Noname manuscript No.
(will be inserted by the editor)

On Cones of Nonnegative Quartic Forms

Bo Jiang · Zhening Li · Shuzhong Zhang

Received: date / Accepted: date

Abstract Historically, much of the theory and practice in nonlinear optimization has revolved around the quadratic models. Though quadratic functions are nonlinear polynomials, they are well structured and many of them are found easy to deal with. Limitations of the quadratics, however, become increasingly binding as higher degree nonlinearity is imperative in modern applications of optimization. In recent years, one observes a surge of research activities in polynomial optimization, and modeling with quartic or higher degree polynomial functions has been more commonly accepted. On the theoretical side, there are also major recent progresses on polynomial functions and optimization. For instance, Ahmadi et al. [2] proved that checking the convexity of a quartic polynomial is strongly NP-hard in general, which settles a long-standing open question. In this paper we proceed to study six fundamentally important convex cones of quartic forms in the space of super-symmetric tensors, including the cone of nonnegative quartic forms, the sums of squared forms, the convex quartic forms, and the sums of fourth-power forms. It turns out that these convex cones coagulate into a chain in a decreasing order with varying complexity status. Potential applications of these results to solve highly nonlinear and/or combinatorial optimization problems are discussed.

Keywords Cone of polynomial functions · Super-symmetric tensors · Nonnegative quartic forms · Sums of squares · SOS-convexity · Polynomial optimization

Mathematics Subject Classification (2000) 15A69 · 08A40 · 90C25 · 49N15

Communicated by Michael Jeremy Todd

B. Jiang
Research Center for Management Science and Data Analytics, School of Information Management and Engineering, Shanghai University of Finance and Economics, Shanghai 200433, China. E-mail: isyebojiang@gmail.com

Z. Li
Department of Mathematics, University of Portsmouth, Portsmouth PO1 3HF, United Kingdom. E-mail: zheningli@gmail.com

S. Zhang
Department of Industrial and Systems Engineering, University of Minnesota, Minneapolis, MN 55455, USA. E-mail: zhangs@umn.edu

1 Introduction

Checking the convexity of a quadratic function boils down to test the positive semidefiniteness of its Hessian matrix in the domain. Since the Hessian matrix is constant, the test can be done easily. A natural question thus arises:

> Given a fourth degree polynomial function in n variables, can one still easily tell if the function is convex or not?

This simple-looking question was first put forward by Shor [39] in 1992, which turned out later to be a very challenging question to answer. For almost two decades, the question remained open. Only until recently Ahmadi et al. [2] proved that checking the convexity of a general quartic polynomial function is actually strongly NP-hard. The result not only settled this particular open problem, but also helped to highlight a crucial difference between quartic and quadratic polynomials, which makes the study of quartic polynomials all the more compelling and interesting.

On the practical side, quartic polynomial optimization has a wide spectrum of applications, including sensor network localization [6], MIMO radar waveform optimization [12], portfolio management with high moments information [27], quantum entanglement [16], among many others. This has stimulated a burst of recent research activities with regard to quartic polynomial optimization. Due to the NP-hardness of quartic polynomial optimization models (see e.g. [36, 19, 31]), there is a considerable amount of recent research work devoted to approximation algorithms for solving various quartic polynomial optimization models. Luo and Zhang [36] proposed an approximation algorithm for optimization of a quartic polynomial with quadratic constraints. Ling et al. [34] considered a special quartic optimization model, which is to minimize a bi-quadratic function over two spheres. He et al. [19, 20] extended the study to arbitrary degree polynomials. So [49] improved some of the approximation bounds presented in [19]. For a comprehensive survey on the topic, one may refer to the monograph of Li et al. [31]. Another well studied approach to cope with general polynomial optimization problems is the so-called SOS method proposed by Lasserre [28] and Parrilo [40]. Theoretically, it can solve any general polynomial optimization model to any given accuracy through resorting to a sequence of semidefinite programs (SDP). However, the size of those SDP problems may grow large very quickly. Interested readers may find more information in the survey paper [29] and the references therein.

There is an intrinsic connection between optimizing a polynomial function and the description of *all* polynomial functions that are nonnegative over a given domain. For the case of quadratic polynomials, this connection was explored by Sturm and Zhang [50], and later for the bi-quadratic case by Luo et al. [35]. Such investigations can be traced back to the 19th century when the relationship between nonnegative polynomials and the sums of squares (SOS) of polynomials was explicitly studied. One concrete question of interest was: *Given a multivariate polynomial function that takes only nonnegative values over the real numbers, can it be represented as a sum of squares of polynomial functions?* Hilbert [22] in 1888 showed that the only three general classes of polynomial functions for which this is true can be explicitly identified: (1) univariate polynomials; (2) multivariate quadratic polynomials; (3) bivariate quartic polynomials. Since polynomial functions with a fixed degree form a vector space, and the *nonnegative* polynomials and the *SOS* polynomials form two convex cones respectively within that vector space, the afore-mentioned results can be understood as a specification of three particular cases where these two convex cones coincide, while in general of course the cone of nonnegative polynomials is larger. There are certainly other interesting convex cones in the same vector space. For instance, the convex polynomial functions form yet another convex cone in that vector space. Helton and Nie [21] introduced the notion of sos-convex polynomials, to indicate the polynomials whose Hessian matrix can be decomposed as a sum of squares of polynomial matrices. All these classes of convex cones are important in their own rights. They are also important for the sake of optimization of polynomial functions. There have been substantial recent progresses along this direction. As we mentioned earlier, e.g. the question of Shor [39] regarding the complexity of deciding the convexity of a quartic polynomial was nicely settled by Ahmadi et al. [2]. It is also natural to inquire if the Hessian matrix of a convex polynomial is SOS. Ahmadi and Parrilo [3] gave an example to show that this is not the case in general. Blekherman proved that a nonnegative convex polynomial is not necessary a sum of squares [7] if the degree of the polynomial is larger than two. However, Blekherman's proof is not constructive, and it remains an open problem to construct a concrete example of convex polynomial which is not a sum of squares. Reznick [45] studied the sum of even powers of linear forms, the sum of squares of forms, and the positive semidefinite forms.

Compared to the quadratic case (cf. Sturm and Zhang [50]), the structure of the quartic forms is far from being clear, and many solvable quadratic problems become NP-hard when the scope of polynomials goes beyond quadratics. We believe that the class of quartic polynomial functions (or the class of fourth order tensors) is an appropriate subject of study on its own right, beyond quadratic functions (or matrices). There are at least three immediate reasons to elaborate on the quartic polynomials, rather than polynomial functions of other (or general) degrees. First of all, nonnegativity is naturally associated with even degree polynomials, and the quartic polynomial is next to quadratic polynomial in that hierarchy. Second, quartic polynomials represent a landscape *after* the 'phase transition' takes place. Moreover, dealing with quartic polynomials is still manageable, as far as notations are concerned. Finally, from an application point of view, quartic polynomial optimization is by far the most relevant polynomial optimization model beyond quadratic polynomials. The afore-mentioned examples such as kurtosis risks in portfolio management [27], the bi-quadratic optimization models [34], and the nonlinear least square formulation of sensor network localization [6] are all such examples.

In view of the cones formed by the quartic polynomials (e.g. the cones of nonnegative quartic forms, the convex quartic forms, the SOS forms and the sos-convex forms), it is natural to inquire about their relational structures, complexity status and the description of their interiors. We aim to conduct a systematic study on these topics in this paper, to bring together many of the known results in the context of our new findings, and to present them in a self-contained manner. For some historical reasons, results in that direction are usually presented in the framework of polynomials; however, in this paper the chosen space is super-symmetric tensors, which is more versatile and also naturally connects to linear algebra. In this paper, due to the one-to-one correspondence between super-symmetric tensors and homogenous polynomials, we provide various characterizations of several important convex cones in the fourth order super-symmetric tensor space, present their relational structures and complexity status. Therefore, our results can be helpful in tensor optimization (see [13,51] for recent development in sparse or low rank tensor optimization) as well. We also motivate the study by some examples from applications. The contributions of this paper are summarized in Section 2.3.

2 Preliminaries

2.1 Notations

Throughout this paper, we use the lower-case letters to denote vectors (e.g. $x \in \mathbf{R}^n$), the capital letters to denote matrices (e.g. $A \in \mathbf{R}^{n^2}$), and the capital calligraphy letters to denote fourth order tensors (e.g. $\mathcal{F} \in \mathbf{R}^{n^4}$), with subscripts of indices being their entries (e.g. $x_1, A_{ij}, \mathcal{F}_{ijk\ell} \in \mathbf{R}$). The boldface capital letters are reserved for sets in the Euclidean space, e.g. various sets of quatric forms to be introduced later, as well as $\mathbf{R}^{n^4}$, the space of n-dimensional fourth order tensors.

A generic quartic *form* is a fourth degree homogeneous polynomial function in n variables, or specifically the function

$$f(x) = \sum_{1 \le i \le j \le k \le \ell \le n} \mathcal{G}_{ijk\ell}\, x_i x_j x_k x_\ell, \tag{1}$$

where $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$. Closely related to a quartic form is a fourth order super-symmetric tensor $\mathcal{F} \in \mathbf{R}^{n^4}$. A tensor is said to be *super-symmetric* if its entries are invariant under all permutations of its indices. The set of n-dimensional super-symmetric fourth order tensors is denoted by $\mathbf{S}^{n^4}$. In fact, super-symmetric tensors are bijectively related to forms. In particular, restricting to fourth order tensors, for a given super-symmetric tensor $\mathcal{F} \in \mathbf{S}^{n^4}$, the quartic form in (1) can be uniquely determined by the following operation:

$$f(x) = \mathcal{F}(x, x, x, x) := \sum_{1 \le i,j,k,\ell \le n} \mathcal{F}_{ijk\ell}\, x_i x_j x_k x_\ell, \tag{2}$$

where $x \in \mathbf{R}^n$, $\mathcal{F}_{ijk\ell} = \mathcal{G}_{ijk\ell}/|\Pi(ijk\ell)|$ and $\Pi(ijk\ell)$ is the set of all distinctive permutations of the indices $\{i, j, k, \ell\}$, and vice versa. This is the same as the one-to-one correspondence between symmetric matrices and quadratic forms. In the remainder of this paper, we shall frequently use a super-symmetric tensor $\mathcal{F} \in \mathbf{S}^{n^4}$ to indicate a quartic form $f(x)$ or $\mathcal{F}(x, x, x, x)$, i.e., the notion of "super-symmetric fourth order tensor" and "quartic form" are used interchangeably in this paper.

Given a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ and a matrix $X \in \mathbf{R}^{n^2}$, we may also define the following operation (in the same spirit as (2)):

$$\mathcal{F}(X, X) := \sum_{1 \le i,j,k,\ell \le n} \mathcal{F}_{ijk\ell}\, X_{ij} X_{k\ell}.$$

We call a fourth order tensor $\mathcal{G} \in \mathbf{R}^{n^4}$ *partial-symmetric*, if

$$\mathcal{G}_{ijk\ell} = \mathcal{G}_{jik\ell} = \mathcal{G}_{ij\ell k} = \mathcal{G}_{k\ell ij} \quad \forall\, 1 \le i, j, k, \ell \le n.$$

Essentially this means that the tensor is symmetric for the first and the last two indices respectively, and is also symmetric by swapping the first two and the last two indices. The set of all partial-symmetric fourth order tensors in $\mathbf{R}^{n^4}$ is denoted by $\overrightarrow{\mathbf{S}}^{n^4}$. Obviously $\mathbf{S}^{n^4} \subsetneq \overrightarrow{\mathbf{S}}^{n^4} \subsetneq \mathbf{R}^{n^4}$ if $n \geq 2$.

