. (i) By Definition 4.2, Definition 3.1 and Lemma 3.4, we have

$(xy\,[z, u])^d + (x\,[y, u]\,z)^d + ([x, u]\,yz)^d = [a, xy\,[z, u]] + [a, x\,[y, u]\,z] + [a, [x, u]\,yz]$
$= axyzu + aux^{'}yz + xyzua^{'} + ux^{'}yza^{'} + axyuz_0 + xyuz_0a^{'} + ax_0uyz + x_0uyza^{'}$
$= axyzu + aux^{'}yz + xyzua^{'} + ux^{'}yza^{'} + axyz_0u + xyz_0ua^{'} + aux_0yz + ux_0yza^{'}$
$= axy(z_0 + z)u + xy(z_0 + z)ua^{'} + au(x_0 + x^{'})yz + u(x_0 + x^{'})yza^{'}$
$= axyzu + aux^{'}yz + xyzua^{'} + ux^{'}yza^{'} = a(xyzu + u^{'}xyz) + (xyzu + u^{'}xyz)a^{'}$
$= \left[a, xyzu + u^{'}xyz\right] = [a, [xyz, u]] = [xyz, u]^d$.

(ii). By Definition 4.2, Definition 3.1 and Lemma 3.4, the right hand side becomes.

$[a, [xyz, u]] + [a, [yzu, x]] + [a, [zux, y]]$
$= au^{'}xyz + azuxy + u^{'}xyza^{'} + zuxya^{'} + axyzu_0 + xyzu_0a^{'} + ay_0zux + y_0zuxa^{'}$
$= au^{'}xyz + azuxy + u^{'}xyza^{'} + zuxya^{'} + au_0xyz + u_0xyza^{'} + azuxy_0 + zuxy_0a^{'}$
$= a(u_0 + u^{'})xyz + (u_0 + u^{'})xyza^{'} + azux(y_0 + y) + zux(y_0 + y)a^{'}$
$= au^{'}xyz + u^{'}xyza^{'} + azuxy + zuxya^{'} = a(zuxy + uxyz^{'}) + (zuxy + uxyz^{'})a^{'}$
$= a\,[z, uxy] + [z, uxy]\,a^{'} = [a, [z, uxy]] = [z, uxy]^d$.

The following Jordan identities hold:

Proposition 4.6. Let R be an MA-semiring, $a \in R$. Let d be an inner derivation, that is, $d(x) = [a, x]$ for all $x \in R$. Then for all $x, y, z \in R$,the Jordan identities hold: (i) $[x \circ y, z]^d + [y \circ z, x]^d = [y, z \circ x]^d$

(ii)$(x \circ [y, z])^d + ([x, z] \circ y)^d = [x \circ y, z]^d$.

Proof. (i). By Definition 4.2, Definition 3.1 and Lemma 3.4, we have

$[x \circ y, z]^d + [y \circ z, x]^d = [y, z \circ x]^d = [a, [x \circ y, z]] + [a, [y \circ z, x]]$
$= ayzx + ayxz + az^{'}xy + ax^{'}zy + yzxa^{'} + yxza^{'} + z^{'}xya^{'} + x^{'}zya^{'} + ax_0yz + az_0yx + x_0yza^{'} + z_0yxa^{'} = ayzx + ayxz + az^{'}xy + ax^{'}zy + yzxa^{'} + yxza^{'} + z^{'}xya^{'} + x^{'}zya^{'} + ayzx_0 + ayxz_0 + yzx_0a^{'} + yxz_0a^{'} = ayz(x_0 + x) + ayx(z_0 +$

$z) + az'xy + ax'zy + yz(x_0 + x)a' + yx(z_0 + z)a' + z'xya' + x'zya' = ayzx + ayxz + az'xy + ax'zy + yzxa' + yxza' + z'xya' + x'zya'$
$= a(y(zx + xz) + (zx + xz)y') + (y(zx + xz) + (zx + xz)y')a'$
$= a(y(z \circ x) + (z \circ x)y') + (y(z \circ x) + (z \circ x)y')a'$
$= a\,[y, z \circ x] + [y, z \circ x]\,a' = [a, [y, z \circ x]] = [y, z \circ x]^d$.