For any fourth order tensor $\mathcal{G} \in \mathbf{R}^{n^4}$, we introduce a *symmetrization* mapping $\mathrm{sym}\colon \mathbf{R}^{n^4} \mapsto \mathbf{S}^{n^4}$, which is $\mathcal{F} = \mathrm{sym}\,\mathcal{G}$ with

$$\mathcal{F}_{ijk\ell} = \frac{1}{|\Pi(ijk\ell)|} \sum_{\pi \in \Pi(ijk\ell)} \mathcal{G}_\pi \quad \forall\, 1 \leq i,j,k,\ell \leq n,$$

which is the average of all the entries within the same set of indices. Note that this is different from the tensor symmetrization mapping proposed in [43], where the symmetry is realized by carefully imbedding the original tensor and its 'transposes' into a tensor in a larger dimension.

For any given set S, $\mathrm{Int}\,(S)$ denotes the interior of S. The symbol '$\otimes$' represents the outer product of vectors or matrices. If $\mathcal{F} = x \otimes x \otimes x \otimes x$ for some $x \in \mathbf{R}^n$, then $\mathcal{F}_{ijk\ell} = x_i x_j x_k x_\ell$; and if $\mathcal{G} = X \otimes X$ for some $X \in \mathbf{R}^{n^2}$, then $\mathcal{G}_{ijk\ell} = X_{ij} X_{k\ell}$. The symbol '$\bullet$' denotes the operation of inner product. As a result, we have $\mathcal{F}(x,x,x,x) = \mathcal{F} \bullet (x \otimes x \otimes x \otimes x)$.

2.2 Introducing the Quartic Forms

In this subsection we shall formally introduce the quartic forms in the super-symmetric fourth order tensor space. Let us start with the well known notion of positive semidefinite (PSD) and the sum of squares (SOS) of polynomials.

Definition 2.1 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called PSD if*

$$\mathcal{F}(x,x,x,x) \geq 0 \quad \forall\, x \in \mathbf{R}^n. \tag{3}$$

The set of all PSD forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{S}^{n^4}_+$.

If a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ can be written as a sum of squares of polynomial functions, then these polynomials must be quadratic forms, i.e.,

$$\mathcal{F}(x,x,x,x) = \sum_{i=1}^m \left(x^{\mathrm{T}} A^i x\right)^2 = (x \otimes x \otimes x \otimes x) \bullet \sum_{i=1}^m A^i \otimes A^i,$$

where $A^i \in \mathbf{S}^{n^2}$, the set of symmetric matrices. However, $\sum_{i=1}^m \left(A^i \otimes A^i\right) \in \overrightarrow{\mathbf{S}}^{n^4}$ is only partial-symmetric, and may not be exactly $\mathcal{F}$, which must be super-symmetric. To place it in the family $\mathbf{S}^{n^4}$, a symmetrization operation is required. Since $x \otimes x \otimes x \otimes x$ is super-symmetric, we still have

$$(x \otimes x \otimes x \otimes x) \bullet \mathrm{sym} \left(\sum_{i=1}^m A^i \otimes A^i\right) = (x \otimes x \otimes x \otimes x) \bullet \sum_{i=1}^m A^i \otimes A^i = \mathcal{F}(x,x,x,x).$$

Definition 2.2 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called SOS if $\mathcal{F}(x,x,x,x)$ is a sum of squares of quadratic forms, i.e., there exist m symmetric matrices $A^1,\dots,A^m \in \mathbf{S}^{n^2}$ such that*

$$\mathcal{F} = \mathrm{sym}\left(\sum_{i=1}^m A^i \otimes A^i\right) = \sum_{i=1}^m \mathrm{sym}\left(A^i \otimes A^i\right).$$

The set of SOS forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{\Sigma}^2_{n,4}$.

As all SOS forms constitute a convex cone, we have

$$\mathbf{\Sigma}^2_{n,4} = \mathrm{sym}\,\mathrm{cone}\left\{A \otimes A \mid A \in \mathbf{S}^{n^2}\right\}.$$

In general, for a given $\mathcal{F} = \mathrm{sym}\left(\sum_{i=1}^m A^i \otimes A^i\right)$ it may be a challenge to write it explicitly as a sum of squares, although the construction can be done in principle through SDP, which however may be costly. In this sense, having an SOS form expressed as a super-symmetric tensor may not always be beneficial, since the super-symmetry can hide the SOS structure. It is possible that $\mathcal{F} \in \mathbf{\Sigma}^2_{n,4}$ cannot be written in the form of $\sum_{i=1}^m A^i \otimes A^i$ for any A^i's (without the symmetrization mapping). For instance, consider an SOS quartic form in two variables: $6{x_1}^2{x_2}^2$. After symmetrization $\mathcal{F}_{ijk\ell} = 1$ for $(ijk\ell) = (1122), (1212), (1221), (2112), (2121), (2211)$ and 0 for all other entries, and the corresponding matrix in the variable $X = (X_{11}, X_{12}, X_{21}, X_{22})$ is given by $\begin{bmatrix} 0&0&0&1\\ 0&1&1&0\\ 0&1&1&0\\ 1&0&0&0 \end{bmatrix}$, which obviously is not SOS in terms of the matrix variable X.

The difference in fact leads to the next definition of nonnegativity. Since $\mathcal{F}(X,X)$ is a quadratic form, the nonnegativity for quadratic functions carries over. Formally we introduce this notion below.

Definition 2.3 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called matrix PSD if*

$$\mathcal{F}(X,X) \ge 0 \quad \forall\, X \in \mathbf{R}^{n^2}.$$

The set of matrix PSD forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{S}_+^{n^2 \times n^2}$.

We remark that the matrix PSD forms is essentially equivalent to the cone of PSD moment matrices; see e.g. [29]. However, our definition here is more straightforward.

Related to the sum of squares for quartic forms, we now introduce the notion of *sum of powers of linear forms* (SOP): If a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is SOP, then there are m vectors $a^1,\dots,a^m \in \mathbf{R}^n$ such that

$$\mathcal{F}(x,x,x,x) = \sum_{i=1}^m \left(x^{\mathrm{T}} a^i\right)^4 = (x \otimes x \otimes x \otimes x) \bullet \sum_{i=1}^m a^i \otimes a^i \otimes a^i \otimes a^i.$$

Definition 2.4 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called SOP if $\mathcal{F}(x,x,x,x)$ is a sum of powers of linear forms, i.e., there exist m vectors $a^1,\dots,a^m \in \mathbf{R}^n$ such that*

$$\mathcal{F} = \sum_{i=1}^m a^i \otimes a^i \otimes a^i \otimes a^i.$$

The set of SOP forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{\Sigma}^4_{n,4}$.

As all SOP forms also constitute a convex cone, we denote

$$\mathbf{\Sigma}^4_{n,4} = \text{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a\in \mathbf{R}^n\} \subseteq \mathbf{\Sigma}^2_{n,4}.$$

In the case of quadratic functions, it is well known that for a given homogeneous form (i.e., a symmetric matrix, for that matter) $A\in\mathbf{S}^{n^2}$ the following two statements are equivalent:

- A is positive semidefinite (PSD): $A(x,x) := x^{\mathrm{T}}Ax \ge 0$ for all $x\in\mathbf{R}^n$.
- A is a sum of squares (SOS): $A(x,x) = \sum_{i=1}^m (x^{\mathrm{T}}a^i)^2$ (or equivalently $A = \sum_{i=1}^m a^i\otimes a^i$) for some $a^1,\ldots,a^m\in\mathbf{R}^n$.

It is therefore clear that the four types of quartic forms defined above are actually different extensions of the nonnegativity. In particular, PSD forms and matrix PSD forms are extended from quadratic PSD, while SOS and SOP forms are in the form of summation of nonnegative polynomials, and are extended from quadratic SOS. We will present later that there is an interesting hierarchical relationship for general n:

$$\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}_+^{n^2\times n^2} \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}_+^{n^4}. \tag{4}$$

Recently, a class of polynomials termed the *sos-convex polynomials* (cf. Helton and Nie [21]) has been brought to attention, which is defined as follows (see [4] for two other equivalent formulations of sos-convexity):

> A multivariate polynomial function $f(x)$ is sos-convex if its Hessian matrix $H(x)$ can be factorized as $H(x) = (M(x))^{\mathrm{T}}M(x)$ with a polynomial matrix $M(x)$.

The reader is referred to [3] for applications of the sos-convex polynomials. In this paper, we shall focus on $\mathbf{S}^{n^4}$ and investigate sos-convex quartic forms with the hierarchy (4). For a quartic form $\mathcal{F}\in\mathbf{S}^{n^4}$, it is straightforward to compute its Hessian matrix $H(x) = 12\mathcal{F}(x,x,\cdot,\cdot)$, i.e.,

$$(H(x))_{ij} = 12\mathcal{F}(x,x,e^i,e^j) \quad \forall\, 1\le i,j\le n,$$

where $e^i\in\mathbf{R}^n$ is the vector whose i-th entry is 1 and other entries are zeros. Therefore $H(x)$ is a quadratic matrix of x. If $H(x)$ can be decomposed as $H(x) = (M(x))^{\mathrm{T}}M(x)$ with $M(x)$ being a polynomial matrix, then $M(x)$ must be linear with respect to x.

Definition 2.5 *A quartic form $\mathcal{F}\in\mathbf{S}^{n^4}$ is called sos-convex, if there exists a linear matrix $M(x)$ of x, such that its Hessian matrix*

$$12\mathcal{F}(x,x,\cdot,\cdot) = (M(x))^{\mathrm{T}}M(x).$$

The set of sos-convex forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$.

Helton and Nie [21] proved that if a nonnegative polynomial is sos-convex, then it must be SOS. In particular, if the polynomial is a quartic form, by denoting the i-th row of the linear matrix $M(x)$ to be $x^{\mathrm{T}}A^i$ for $i=1,\ldots,m$ and some matrices $A^1,\ldots,A^m\in\mathbf{R}^{n^2}$, then $(M(x))^{\mathrm{T}}M(x) = \sum_{i=1}^m (A^i)^{\mathrm{T}}xx^{\mathrm{T}}A^i$. Therefore

$$\mathcal{F}(x,x,x,x) = x^{\mathrm{T}}\mathcal{F}(x,x,\cdot,\cdot)x = \frac{1}{12}x^{\mathrm{T}}(M(x))^{\mathrm{T}}M(x)x = \frac{1}{12}\sum_{i=1}^m \left(x^{\mathrm{T}}A^ix\right)^2 \in \mathbf{\Sigma}^2_{n,4}.$$

In addition, the Hessian matrix for an sos-convex form is obviously positive semidefinite for any $x \in \mathbf{R}^n$. Hence sos-convexity implies convexity. Combining these two facts, we conclude that an sos-convex form is both SOS and convex, which motivates us to study the last quartic forms in this paper.

Definition 2.6 *A quartic form $\mathcal{F} \in \mathbf{S}^{n^4}$ is called convex and SOS, if it is both SOS and convex. The set of quartic convex and SOS forms in $\mathbf{S}^{n^4}$ is denoted by $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\text{cvx}}$.*

Here $\mathbf{S}^{n^4}_{\text{cvx}}$ is denoted to be the set of all convex quartic forms in $\mathbf{S}^{n^4}$.

2.3 Our Contributions and the Organization of the Paper

All sets of quartic forms defined in Section 2.2 are clearly convex cones. The remainder of this paper is organized as follows. In Section 3, we start by studying the cones: $\mathbf{S}^{n^4}_+$, $\mathbf{\Sigma}^2_{n,4}$, $\mathbf{S}^{n^2 \times n^2}_+$, and $\mathbf{\Sigma}^4_{n,4}$. We first show that they are all closed, and that they can be presented in different formulations. As an example, the cone of SOP forms is

$$\mathbf{\Sigma}^4_{n,4} = \text{cone}\,\{a \otimes a \otimes a \otimes a \,|\, a \in \mathbf{R}^n\} = \text{sym cone}\,\left\{A \otimes A \,|\, A \in \mathbf{S}^{n^2}_+,\, \text{rank}(A) = 1\right\},$$

which can also be written as

$$\text{sym cone}\,\left\{A \otimes A \,|\, A \in \mathbf{S}^{n^2}_+\right\},$$

meaning that the rank-one constraint can be removed without affecting the cone itself. We know that among these four cones there are two primal-dual pairs: $\mathbf{S}^{n^4}_+$ is dual to $\mathbf{\Sigma}^4_{n,4}$, and $\mathbf{\Sigma}^2_{n,4}$ is dual to $\mathbf{S}^{n^2 \times n^2}_+$, and a hierarchical relationship $\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}^{n^2 \times n^2}_+ \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}^{n^4}_+$ exists. Although all these results can be found in [45, 29] thanks to various representations of quartic forms, it is beneficial to present them in a unified manner in the super-symmetric tensor space. Moreover, the tensor representation of quartic forms is interesting on its own. For instance, it sheds some light on how symmetric property changes the nature of quartic cones. To see this, let us consider an SOS quartic form $\sum_{i=1}^m (x^{\mathrm{T}} A^i x)^2$, which will become matrix PSD if $\sum_{i=1}^m A^i \otimes A^i$ is already a super-symmetric tensor (Theorem 3.3). If we further assume $m = 1$, then we have $\text{rank}(A^1) = 1$ (Theorem 2.4 in [26]) meaning that $A^1 \otimes A^1 = a \otimes a \otimes a \otimes a$ for some a, is SOP. Besides, explicit examples are also very important for people to get some concrete feelings about quartic forms. It is worth mentioning that the main work of Ahmadi and Parrilo [3] is to provide a polynomial which is convex but not sos-convex. Here we present an explicit instance of quartic form, which is matrix PSD but not SOP; see Example 3.2.

In Section 4, we further study two more cones: $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ and $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\text{cvx}}$. Interestingly, these two new cones can be nicely placed in the hierarchical scheme (4) for general n:

$$\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}^{n^2 \times n^2}_+ \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subsetneq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\text{cvx}}\right) \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}^{n^4}_+. \tag{5}$$

The complexity status of all these cones are summarized in Section 5, including some well known results in the literature, and our new finding is that testing the convexity is still NP-hard even for SOS quartic forms (Theorem 5.4). The low dimensional cases of these cones are also discussed in Section 5. Specially, for the case $n = 2$, all the six cones reduce to only two distinctive ones, and for the case $n = 3$, they reduce to exactly three distinctive cones. In addition, we study two particular simple quartic forms: $\left(x^{\mathrm{T}}x\right)^2$ and $\sum_{i=1}^n {x_i}^4$. Since they both belong to $\mathbf{\Sigma}^4_{n,4}$, which is the smallest cone in our hierarchy, one may ask whether or not they belong to the interior of $\mathbf{\Sigma}^4_{n,4}$. It may appear plausible that $\sum_{i=1}^n {x_i}^4$ is in the interior of $\mathbf{\Sigma}^4_{n,4}$, since it is the quartic extension of the quadratic form $\sum_{i=1}^n {x_i}^2$. However, it can be shown that $\sum_{i=1}^n {x_i}^4$ is not in $\mathrm{Int}\,(\mathbf{S}^{n^4}_{\mathrm{cvx}}) \supsetneq \mathrm{Int}\,(\mathbf{\Sigma}^4_{n,4})$ but in $\mathrm{Int}\,(\mathbf{\Sigma}^2_{n,4})$ (Theorem 5.8), and $\left(x^{\mathrm{T}}x\right)^2$ is actually in $\mathrm{Int}\,(\mathbf{\Sigma}^4_{n,4})$ (Theorem 5.9), implying that $\left(x^{\mathrm{T}}x\right)^2$ is more 'positive' than $\sum_{i=1}^n {x_i}^4$.

Finally, in Section 6 we discuss applications of quartic conic programming, including bi-quadratic assignment problems and eigenvalues of super-symmetric tensors.

3 PSD Forms, SOS Forms, and the Dual Cones

Let us now consider the first four cones of quartic forms introduced in Section 2.2: $\mathbf{\Sigma}^4_{n,4}$, $\mathbf{S}^{n^2\times n^2}_{+}$, $\mathbf{\Sigma}^2_{n,4}$, and $\mathbf{S}^{n^4}_{+}$.

3.1 Closedness

Proposition 3.1 *$\mathbf{\Sigma}^4_{n,4}$, $\mathbf{S}^{n^2\times n^2}_{+}$, $\mathbf{\Sigma}^2_{n,4}$, and $\mathbf{S}^{n^4}_{+}$ are all closed convex cones.*

While $\mathbf{S}^{n^4}_{+}$ and $\mathbf{S}^{n^2\times n^2}_{+}$ are evidently closed, by a similar argument as in [50] it is also easy to see that the cone of SOS forms $\mathbf{\Sigma}^2_{n,4} := \mathrm{sym\,cone}\left\{A\otimes A \,|\, A \in \mathbf{S}^{n^2}\right\}$ is closed. The closedness of $\mathbf{S}^{n^2\times n^2}_{+}$, $\mathbf{\Sigma}^2_{n,4}$ and $\mathbf{S}^{n^4}_{+}$ were also known in polynomial optimization, e.g. [29]. The closedness of the cone of SOP forms $\mathbf{\Sigma}^4_{n,4}$ was proved in Proposition 3.6 of [45] for general even degree forms. In fact, we have a slightly stronger result below:

Lemma 3.2 *If $\mathbf{D} \subseteq \mathbf{R}^n$ is closed, then* $\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a\in\mathbf{D}\}$ *is closed.*

Proof. Suppose that $\mathcal{F} \in \mathrm{cl\,cone}\,\{a\otimes a\otimes a\otimes a \,|\, a\in\mathbf{D}\}$, then there is a sequence of quartic forms $\mathcal{F}^k \in \mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a\in\mathbf{D}\}$ $(k = 1, 2, \dots)$, such that $\mathcal{F} = \lim_{k\to\infty}\mathcal{F}^k$. Since the dimension of $\mathbf{S}^{n^4}$ is $\binom{n+3}{4}$, it follows from Carathéodory's theorem that for any given $\mathcal{F}^k$, there exists an $n \times \left(\binom{n+3}{4}+1\right)$ matrix Z^k, such that

$$\mathcal{F}^k = \sum_{i=1}^{\binom{n+3}{4}+1} z^k(i)\otimes z^k(i)\otimes z^k(i)\otimes z^k(i),$$

where $z^k(i)$ is the i-th column vector of Z^k, and is a positive multiple of a vector in $\mathbf{D}$. Now define $\operatorname{tr}\mathcal{F}^k = \sum_{j=1}^n \mathcal{F}^k_{jjjj}$, then

$$\sum_{i=1}^{\binom{n+3}{4}+1} \sum_{j=1}^{n} (Z^k_{ji})^4 = \operatorname{tr}\mathcal{F}^k \to \operatorname{tr}\mathcal{F}.$$

Thus, the sequence $\{Z^k\}$ is bounded, and have a cluster point Z^*, satisfying $\mathcal{F} = \sum_{i=1}^{\binom{n+3}{4}+1} z^*(i)\otimes z^*(i)\otimes z^*(i)\otimes z^*(i)$. Note that each column of Z^* is also a positive multiple of a vector in $\mathbf{D}$, it follows that $\mathcal{F} \in \operatorname{cone}\{a\otimes a\otimes a\otimes a \mid a \in \mathbf{D}\}$.
□

The cone of SOP forms is closely related to the fourth moment of a multi-dimensional random variable. Given an n-dimensional random variable $\xi = (\xi_1, \ldots, \xi_n)^{\mathrm{T}}$ on the support set $\mathbf{D} \subseteq \mathbf{R}^n$ with density function p, its fourth moment is a super-symmetric fourth order tensor $\mathcal{M} \in \mathbf{S}^{n^4}$, whose (i,j,k,ℓ)-th entry is

$$\mathcal{M}_{ijk\ell} = \mathsf{E}\left[\xi_i\xi_j\xi_k\xi_\ell\right] = \int_{\mathbf{D}} x_i x_j x_k x_\ell \, p(x) dx.$$

Suppose the fourth moment of ξ is finite. By the closedness of $\mathbf{\Sigma}^4_{n,4}$, we have

$$\mathcal{M} = \mathsf{E}\left[\xi\otimes\xi\otimes\xi\otimes\xi\right] = \int_{\mathbf{D}} (x\otimes x\otimes x\otimes x)\, p(x)dx \in \operatorname{cone}\{a\otimes a\otimes a\otimes a \mid a\in\mathbf{R}^n\} = \mathbf{\Sigma}^4_{n,4}.$$

Conversely, for any $\mathcal{M} \in \mathbf{\Sigma}^4_{n,4}$, there exist m vectors $a^1, a^2, \ldots, a^m \in \mathbf{R}^n$ such that $\mathcal{M} = \sum_{i=1}^m a^i\otimes a^i\otimes a^i\otimes a^i$. By defining an n-dimensional random vector ξ with $\operatorname{Prob}\left\{\xi = m^{1/4}a^i\right\} = 1/m$ for $i = 1, \ldots, m$, it is easy to verify that the fourth moment of ξ is exactly the tensor $\mathcal{M}$. Therefore, the set of all finite fourth moments of n-dimensional random variables is exactly $\mathbf{\Sigma}^4_{n,4}$, similar to the fact that all possible covariance matrices form the cone of positive semidefinite matrices.

3.2 Alternative Representations

In this subsection we present some alternative forms of the same cones that we have discussed. Some of these alternative representations are more convenient to use in various applications.

Theorem 3.3 *For the cones of quartic forms introduced, we have the following equivalent representations:*

1. *For the cone of SOS forms*

$$\begin{aligned}\mathbf{\Sigma}^2_{n,4} &:= \operatorname{sym}\operatorname{cone}\left\{A\otimes A \mid A\in\mathbf{S}^{n^2}\right\}\\ &= \operatorname{sym}\left\{\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4} \mid \mathcal{F}(X,X)\ge 0 \ \forall\, X\in\mathbf{S}^{n^2}\right\}\\ &= \operatorname{sym}\left\{\mathcal{F}\in\mathbf{R}^{n^4} \mid \mathcal{F}(X,X)\ge 0 \ \forall\, X\in\mathbf{S}^{n^2}\right\};\end{aligned}$$

2. For the cone of matrix PSD forms

$$\begin{aligned}\mathbf{S}_+^{n^2\times n^2} &:= \left\{\mathcal{F}\in\mathbf{S}^{n^4}\,|\,\mathcal{F}(X,X)\ge 0\ \forall\, X\in\mathbf{R}^{n^2}\right\}\\ &= \left\{\mathcal{F}\in\mathbf{S}^{n^4}\,|\,\mathcal{F}(X,X)\ge 0\ \forall\, X\in\mathbf{S}^{n^2}\right\}\\ &= \mathbf{S}^{n^4}\bigcap \mathrm{cone}\left\{A\otimes A\,|\,A\in\mathbf{S}^{n^2}\right\};\end{aligned}$$

3. For the cone of SOP forms

$$\mathbf{\Sigma}_{n,4}^4 := \mathrm{cone}\,\{a\otimes a\otimes a\otimes a\,|\,a\in\mathbf{R}^n\} = \mathrm{sym}\,\mathrm{cone}\left\{A\otimes A\,|\,A\in\mathbf{S}_+^{n^2}\right\}.$$

Recall that $\overrightarrow{\mathbf{S}}^{n^4}$ is the set of partial-symmetric fourth order tensors in $\mathbf{R}^{n^4}$, defined in Section 2.1. The remainder of this subsection is devoted to the proof of Theorem 3.3.