(ii). By Definition 4.2, Definition 3.1 and Lemma 3.4
$(x \circ [y, z])^d + ([x, z] \circ y)^d = (x \circ (yz + zy'))^d + ((xz + zx') \circ y)^d$
$= (x(yz + zy') + (yz + zy')x)^d + ((xz + zx')y + y(xz + zx'))^d$
$= axyz + azx'y + azy'x + xyza' + ayxz + zx'ya' + yxza' + zy'xa' + ayzx_0 + xzy_0a' + yzx_0a' + axzy_0$
$= axyz + azx'y + azy'x + xyza' + ayxz + zx'ya' + yxza' + zy'xa' + ax_0yz + y_0xza' + x_0yza' + ay_0xz$
$= azx'y + azy'x + zx'ya' + zy'xa' + a(x_0 + x)yz + (x_0 + x)yza' + a(y_0 + y)xz + (y_0 + y)xza'$
$= azx'y + azy'x + zx'ya' + zy'xa' + axyz + xyza' + ayxz + yxza'$
$= a(xy + yx)z + az'(xy + yx) + (xy + yx)za' + z'(xy + yx)a'$
$= \left[a, (xy + yx)z + z'(xy + yx)\right] = \left[a, (x \circ y)z + z'(x \circ y)\right] = [a, [x \circ y, z]] = [x \circ y, z]^d$.

Definition 4.7. Let R be an MA-semiring. Then a mapping $B : R \times R \to R$ is said to be symmetric, if $B(x, y) = B(y, x)$ for all $x, y \in R$. A mapping $f : R \to R$ defined by $f(x) = B(x, x)$ is called the trace of B.

Definition 4.8. Let R be an MA-semiring and d be a derivation of R in to itself. Then $B_d : R \times R \to R$ is defined by $B_d(x, y) = [d(x), y] + [d(y), x]$.

The following proposition shows that the mapping B_d is symmetric.

Proposition 4.9. If R is an MA-semiring, then the following statements hold:
(a) If $d : R \to R$ be a derivation, then B_d is symmetric.
(b) If f is the trace of B_d, then $f(x + y) = f(x) + f(y) + 2B_d(x, y)$.

Proof: (a) By Definition 4.7, we have $B_d(x, y) = [d(x), y] + [d(y), x] = [d(y), x] + [d(x), y] = B_d(y, x)$ for all $x, y \in R$. This implies that B_d is symmetric.

(b): As $d : R \to R$ be a derivation, therefore by Definition 4.8, we have
$f(x + y) = B_d(x + y, x + y) = [d(x + y), x + y] + [d(x + y), x + y] = [d(x) + d(y), x + y] + [d(x) + d(y), x + y] = [d(x), x] + [d(x), x] + [d(y), y] + [d(y), y] + 2([d(x), y] + [d(y), y]) = B_d(x, x) + B_d(y, y) + 2B_d(x, y)$.
This implies that $f(x + y) = f(x) + f(y) + 2B_d(x, y)$. As an application of Theorem 3.2 we can prove Proposition 4.10 and proposition 4.11.

Proposition 4.10. Let R be an MA-semiring and $d : R \to R$ be a derivation. Then the following identity holds:

$B_d(x, z)y + xB_d(y, z) = [z, d(x)]y^{'} + x^{'}[z, d(y)] + [d(z), xy]$.

Proof. By Definition 4.8, Theorem 3.2, we have
$B_d(x, z)y+xB_d(y, z) = ([d(x), z]+[d(z), x])y+x([d(y), z]+[d(z), y]) = [d(x), z]y+ [d(z), x]y+x[d(y), z]+x[d(z), y] = [d(x), z^{'}]y^{'}+x[d(y), z]+[d(z), x]y+x[d(z), y] = [d(x), z^{'}]y^{'} + x^{'}[d(y), z]^{'} + [d(z), xy] = [z, d(x)]y^{'} + x^{'}[z, d(y)] + [d(z), xy]$.

Theorem 4.11. Let R be an MA-semiring and $d : R \to R$ be a derivation. Then $B_d(xy, z) = B_d(x, z)y + xB_d(y, z) + d(x)[y, z] + [x, z]d(y)$.

Proof. By Definition 4.8, Theorem 3.2, we have
$B_d(xy, z) = [d(xy), z]+[d(z), xy] = [d(x)y+xd(y), z]+[d(z), xy] = [d(x)y, z]+ [xd(y), z] + [d(z), xy] = [z, d(x)y^{'}] + [z, x^{'}d(y)] + [d(z), xy] = [z, d(x)]y^{'} + d(x)[z, y^{'}] + [z, x^{'}]d(y) + x^{'}[z, d(y)] + [d(z), xy]$
$= [z, d(x)]y^{'} + x^{'}[z, d(y)] + [d(z), xy] + d(x)[y, z] + [x, z]d(y) = B_d(x, z)y + xB_d(y, z) + d(x)[y, z] + [x, z]d(y)$.

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