Let us first study the equivalent representations for $\mathbf{\Sigma}_{n,4}^2$ and $\mathbf{S}_+^{n^2\times n^2}$. To verify a matrix PSD form, we should check the operations of quartic forms on matrices. In fact, the matrix PSD forms can be extended to the space of partial-symmetric tensors $\overrightarrow{\mathbf{S}}^{n^4}$. It is not hard to verify that for any $\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4}$, it holds that

$$\mathcal{F}(X,Y)=\mathcal{F}(X^{\mathrm{T}},Y)=\mathcal{F}(X,Y^{\mathrm{T}})=\mathcal{F}(Y,X)\quad \forall\, X,Y\in\mathbf{R}^{n^2}, \tag{6}$$

which implies that $\mathcal{F}(X,Y)$ is invariant under the transpose operation as well as the operation to swap the X and Y matrices. Indeed, it is easy to see that the partial-symmetry of $\mathcal{F}$ is a necessary and sufficient condition for (6) to hold. We have the following property for matrix PSD forms in $\overrightarrow{\mathbf{S}}^{n^4}$, similar to that for $\mathbf{S}_+^{n^2\times n^2}$ in Theorem 3.3.

Lemma 3.4 *For partial-symmetric fourth order tensors, it holds that*

$$\begin{aligned}\overrightarrow{\mathbf{S}}_+^{n^2\times n^2} &:= \left\{\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4}\,|\,\mathcal{F}(X,X)\ge 0\ \forall\, X\in\mathbf{R}^{n^2}\right\}\\ &= \left\{\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4}\,|\,\mathcal{F}(X,X)\ge 0\ \forall\, X\in\mathbf{S}^{n^2}\right\} \qquad (7)\\ &= \mathrm{cone}\left\{A\otimes A\,|\,A\in\mathbf{S}^{n^2}\right\}. \qquad (8)\end{aligned}$$

Proof. Observe that for any skew-symmetric $Y\in\mathbf{R}^{n^2}$, i.e., $Y^{\mathrm{T}}=-Y$, we have

$$\mathcal{F}(X,Y)=-\mathcal{F}(X,-Y)=-\mathcal{F}(X,Y^{\mathrm{T}})=-\mathcal{F}(X,Y)\quad\forall\, X\in\mathbf{R}^{n^2},$$

which implies that $\mathcal{F}(X,Y)=0$. As any square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix, say for $Z\in\mathbf{R}^{n^2}$, by letting $X=(Z+Z^{\mathrm{T}})/2$ which is symmetric, and $Y=(Z-Z^{\mathrm{T}})/2$ which is skew-symmetric, we have $Z=X+Y$. Therefore,

$$\mathcal{F}(Z,Z)=\mathcal{F}(X+Y,X+Y)=\mathcal{F}(X,X)+2\mathcal{F}(X,Y)+\mathcal{F}(Y,Y)=\mathcal{F}(X,X).$$

This implies the equivalence between $\mathcal{F}(X,X)\ge 0\ \forall\, X\in\mathbf{R}^{n^2}$ and $\mathcal{F}(X,X)\ge 0\ \forall\, X\in\mathbf{S}^{n^2}$, which proves (7).

To prove (8), first note that

$$\operatorname{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}\right\} \subseteq \{\mathcal{F}\in \overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X)\ge 0\ \forall\, X\in \mathbf{R}^{n^2}\}.$$

Conversely, given any $\mathcal{G}\in \overrightarrow{\mathbf{S}}^{n^4}$ with $\mathcal{G}(X,X)\ge 0\ \forall\, X\in\mathbf{R}^{n^2}$, we may rewrite $\mathcal{G}$ as an $n^2\times n^2$ symmetric matrix $M_{\mathcal{G}}$. Therefore

$$(\operatorname{vec}(X))^{\mathrm{T}} M_{\mathcal{G}} \operatorname{vec}(X) = \mathcal{G}(X,X)\ge 0 \quad \forall\, X\in\mathbf{R}^{n^2},$$

which implies that $M_{\mathcal{G}}$ is positive semidefinite. Let $M_{\mathcal{G}}=\sum_{i=1}^m z^i(z^i)^{\mathrm{T}}$, where

$$z^i=\left(z^i_{11},\ldots,z^i_{1n},\ldots,z^i_{n1},\ldots,z^i_{nn}\right)^{\mathrm{T}} \quad \forall\, 1\le i\le m.$$

Note that for any $1\le k,\ell\le n$, $\mathcal{G}_{k\ell\ell k}=\sum_{i=1}^m z^i_{k\ell}z^i_{\ell k}$, $\mathcal{G}_{k\ell k\ell}=\sum_{i=1}^m (z^i_{k\ell})^2$ and $\mathcal{G}_{\ell k\ell k}=\sum_{i=1}^m (z^i_{\ell k})^2$, as well as $\mathcal{G}_{k\ell\ell k}=\mathcal{G}_{k\ell k\ell}=\mathcal{G}_{\ell k\ell k}$ by partial-symmetry of $\mathcal{G}$. We have

$$\sum_{i=1}^m (z^i_{k\ell}-z^i_{\ell k})^2=\sum_{i=1}^m (z^i_{k\ell})^2+\sum_{i=1}^m (z^i_{\ell k})^2-2\sum_{i=1}^m z^i_{k\ell}z^i_{\ell k}=\mathcal{G}_{k\ell k\ell}+\mathcal{G}_{\ell k\ell k}-2\mathcal{G}_{k\ell\ell k}=0,$$

which implies that $z^i_{k\ell}=z^i_{\ell k}$ for any $1\le k,\ell\le n$. Therefore, we may construct a symmetric matrix $Z^i\in\mathbf{S}^{n^2}$, such that $\operatorname{vec}(Z^i)=z^i$ for all $1\le i\le m$. We have $\mathcal{G}=\sum_{i=1}^m Z^i\otimes Z^i$, and so (8) is proven. □

For the first part of Theorem 3.3, the first identity follows from (8) by applying the symmetrization operation on both sides. The second identity is quite obvious. Essentially, for any $\mathcal{F}\in\mathbf{R}^{n^4}$, we may make it partial-symmetric by averaging the corresponding entries, to be denoted by $\mathcal{F}_0\in\overrightarrow{\mathbf{S}}^{n^4}$. It is easy to see that $\mathcal{F}_0(X,X)=\mathcal{F}(X,X)$ for all $X\in\mathbf{S}^{n^2}$ since $X\otimes X\in\overrightarrow{\mathbf{S}}^{n^4}$, which implies that

$$\operatorname{sym}\left\{\mathcal{F}\in\mathbf{R}^{n^4} \,|\, \mathcal{F}(X,X)\ge 0\ \forall\, X\in\mathbf{S}^{n^2}\right\}\subseteq \operatorname{sym}\left\{\mathcal{F}\in\overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X)\ge 0\ \forall\, X\in\mathbf{S}^{n^2}\right\}.$$

The reverse inclusion is trivial.

For the second part of Theorem 3.3, it follows from (7) and (8) by restricting to $\mathbf{S}^{n^4}$. Let us now turn to prove the last part of Theorem 3.3, which is an alternative representation of the SOP forms. Obviously we need only to show that

$$\operatorname{sym}\operatorname{cone}\left\{A\otimes A \,|\, A\in\mathbf{S}^{n^2}_+\right\}\subseteq \operatorname{cone}\left\{a\otimes a\otimes a\otimes a \,|\, a\in\mathbf{R}^n\right\}.$$

Since there is a one-to-one mapping from quartic forms to fourth order super-symmetric tensors, it suffices to show that for any $A\in\mathbf{S}^{n^2}_+$, the function $(x^{\mathrm{T}}Ax)^2$ can be written as a form of $\sum_{i=1}^m (x^{\mathrm{T}}a^i)^4$ for some $a^1,\ldots,a^m\in\mathbf{R}^n$. Note that the so-called Hilbert's identity (see e.g. Barvinok [5]) asserts the following:

For any fixed positive integers d and n, there always exist m real vectors $a^1,\ldots,a^m\in\mathbf{R}^n$ such that $(x^{\mathrm{T}}x)^d=\sum_{i=1}^m (x^{\mathrm{T}}a^i)^{2d}$.

In the original Hilbert's identity, m is exponential in n. However, by Caratheodory's theorem m can be bounded above by $\binom{n+2d-1}{2d}+1$. We refer the interested readers to Chapters 8 and 9 of [45] for more details. Recently, Jiang et al. [25] proposed a polynomial-time algorithm to find such polynomial-size representations when $d=2$. Since we have $A\in\mathbf{S}^{n^2}_+$, replacing x by $A^{1/2}y$ in Hilbert's identity when $d=2$, one gets $(y^{\mathrm{T}}Ay)^2=\sum_{i=1}^m (y^{\mathrm{T}}A^{1/2}a^i)^4$. The desired decomposition follows, and this proves the last part of Theorem 3.3.

3.3 Duality

In this subsection, we shall discuss the duality relationships among these four cones of quartic forms. Note that $\mathbf{S}^{n^4}$ is the ground tensor space within which the duality is defined, unless otherwise specified.

Theorem 3.5 *The cone of PSD forms and the cone of SOP forms are a primal-dual pair, i.e.,* $\mathbf{\Sigma}^4_{n,4} = \left(\mathbf{S}^{n^4}_+\right)^*$ *and* $\mathbf{S}^{n^4}_+ = \left(\mathbf{\Sigma}^4_{n,4}\right)^*$. *The cone of SOS forms and the cone of matrix PSD forms are a primal-dual pair, i.e.,* $\mathbf{S}^{n^2\times n^2}_+ = \left(\mathbf{\Sigma}^2_{n,4}\right)^*$ *and* $\mathbf{\Sigma}^2_{n,4} = \left(\mathbf{S}^{n^2\times n^2}_+\right)^*$.

Remark that the primal-dual relationship between $\mathbf{\Sigma}^4_{n,4}$ and $\mathbf{S}^{n^4}_+$ was already proved in Theorem 3.7 of [45] for general even degree forms. The primal-dual relationship between $\mathbf{S}^{n^2\times n^2}_+$ and $\mathbf{\Sigma}^2_{n,4}$ was also mentioned in Theorem 3.16 of [45] for general even degree forms. Here we give the proof in the language of tensors. Let us start by discussing the primal-dual pair $\mathbf{\Sigma}^4_{n,4}$ and $\mathbf{S}^{n^4}_+$. In Proposition 1 of [50], Sturm and Zhang proved that for the quadratic forms, $\{A \in \mathbf{S}^{n^2} \,|\, x^{\mathrm{T}}Ax \geq 0 \;\forall\, x \in \mathbf{D}\}$ and $\mathrm{cone}\,\{aa^{\mathrm{T}} \,|\, a \in \mathbf{D}\}$ are a primal-dual pair for any closed $\mathbf{D} \subseteq \mathbf{R}^n$. We observe that a similar structure holds for the quartic forms as well. The first part of Theorem 3.5 then follows from next lemma.

Lemma 3.6 *If* $\mathbf{D} \subseteq \mathbf{R}^n$ *is closed, then* $\mathbf{S}^{n^4}_+(\mathbf{D}) := \{\mathcal{F} \in \mathbf{S}^{n^4} \,|\, \mathcal{F}(x,x,x,x) \geq 0 \;\forall\, x \in \mathbf{D}\}$ *and* $\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}$ *are a primal-dual pair, i.e.,*

$$\mathbf{S}^{n^4}_+(\mathbf{D}) = \left(\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}\right)^* \tag{9}$$

and

$$\left(\mathbf{S}^{n^4}_+(\mathbf{D})\right)^* = \mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}.$$

Proof. Since $\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}$ is closed by Lemma 3.2, we only need to show (9). In fact, if $\mathcal{F} \in \mathbf{S}^{n^4}_+(\mathbf{D})$, then $\mathcal{F}\bullet(a\otimes a\otimes a\otimes a) = \mathcal{F}(a,a,a,a) \geq 0$ for all $a \in \mathbf{D}$. Thus $\mathcal{F}\bullet\mathcal{G} \geq 0$ for all $\mathcal{G} \in \mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}$, which implies that $\mathcal{F} \in (\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\})^*$. Conversely, if $\mathcal{F} \in (\mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\})^*$, then $\mathcal{F}\bullet\mathcal{G} \geq 0$ for all $\mathcal{G} \in \mathrm{cone}\,\{a\otimes a\otimes a\otimes a \,|\, a \in \mathbf{D}\}$. In particular, by letting $\mathcal{G} = x\otimes x\otimes x\otimes x$, we have $\mathcal{F}(x,x,x,x) = \mathcal{F}\bullet(x\otimes x\otimes x\otimes x) \geq 0$ for all $x \in \mathbf{D}$, which implies that $\mathcal{F} \in \mathbf{S}^{n^4}_+(\mathbf{D})$. □

Let us turn to the primal-dual pair of $\mathbf{S}^{n^2\times n^2}_+$ and $\mathbf{\Sigma}^2_{n,4}$. For technical reasons, we shall momentarily lift the ground space from $\mathbf{S}^{n^4}$ to the space of partial-symmetric tensors $\overrightarrow{\mathbf{S}}^{n^4}$. This enlarges all the dual objects. To distinguish these two dual objects, let us use the notation '$\mathbf{K}^{\overrightarrow{*}}$' to indicate the dual of convex cone $\mathbf{K} \in \mathbf{S}^{n^4} \subseteq \overrightarrow{\mathbf{S}}^{n^4}$ generated in the space $\overrightarrow{\mathbf{S}}^{n^4}$, while '$\mathbf{K}^*$' is the dual of $\mathbf{K}$ generated in the space $\mathbf{S}^{n^4}$.

Lemma 3.7 *For partial-symmetric tensors, the cone* $\overrightarrow{\mathbf{S}}^{n^2\times n^2}_+$ *is self-dual with respect to the space* $\overrightarrow{\mathbf{S}}^{n^4}$, *i.e.,* $\overrightarrow{\mathbf{S}}^{n^2\times n^2}_+ = \left(\overrightarrow{\mathbf{S}}^{n^2\times n^2}_+\right)^{\overrightarrow{*}}$.

Proof. According to Proposition 1 of [50] and the partial-symmetry of $\overrightarrow{\mathbf{S}}^{n^4}$, we have

$$\left(\text{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}\right\}\right)^{\overrightarrow{*}} = \left\{\mathcal{F}\in \overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X)\geq 0\ \forall\, X\in \mathbf{S}^{n^2}\right\}.$$

By Lemma 3.4, we have

$$\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2} = \left\{\mathcal{F}\in \overrightarrow{\mathbf{S}}^{n^4} \,|\, \mathcal{F}(X,X)\geq 0\ \forall\, X\in \mathbf{S}^{n^2}\right\} = \text{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}\right\}.$$

Thus $\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2}$ is self-dual with respect to the space $\overrightarrow{\mathbf{S}}^{n^4}$. □

Notice that by definition and Lemma 3.7, we have

$$\mathbf{\Sigma}_{n,4}^2 = \text{sym}\,\text{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}\right\} = \text{sym}\,\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2} = \text{sym}\,\left(\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2}\right)^{\overrightarrow{*}},$$

and by the alternative representation in Theorem 3.3 we have

$$\mathbf{S}_{+}^{n^2\times n^2} = \mathbf{S}^{n^4}\bigcap\text{cone}\left\{A\otimes A \,|\, A\in \mathbf{S}^{n^2}\right\} = \mathbf{S}^{n^4}\bigcap\overrightarrow{\mathbf{S}}_{+}^{n^2\times n^2}.$$

Therefore the duality between $\mathbf{S}_{+}^{n^2\times n^2}$ and $\mathbf{\Sigma}_{n,4}^2$ follows immediately from the following lemma.

Lemma 3.8 *If* $\mathbf{K}\subseteq \overrightarrow{\mathbf{S}}^{n^4}$ *is a closed convex cone and* $\mathbf{K}^{\overrightarrow{*}}$ *is its dual with respect to the space* $\overrightarrow{\mathbf{S}}^{n^4}$*, then* $\mathbf{K}\bigcap\mathbf{S}^{n^4}$ *and* $\text{sym}\,\mathbf{K}^{\overrightarrow{*}}$ *are a primal-dual pair with respect to the space* $\mathbf{S}^{n^4}$*, i.e.,* $\left(\mathbf{K}\bigcap\mathbf{S}^{n^4}\right)^{*} = \text{sym}\,\mathbf{K}^{\overrightarrow{*}}$ *and* $\mathbf{K}\bigcap\mathbf{S}^{n^4} = \left(\text{sym}\,\mathbf{K}^{\overrightarrow{*}}\right)^{*}$.

Proof. For any $\mathcal{G}\in\text{sym}\,\mathbf{K}^{\overrightarrow{*}}\subseteq\mathbf{S}^{n^4}$, there is a $\mathcal{G}'\in\mathbf{K}^{\overrightarrow{*}}\subseteq\overrightarrow{\mathbf{S}}^{n^4}$, such that $\mathcal{G}=\text{sym}\,\mathcal{G}'\in\mathbf{S}^{n^4}$. We then have $\mathcal{G}_{ijk\ell}=\frac{1}{3}(\mathcal{G}'_{ijk\ell}+\mathcal{G}'_{ikj\ell}+\mathcal{G}'_{i\ell jk})$. Thus for any $\mathcal{F}\in\mathbf{K}\bigcap\mathbf{S}^{n^4}\subseteq\mathbf{S}^{n^4}$, it follows that

$$\begin{aligned}
\mathcal{F}\bullet\mathcal{G} &= \sum_{1\leq i,j,k,\ell\leq n}\frac{\mathcal{F}_{ijk\ell}(\mathcal{G}'_{ijk\ell}+\mathcal{G}'_{ikj\ell}+\mathcal{G}'_{i\ell jk})}{3}\\
&= \sum_{1\leq i,j,k,\ell\leq n}\frac{\mathcal{F}_{ijk\ell}\mathcal{G}'_{ijk\ell}+\mathcal{F}_{ikj\ell}\mathcal{G}'_{ikj\ell}+\mathcal{F}_{i\ell jk}\mathcal{G}'_{i\ell jk}}{3}\\
&= \mathcal{F}\bullet\mathcal{G}'\geq 0.
\end{aligned}$$

Therefore $\mathcal{G}\in\left(\mathbf{K}\bigcap\mathbf{S}^{n^4}\right)^{*}$, implying that $\text{sym}\,\mathbf{K}^{\overrightarrow{*}}\subseteq\left(\mathbf{K}\bigcap\mathbf{S}^{n^4}\right)^{*}$.

Moreover, if $\mathcal{F}\in\left(\text{sym}\,\mathbf{K}^{\overrightarrow{*}}\right)^{*}\subseteq\mathbf{S}^{n^4}$, then for any $\mathcal{G}'\in\mathbf{K}^{\overrightarrow{*}}\subseteq\overrightarrow{\mathbf{S}}^{n^4}$, we have $\mathcal{G}=\text{sym}\,\mathcal{G}'\in\text{sym}\,\mathbf{K}^{\overrightarrow{*}}$, and $\mathcal{G}'\bullet\mathcal{F}=\mathcal{G}\bullet\mathcal{F}\geq 0$. Therefore $\mathcal{F}\in\left(\mathbf{K}^{\overrightarrow{*}}\right)^{\overrightarrow{*}}=\text{cl}\,\mathbf{K}=\mathbf{K}$, which implies that $\left(\text{sym}\,\mathbf{K}^{\overrightarrow{*}}\right)^{*}\subseteq\left(\mathbf{K}\bigcap\mathbf{S}^{n^4}\right)$. Finally, the duality relationship holds by the bipolar theorem and the closedness of these cones. □

3.4 The Hierarchical Structure

The last part of this section is to present a hierarchy among these four cones of quartic forms. The main result is summarized in the theorem below.

Theorem 3.9 *If $n \geq 4$, then*

$$\mathbf{\Sigma}_{n,4}^4 \subsetneq \mathbf{S}_+^{n^2 \times n^2} \subsetneq \mathbf{\Sigma}_{n,4}^2 \subsetneq \mathbf{S}_+^{n^4}.$$

For the low dimension cases ($n \leq 3$), we shall present it in Section 5.2. Evidently an SOS form is PSD, implying $\mathbf{\Sigma}_{n,4}^2 \subseteq \mathbf{S}_+^{n^4}$. By invoking the duality operation and using Theorem 3.5 we have $\mathbf{\Sigma}_{n,4}^4 \subseteq \mathbf{S}_+^{n^2 \times n^2}$, while by the alternative representation in Theorem 3.3 we have $\mathbf{S}_+^{n^2 \times n^2} = \mathbf{S}^{n^4} \bigcap \operatorname{cone} \left\{ A \otimes A \,|\, A \in \mathbf{S}^{n^2} \right\}$, and by the very definition we have $\mathbf{\Sigma}_{n,4}^2 = \operatorname{sym} \operatorname{cone} \left\{ A \otimes A \,|\, A \in \mathbf{S}^{n^2} \right\}$. Therefore $\mathbf{S}_+^{n^2 \times n^2} \subseteq \mathbf{\Sigma}_{n,4}^2$. Finally, the strict containing relationship is a result of the following examples.

Example 3.1 *(Quartic forms in $\mathbf{S}_+^{n^4} \setminus \mathbf{\Sigma}_{n,4}^2$ when $n = 4$) Let $g_1(x) = {x_1}^2(x_1 - x_4)^2 + {x_2}^2(x_2 - x_4)^2 + {x_3}^2(x_3 - x_4)^2 + 2x_1x_2x_3(x_1 + x_2 + x_3 - 2x_4)$ and $g_2(x) = {x_1}^2{x_2}^2 + {x_2}^2{x_3}^2 + {x_3}^2{x_1}^2 + {x_4}^4 - 4x_1x_2x_3x_4$, then both $g_1(x)$ and $g_2(x)$ are in $\mathbf{S}_+^{4^4} \setminus \mathbf{\Sigma}_{4,4}^2$.*

Historically, $g_1(x)$ is called Robinson form [48] and $g_2(x)$ is due to Choi and Lam [14, 15]. We refer the interested readers to [46] for more information.

Example 3.2 *(A quartic form in $\mathbf{S}_+^{n^2 \times n^2} \setminus \mathbf{\Sigma}_{n,4}^4$ when $n = 4$) Construct $\mathcal{F} \in \mathbf{S}^{4^4}$, whose only nonzero entries (taking into account the super-symmetry) are $\mathcal{F}_{1122} = 4$, $\mathcal{F}_{1133} = 4$, $\mathcal{F}_{2233} = 4$, $\mathcal{F}_{1144} = 9$, $\mathcal{F}_{2244} = 9$, $\mathcal{F}_{3344} = 9$, $\mathcal{F}_{1234} = 6$, $\mathcal{F}_{1111} = 29$, $\mathcal{F}_{2222} = 29$, $\mathcal{F}_{3333} = 29$, and $\mathcal{F}_{4444} = 3 + \frac{25}{7}$. One may verify straightforwardly that $\mathcal{F}$ can be decomposed as $\sum_{i=1}^7 A^i \otimes A^i$, with $A^1 = \begin{bmatrix} \sqrt{7} & 0 & 0 & 0 \\ 0 & \sqrt{7} & 0 & 0 \\ 0 & 0 & \sqrt{7} & 0 \\ 0 & 0 & 0 & \frac{5}{\sqrt{7}} \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$, $A^4 = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$, $A^5 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $A^6 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $A^7 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. According to Theorem 3.3, we have $\mathcal{F} \in \mathbf{S}_+^{4^2 \times 4^2}$. Recall $g_2(x)$ in Example 3.1, which is PSD. Denote $\mathcal{G}$ to be the super-symmetric tensor associated with $g_2(x)$, thus $\mathcal{G} \in \mathbf{S}_+^{4^4}$. One computes that $\mathcal{G} \bullet \mathcal{F} = 4 + 4 + 4 + 3 + \frac{25}{7} - 24 < 0$. By the duality result as stipulated in Theorem 3.5, we conclude that $\mathcal{F} \notin \mathbf{\Sigma}_{4,4}^4$.*

Example 3.3 *(A quartic form in $\mathbf{\Sigma}_{n,4}^2 \setminus \mathbf{S}_+^{n^2 \times n^2}$ when $n = 3$). Let $g_3(x) = 2{x_1}^4 + 2{x_2}^4 + \frac{1}{2}{x_3}^4 + 6{x_1}^2{x_3}^2 + 6{x_2}^2{x_3}^2 + 6{x_1}^2{x_2}^2$, which is obviously SOS. Now recycle the notation and denote $\mathcal{G} \in \mathbf{\Sigma}_{3,4}^2$ to be the super-symmetric tensor associated with $g_3(x)$, and we have $\mathcal{G}_{1111} = 2$, $\mathcal{G}_{2222} = 2$, $\mathcal{G}_{3333} = \frac{1}{2}$, $\mathcal{G}_{1122} = 1$, $\mathcal{G}_{1133} = 1$, and $\mathcal{G}_{2233} = 1$. If we let $X = \operatorname{Diag}(1, 1, -4) \in \mathbf{S}^{3^2}$, then*

$$\mathcal{G}(X, X) = \sum_{1 \leq i,j,k,\ell \leq 3} \mathcal{G}_{ijk\ell} X_{ij} X_{k\ell} = \sum_{1 \leq i,k \leq 3} \mathcal{G}_{iikk} X_{ii} X_{kk} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = -2,$$

implying that $\mathcal{G} \notin \mathbf{S}_+^{3^2 \times 3^2}$.

We remark that an example of $\mathbf{\Sigma}^2_{n,4} \backslash \mathbf{S}_+^{n^2\times n^2}$ even exists for $n=2$, e.g. $({x_1}^2 - {x_2}^2)^2$. However, the above example serves another purpose; see Example 4.2.

4 Cones Related to Convex Quartic Forms

In this section we study the cone of sos-convex quartic forms $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$, and the cone of quartic forms which are both SOS and convex $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\text{cvx}}$. The aim is to incorporate these two new cones into the hierarchical structure as depicted in Theorem 3.9.

Theorem 4.1 *If $n \geq 6$, then*

$$\mathbf{\Sigma}^4_{n,4} \subsetneq \mathbf{S}_+^{n^2\times n^2} \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subsetneq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\text{cvx}}\right) \subsetneq \mathbf{\Sigma}^2_{n,4} \subsetneq \mathbf{S}^{n^4}_+. \tag{10}$$

First, it is obvious that $\left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\text{cvx}}\right) \subseteq \mathbf{\Sigma}^2_{n,4}$. Moreover, the following example shows that an SOS form is not necessarily convex, which suggests that $\left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\text{cvx}}\right) \subsetneq \mathbf{\Sigma}^2_{n,4}$ when $n \geq 2$.

Example 4.1 *(A quartic form in $\mathbf{\Sigma}^2_{n,4} \setminus \mathbf{S}^{n^4}_{\text{cvx}}$ when $n=2$) Let $g_4(x) = (x^{\mathrm{T}}Ax)^2$ with $A \in \mathbf{S}^{n^2}$, and its Hessian matrix is $\nabla^2 g_4(x) = 8Axx^{\mathrm{T}}A + 4x^{\mathrm{T}}AxA$. In particular, by letting $A = \left[\begin{smallmatrix} -3 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $x = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, we have $\nabla^2 f(x) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 8 \end{smallmatrix}\right] + \left[\begin{smallmatrix} -12 & 0 \\ 0 & 4 \end{smallmatrix}\right] \not\succeq 0$, implying that $g_4(x)$ is not convex.*

Next we prove the assertion that $\mathbf{S}_+^{n^2\times n^2} \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ when $n \geq 3$. To this end, let us first quote a result on the sos-convex functions due to Ahmadi and Parrilo [3]:

> If $f(x)$ is a polynomial with its Hessian matrix being $\nabla^2 f(x)$, then $f(x)$ is sos-convex if and only if $y^{\mathrm{T}}\nabla^2 f(x) y$ is a sum of squares in (x,y).

For a quartic form $\mathcal{F}(x,x,x,x)$, its Hessian matrix is $12\mathcal{F}(x,x,\cdot,\cdot)$. Therefore, $\mathcal{F}$ is sos-convex if and only if $\mathcal{F}(x,x,y,y)$ is is a sum of squares in (x,y). Now if $\mathcal{F} \in \mathbf{S}_+^{n^2\times n^2}$, then by Theorem 3.3 we may find matrices $A^1,\ldots,A^m \in \mathbf{S}^{n^2}$ such that $\mathcal{F} = \sum_{t=1}^m A^t \otimes A^t$. We have

$$\begin{aligned}
\mathcal{F}(x,x,y,y) = \mathcal{F}(x,y,x,y) &= \sum_{t=1}^m \sum_{1\leq i,j,k,\ell\leq n} x_i y_j x_k y_\ell A^t_{ij} A^t_{k\ell} \\
&= \sum_{t=1}^m \left(\sum_{1\leq i,j\leq n} x_i y_j A^t_{ij}\right)\left(\sum_{1\leq k,\ell\leq n} x_k y_\ell A^t_{k\ell}\right) \\
&= \sum_{t=1}^m \left(x^{\mathrm{T}} A^t y\right)^2,
\end{aligned}$$

which is a sum of squares in (x,y), hence sos-convex. This proves $\mathbf{S}_+^{n^2\times n^2} \subseteq \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$, and the example below rules out the equality when $n \geq 3$.

Example 4.2 *(A quartic form in $\mathbf{\Sigma}^2_{\nabla^2_{n,4}} \setminus \mathbf{S}_+^{n^2\times n^2}$ when $n = 3$) Recall $g_3(x) = 2{x_1}^4 + 2{x_2}^4 + \frac{1}{2}{x_3}^4 + 6{x_1}^2{x_3}^2 + 6{x_2}^2{x_3}^2 + 6{x_1}^2{x_2}^2$ in Example 3.3, which is shown not to be matrix PSD. Moreover, it is straightforward to compute that*

$$\nabla^2 g_3(x) = 24 \begin{pmatrix} x_1 \\ x_2 \\ \frac{x_3}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \frac{x_3}{2} \end{pmatrix}^{\mathrm{T}} + 12 \begin{pmatrix} 0 \\ x_3 \\ x_2 \end{pmatrix} \begin{pmatrix} 0 \\ x_3 \\ x_2 \end{pmatrix}^{\mathrm{T}} + 12 \begin{pmatrix} x_3 \\ 0 \\ x_1 \end{pmatrix} \begin{pmatrix} x_3 \\ 0 \\ x_1 \end{pmatrix}^{\mathrm{T}} + 12 \begin{bmatrix} {x_2}^2 & 0 & 0 \\ 0 & {x_1}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq 0,$$

which implies that $g_3(x)$ is sos-convex.

Finally, we shall discuss $\mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subsetneq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\mathrm{cvx}}\right)$ in (10). Recall in Section 2.2, an sos-convex homogeneous quartic polynomial function is both SOS and convex (see also [3]), which implies that $\mathbf{\Sigma}^2_{\nabla^2_{n,4}} \subseteq \left(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\mathrm{cvx}}\right)$. However, the gap between $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ and $\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\mathrm{cvx}}$ was not clear until recently Ahmadi and Parrilo [4] completely characterized the gap between convexity and sos-convexity. In particular, the following example in [4] rules out the possibility of their equivalence for $n \geq 6$.

Example 4.3 *(A quartic form in $(\mathbf{\Sigma}^2_{n,4} \bigcap \mathbf{S}^{n^4}_{\mathrm{cvx}}) \setminus \mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ when $n = 6$) Let*

$$\begin{aligned} g_5(x) = {} & {x_1}^4 + {x_2}^4 + {x_3}^4 + {x_4}^4 + {x_5}^4 + {x_6}^4 + {x_1}^2{x_6}^2 + {x_2}^2{x_4}^2 + {x_3}^2{x_5}^2 \\ & + 2\left({x_1}^2{x_2}^2 + {x_1}^2{x_3}^2 + {x_2}^2{x_3}^2 + {x_4}^2{x_5}^2 + {x_4}^2{x_6}^2 + {x_5}^2{x_6}^2\right) \\ & + \frac{1}{2}\left({x_1}^2{x_4}^2 + {x_2}^2{x_5}^2 + {x_3}^2{x_6}^2\right) - (x_1x_2x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_5x_6)\,. \end{aligned}$$

It is easy to see that $g_5(x)$ is SOS. Moreover, it was shown in [4] that $g_5(x)$ is convex but not sos-convex. Thus $g_5(x)$ is both convex and SOS while not sos-convex.

This completes the proof for Theorem 4.1. The relationship among these six cones of quartic forms is depicted in Fig. 1, where a primal-dual pair is painted by the same color.

The two newly introduced cones in this section are related to the convexity properties. In fact, the relationship among convexity, sos-convexity and SOS is an interesting subject which attracted many speculations recently. Prior to $g_5(x)$ in Example 4.3 by Ahmadi and Parrilo [4], in [3] the same authors first gave an explicit example of a degree eight form in three variables, which was shown to be both convex and SOS while not sos-convex by means of numerical verification. For a complete characterization of all dimensions and degrees for convex forms that are not sos-convex, the readers are refereed to [4].

For the relationship between the cone of convex forms and the cone of SOS forms, Example 4.1 has ruled out the possibility that $\mathbf{\Sigma}^2_{n,4} \subseteq \mathbf{S}^{n^4}_{\mathrm{cvx}}$, while Blekherman [7] proved that $\mathbf{S}^{n^4}_{\mathrm{cvx}}$ is not contained in $\mathbf{\Sigma}^2_{n,4}$ either. Therefore these two cones are indeed distinctive. According to Blekherman [7], the cone of convex forms is actually much bigger than the cone of SOS forms for quartic forms when n is sufficiently large. However, at this point we are not aware of any explicit instance of $\mathbf{S}^{n^4}_{\mathrm{cvx}} \setminus \mathbf{\Sigma}^2_{n,4}$. In fact, according to a recent working paper of Ahmadi et al. [1], such instances exist only when $n \geq 4$. Therefore, the following challenge remains:

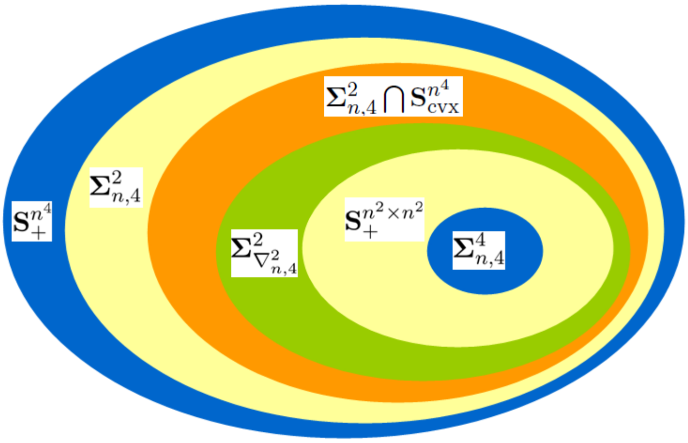

Fig. 1 Hierarchy for the cones of nonnegative quartic forms

Question 4.1 *Find an explicit instance of a form in* $\mathbf{S}^{n^4}_{\mathrm{cvx}} \setminus \mathbf{\Sigma}^2_{n,4}$*, i.e., a quartic convex form that is not SOS.*

Some more words on convex quartic forms are in order here. As mentioned in Section 2.2, for a quartic form $\mathcal{F} \in \mathbf{S}^{n^4}_{\mathrm{cvx}}$, its Hessian matrix is $12\,\mathcal{F}(x, x, \cdot, \cdot)$. Therefore, $\mathcal{F}$ is convex if and only if $\mathcal{F}(x, x, \cdot, \cdot) \succeq 0$ for all $x \in \mathbf{R}^n$, which is equivalent to $\mathcal{F}(x, x, y, y) \geq 0$ for all $x, y \in \mathbf{R}^n$. In fact, it is also equivalent to $\mathcal{F}(X, Y) \geq 0$ for all $X, Y \in \mathbf{S}^{n^2}_+$. To see why, we first decompose the positive semidefinite matrices X and Y, and let $X = \sum_{i=1}^n x^i (x^i)^{\mathrm{T}}$ and $Y = \sum_{j=1}^n y^j (y^j)^{\mathrm{T}}$ (see e.g. Sturm and Zhang [50]). Then

$$\begin{aligned}
\mathcal{F}(X, Y) &= \mathcal{F}\left(\sum_{i=1}^n x^i (x^i)^{\mathrm{T}}, \sum_{j=1}^n y^j (y^j)^{\mathrm{T}}\right) \\
&= \sum_{1 \leq i,j \leq n} \mathcal{F}\left(x^i (x^i)^{\mathrm{T}}, y^j (y^j)^{\mathrm{T}}\right) \\
&= \sum_{1 \leq i,j \leq n} \mathcal{F}\left(x^i, x^i, y^j, y^j\right) \geq 0,
\end{aligned}$$

if $\mathcal{F}(x, x, y, y) \geq 0$ for all $x, y \in \mathbf{R}^n$. Note that the converse is trivial, as it reduces to let X and Y be rank-one positive semidefinite matrices. Thus we have the following equivalence for the quartic convex forms.

Proposition 4.2 *For a given quartic form* $\mathcal{F} \in \mathbf{S}^{n^4}$*, the following statements are equivalent:*

- $\mathcal{F}(x, x, x, x)$ *is convex;*
- $\mathcal{F}(x, x, \cdot, \cdot)$ *is positive semidefinite for all* $x \in \mathbf{R}^n$*;*
- $\mathcal{F}(x, x, y, y) \geq 0$ *for all* $x, y \in \mathbf{R}^n$*;*
- $\mathcal{F}(X, Y) \geq 0$ *for all* $X, Y \in \mathbf{S}^{n^2}_+$*.*

Before concluding this section, we would like to mention the dual of the cone of convex quartic forms, which was studied earlier in [7,47]. According to Theorem 3.10 in [47], the dual of $\mathbf{S}^{n^4}_{\mathrm{cvx}}$ can be described in the super-symmetric tensor format as follows.

Proposition 4.3 *The cone* $\mathbf{S}^{n^4}_{\mathrm{cvx}}$ *and the cone*

$$\operatorname{sym}\operatorname{cone}\{a\otimes a\otimes b\otimes b \,|\, a,b\in\mathbf{R}^n\}$$

are a primal-dual pair.

Its proof can be constructed similarly to that of Theorem 3.5 by paying attention to the super-symmetry and that $\mathcal{F}(x,x,y,y)\ge 0$ for all $x,y\in\mathbf{R}^n$ stipulated in Proposition 4.2 for any convex form $\mathcal{F}$. Alternatively, one may consult [47] for a presentation involving only polynomials.

5 Complexities, Low-Dimensional Cases, and the Interiors of the Quartic Cones

In this section, we study the computational complexity issues for the membership queries regarding these cones of quartic forms. Unlike their quadratic counterparts where the positive semidefiniteness can be checked in polynomial-time, the case for the quartic cones are substantially subtler. We also study the low dimension cases of these cones, as a complement to the result on the hierarchic relationship displayed in Theorem 4.1. Finally, the interiors for some quartic cones are studied.

5.1 Complexity

Let us start with some easy cases. It is well known that deciding whether or not a polynomial function is SOS can be done by resorting to checking the feasibility of an SDP problem. As we all know, an SDP problem can be solved to arbitrary accuracy in polynomial-time. Therefore, defining the ϵ-weak-member of the SOS cone as the ϵ-optimal solution of the associated SDP feasibility problem, the weak membership query for $\mathbf{\Sigma}^2_{n,4}$ can be done in polynomial-time. Moreover, the strong membership for $\mathbf{S}_+^{n^2\times n^2}$ can be verified in polynomial-time. In fact, for any quartic form $\mathcal{F}\in\mathbf{S}^{n^4}$, we may rewrite $\mathcal{F}$ as an $n^2\times n^2$ matrix, to be denoted by $M_{\mathcal{F}}$, and then Theorem 3.3 assures that $\mathcal{F}\in\mathbf{S}_+^{n^2\times n^2}$ if and only if $M_{\mathcal{F}}$ is positive semidefinite, which can be checked in polynomial-time by computing the characteristic polynomial of $M_{\mathcal{F}}$ and then checking if the signs of its coefficients alternate [24]. Furthermore, as discussed in Section 4, a quartic form $\mathcal{F}$ is sos-convex if and only if $y^{\mathrm{T}}\left(\nabla^2\mathcal{F}(x,x,x,x)\right)y = 12\mathcal{F}(x,x,y,y)$ is SOS in (x,y), which can be again reduced to the feasibility of an SDP. Therefore, the weak membership checking problem for $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ can be carried out in polynomial-time as well. Summarizing, we have:

Proposition 5.1 *The strong membership query for* $\mathbf{S}_+^{n^2\times n^2}$ *can be done in polynomial-time, while the weak membership for the cones* $\mathbf{\Sigma}^2_{n,4}$ *and* $\mathbf{\Sigma}^2_{\nabla^2_{n,4}}$ *can be verified in polynomial-time.*

Unfortunately, the membership checking problems for all the other cones that we have discussed so far are difficult. To see why, let us introduce a famous cone of quadratic functions: the copositive cone

$$\mathbf{C} := \left\{ A \in \mathbf{S}^{n^2} \mid x^{\mathrm{T}} A x \geq 0 \;\forall\, x \in \mathbf{R}^n_+ \right\},$$

whose membership query is known to be co-NP-complete. The dual of the copositive cone is the cone of completely positive matrices, defined as

$$\mathbf{C}^* := \operatorname{cone} \left\{ x x^{\mathrm{T}} \mid x \in \mathbf{R}^n_+ \right\}.$$

Recently, Dickinson and Gijben [17] provided a formal proof for the NP-hardness of the membership problem for $\mathbf{C}^*$. The following result on the membership checking problem on $\mathbf{S}_+^{n^4}$ is well-known in the literature (see e.g. [30]). Here we present a proof based on a reduction using the membership query of the copositive cone $\mathbf{C}$. This reduction method can also be found in Chapter 5 of [40].

Proposition 5.2 *It is NP-hard to check if a quartic form belongs to* $\mathbf{S}_+^{n^4}$ *(the cone of PSD forms).*

Proof. Given a matrix $A \in \mathbf{S}^{n^2}$, we construct an $\mathcal{F} \in \mathbf{S}^{n^4}$, whose only nonzero entries are

$$\mathcal{F}_{iikk} = \mathcal{F}_{ikik} = \mathcal{F}_{ikki} = \mathcal{F}_{kiik} = \mathcal{F}_{kiki} = \mathcal{F}_{kkii} = \begin{cases} \frac{A_{ik}}{3} & i \neq k \\ A_{ik} & i = k. \end{cases} \tag{11}$$

For any $x \in \mathbf{R}^n$,

$$\begin{aligned} \mathcal{F}(x,x,x,x) &= \sum_{1 \leq i < k \leq n} \left(\mathcal{F}_{iikk} + \mathcal{F}_{ikik} + \mathcal{F}_{ikki} + \mathcal{F}_{kiik} + \mathcal{F}_{kiki} + \mathcal{F}_{kkii} \right) {x_i}^2 {x_k}^2 + \sum_{i=1}^n \mathcal{F}_{iiii} {x_i}^4 \\ &= \sum_{1 \leq i,k \leq n} A_{ik} {x_i}^2 {x_k}^2 \\ &= (x \circ x)^{\mathrm{T}} A (x \circ x), \end{aligned} \tag{12}$$

where the symbol '$\circ$' represents the Hadamard product. Denote $y = x \circ x \geq 0$, and then $\mathcal{F}(x,x,x,x) \geq 0$ if and only if $y^{\mathrm{T}} A y \geq 0$. Therefore $A \in \mathbf{C}$ if and only if $\mathcal{F} \in \mathbf{S}_+^{n^4}$ and the reduction is complete. □

Proposition 5.3 *It is NP-hard to check if a quartic form belongs to* $\mathbf{\Sigma}^4_{n,4}$ *(the cone of SOP forms).*

Proof. Similarly, the problem can be reduced to checking the membership of the completely positive cone $\mathbf{C}^*$. In particular, given any matrix $A \in \mathbf{S}^{n^2}$, construct an $\mathcal{F} \in \mathbf{S}^{n^4}$, whose only nonzero entries are defined exactly as in (11). If $A \in \mathbf{C}^*$, then $A = \sum_{t=1}^m a^t (a^t)^{\mathrm{T}}$ for some $a^1, \ldots, a^m \in \mathbf{R}^n_+$. By the construction of $\mathcal{F}$, we have

$$\mathcal{F}_{iikk} = \mathcal{F}_{ikik} = \mathcal{F}_{ikki} = \mathcal{F}_{kiik} = \mathcal{F}_{kiki} = \mathcal{F}_{kkii} = \begin{cases} \sum_{t=1}^m \frac{a_i^t a_k^t}{3} & i \neq k \\ \sum_{t=1}^m \left(a_i^t\right)^2 & i = k. \end{cases}$$

Denote $A^t = \mathrm{Diag}\,(a^t) \in \mathbf{S}_+^{n^2}$ for all $1 \le t \le m$. It is straightforward to verify that

$$\mathcal{F} = \sum_{t=1}^{m} \mathrm{sym}\,\left(A^t \otimes A^t\right) = \mathrm{sym}\,\left(\sum_{t=1}^{m} A^t \otimes A^t\right).$$

Therefore by Theorem 3.3 we have $\mathcal{F} \in \mathbf{\Sigma}_{n,4}^4$.

Conversely, if $A \notin \mathbf{C}^*$, then there exits a vector $y \in \mathbf{R}_+^n$, such that $y^{\mathrm{T}}Ay < 0$. Define a vector $x \in \mathbf{R}_+^n$ with $x_i = \sqrt{y_i}$ for all $1 \le i \le n$. By (12), we have

$$\mathcal{F} \bullet (x \otimes x \otimes x \otimes x) = \mathcal{F}(x,x,x,x) = (x \circ x)^{\mathrm{T}} A (x \circ x) = y^{\mathrm{T}} A y < 0.$$

Therefore, by the duality relationship in Theorem 3.5, we have $\mathcal{F} \notin \mathbf{\Sigma}_{n,4}^4$. Since $A \in \mathbf{C}^*$ if and only if $\mathcal{F} \in \mathbf{\Sigma}_{n,4}^4$ and so it follows that $\mathbf{\Sigma}_{n,4}^4$ is a hard cone. $\square$

Proposition 5.3 and its variations were also well-known in the literature in different contexts; see Section 11 of [23]. The above proof, however, emphasizes the representation in the space of super-symmetric tensors. Related to the membership query, Nie [38] recently proposed some numerical methods to actually compute an SOP-decomposition.

In recent years, Burer [8] showed that a large class of mixed-binary quadratic programs can be formulated as copositive programs where a linear function is minimized over a linearly constrained subset of the cone of completely positive matrices. Later, Burer and Dong [9] extended this equivalence to general nonconvex quadratically constrained quadratic program whose feasible region is nonempty and bounded. From the proof of Proposition 5.3, the cone of completely positive matrices can be imbedded into the cone of SOP forms. Evidently, these mixed-binary quadratic programs can also be formulated as linear conic program with the cone $\mathbf{\Sigma}_{n,4}^4$. In fact, the modeling power of $\mathbf{\Sigma}_{n,4}^4$ is much greater, which we shall discuss in Section 6 for further illustration.

Before concluding this subsection, a final remark on the cone $\mathbf{\Sigma}_{n,4}^2 \bigcap \mathbf{S}_{\mathrm{cvx}}^{n^4}$ is in order. Recall the recent breakthrough [2] mentioned in Section 1, that checking the convexity of a quartic form is strongly NP-hard. However, if we are given more information, that the quartic form to be considered is a sum of squares, will this make the membership easier? The answer is still no, as the following theorem asserts.

Theorem 5.4 *Deciding the convexity of an SOS form is strongly NP-hard. In particular, it is strongly NP-hard to check if a quartic form belongs to* $\mathbf{\Sigma}_{n,4}^2 \bigcap \mathbf{S}_{\mathrm{cvx}}^{n^4}$.

Proof. Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a graph with $\mathbf{V}$ being the set of n vertices and $\mathbf{E}$ being the set of edges. Define the following bi-quadratic form associated with graph $\mathbf{G}$ as follows:

$$b_{\mathbf{G}}(x,y) := 2 \sum_{(i,j)\in\mathbf{E}} x_i x_j y_i y_j.$$

Ling et al. [34] showed that $\max_{\|x\|_2=\|y\|_2=1} b_{\mathbf{G}}(x,y) = 1 - \frac{1}{\alpha(\mathbf{G})}$, where $\alpha(\mathbf{G})$ is the stability number of the graph $\mathbf{G}$. Therefore, maximizing $b_{\mathbf{G}}(x,y)$ subject to $\|x\|_2 = 1$ and $\|y\|_2 = 1$ is strongly NP-hard. Let us define

$$b_{\mathbf{G},\lambda}(x,y) := \lambda(x^{\mathrm{T}}x)(y^{\mathrm{T}}y) - b_{\mathbf{G}}(x,y) = \lambda(x^{\mathrm{T}}x)(y^{\mathrm{T}}y) - 2 \sum_{(i,j)\in\mathbf{E}} x_i x_j y_i y_j.$$

Then determining the nonnegativity of $b_{\mathbf{G},\lambda}(x,y)$ in (x,y) is also strongly NP-hard, due to the fact that the problem $\max_{\|x\|_2=\|y\|_2=1} b_{\mathbf{G}}(x,y)$ can be polynomially reduced to it. Let us now construct a quartic form in (x,y) as

$$f_{\mathbf{G},\lambda}(x,y) := b_{\mathbf{G},\lambda}(x,y)+n^2\left(\sum_{i=1}^n {x_i}^4+\sum_{i=1}^n {y_i}^4+\sum_{1\le i<j\le n} {x_i}^2{x_j}^2+\sum_{1\le i<j\le n} {y_i}^2{y_j}^2\right).$$

Observe that

$$f_{\mathbf{G},\lambda}(x,y) = g_{\mathbf{G},\lambda}(x,y)+\sum_{(i,j)\in\mathbf{E}}(x_ix_j-y_iy_j)^2+(n^2-1)\sum_{(i,j)\in\mathbf{E}}({x_i}^2{x_j}^2+{y_i}^2{y_j}^2),$$

where $g_{\mathbf{G},\lambda}(x,y) := \lambda(x^{\mathrm{T}}x)(y^{\mathrm{T}}y)+n^2\left(\sum_{i=1}^n({x_i}^4+{y_i}^4)+\sum_{(i,j)\notin\mathbf{E}}({x_i}^2{x_j}^2+{y_i}^2{y_j}^2)\right)$. Therefore $f_{\mathbf{G},\lambda}(x,y)$ is SOS in (x,y). Moreover, according to Theorem 2.3 of [2] with $\gamma=2$, we know that $f_{\mathbf{G},\lambda}(x,y)$ is convex if and only if $b_{\mathbf{G},\lambda}(x,y)$ is nonnegative. The latter being strongly NP-hard, therefore checking the convexity of the SOS form $f_{\mathbf{G},\lambda}(x,y)$ is also strongly NP-hard. $\square$

We remark that the construction in the above proof is similar to that of [2] except that we chose the bi-quadratic form $g_{\mathbf{G},\lambda}(x,y)$ to ensure the resulting quartic form to be SOS. Theorem 5.4 also implies that the reduction in [2] cannot produce convex forms that are not SOS, although it can produce convex forms that are not sos-convex.

To conclude this part, the chain of containing relationship as shown in Fig. 1 is useful especially when some of the cones are hard while others are 'easy'. One obvious possible application is to use an easy cone either as restriction or as relaxation of a hard one. Such scheme is likely to be useful in the design of approximation algorithms.

5.2 The Low Dimensional Cases

The chain of containing relations (10) holds for general dimension n. However, for some particular choices of n these relations may appear to be slightly different. In this subsection we discuss quartic forms in low dimensional cases: $n=2$ and $n=3$. Specifically, when $n=2$, the six cones of quartic forms reduce to two distinctive ones; while $n=3$, they reduce to three distinctive cones. Most of the results in this subsection are found to scatter in the literature (e.g. Proposition 6.1 of [47]); the aim here is to bring them under one theme.

Proposition 5.5 *For the cone of bi-variate quartic forms, it holds that*

$$\mathbf{\Sigma}_{2,4}^4=\mathbf{S}_+^{2^2\times 2^2}=\mathbf{\Sigma}_{\nabla_{2,4}^2}^2=\left(\mathbf{\Sigma}_{2,4}^2\bigcap\mathbf{S}_{\mathrm{cvx}}^{2^4}\right)\subsetneq\mathbf{\Sigma}_{2,4}^2=\mathbf{S}_+^{2^4}.$$

Proof. One well known fact in algebra is the equivalence between nonnegative univariate polynomials and the SOS univariate polynomials. In the homogenization setting, the result extends to the bivariate forms in the quartic case, which is exactly $\mathbf{\Sigma}_{2,4}^2=\mathbf{S}_+^{2^4}$. This is also an obvious consequence of Hilbert's [22] result on the equivalence between nonnegativity and SOS for bivariate quartic polynomials.

Now, the duality relationship in Theorem 3.5 leads to $\mathbf{\Sigma}^4_{2,4} = \mathbf{S}_+^{2^2\times 2^2}$. Next let us focus on the relationship between $\mathbf{S}_+^{2^2\times 2^2}$ and $\mathbf{\Sigma}^2_{2,4}\bigcap \mathbf{S}^{2^4}_{\mathrm{cvx}}$. In fact we shall prove below that $\mathbf{S}^{2^4}_{\mathrm{cvx}} \subseteq \mathbf{S}_+^{2^2\times 2^2}$, i.e., any bi-variate convex quartic form is matrix PSD.

For bi-variate convex quartic form $\mathcal{F}$ with

$$\mathcal{F}_{1111} = a_1, \mathcal{F}_{1112} = a_2, \mathcal{F}_{1122} = a_3, \mathcal{F}_{1222} = a_4, \mathcal{F}_{2222} = a_5,$$

we have $f(x) = \mathcal{F}(x,x,x,x) = a_1{x_1}^4 + 4a_2{x_1}^3x_2 + 6a_3{x_1}^2{x_2}^2 + 4a_4x_1{x_2}^3 + a_5{x_2}^4$, and

$$\nabla^2 f(x) = 12 \begin{bmatrix} a_1{x_1}^2 + 2a_2x_1x_2 + a_3{x_2}^2 & a_2{x_1}^2 + 2a_3x_1x_2 + a_4{x_2}^2 \\ a_2{x_1}^2 + 2a_3x_1x_2 + a_4{x_2}^2 & a_3{x_1}^2 + 2a_4x_1x_2 + a_5{x_2}^2 \end{bmatrix} \succeq 0 \quad \forall\, x_1, x_2 \in \mathbf{R}. \tag{13}$$

Denote $A^1 = \left[\begin{smallmatrix} a_1 & a_2 \\ a_2 & a_3 \end{smallmatrix}\right]$, $A^2 = \left[\begin{smallmatrix} a_2 & a_3 \\ a_3 & a_4 \end{smallmatrix}\right]$ and $A^3 = \left[\begin{smallmatrix} a_3 & a_4 \\ a_4 & a_5 \end{smallmatrix}\right]$, and (13) is equivalent to

$$\begin{bmatrix} x^{\mathrm{T}}A^1x & x^{\mathrm{T}}A^2x \\ x^{\mathrm{T}}A^2x & x^{\mathrm{T}}A^3x \end{bmatrix} \succeq 0 \quad \forall\, x \in \mathbf{R}^2. \tag{14}$$

According to Theorem 4.8 and the subsequent discussions in [35], it follows that (14) is equivalent to $\left[\begin{smallmatrix} A^1 & A^2 \\ A^2 & A^3 \end{smallmatrix}\right] \succeq 0$. Therefore,

$$\mathcal{F}(X,X) = (\mathrm{vec}\,(X))^{\mathrm{T}} \begin{bmatrix} A^1 & A^2 \\ A^2 & A^3 \end{bmatrix} \mathrm{vec}\,(X) \geq 0 \quad \forall\, X \in \mathbf{R}^{2^2},$$

implying that $\mathcal{F}$ is matrix PSD. This proves $\mathbf{S}_+^{2^2\times 2^2} = \mathbf{\Sigma}^2_{2,4}\bigcap \mathbf{S}^{2^4}_{\mathrm{cvx}}$. Finally, Example 4.1 for $\mathbf{\Sigma}^2_{2,4} \setminus \mathbf{S}^{2^4}_{\mathrm{cvx}}$ leads to $\mathbf{\Sigma}^2_{2,4}\bigcap \mathbf{S}^{2^4}_{\mathrm{cvx}} \neq \mathbf{\Sigma}^2_{2,4}$. $\square$

We remark that the relation $\mathbf{\Sigma}^2_{\nabla^2_{2,4}} = \left(\mathbf{\Sigma}^2_{2,4}\bigcap \mathbf{S}^{2^4}_{\mathrm{cvx}}\right)$ can be generalized to any even degree bivariate forms other than quartics; see Theorem 5.4 in [4] and the fact that every sos-convex form is SOS (Lemma 8 in [21]). In addition, the proof of Proposition 5.5 actually implies a stronger statement $\mathbf{\Sigma}^4_{2,4} = \mathbf{S}^{2^4}_{\mathrm{cvx}}$, which was previously shown in [18] and [44]. Our proof here takes along a different and simpler route by using matrix PSD forms as a bridge to establish the equivalence between convexity and SOP for quartic forms.

It remains to consider the case $n = 3$. Our previous discussion concluded that $\mathbf{\Sigma}^2_{3,4} = \mathbf{S}^{3^4}_+$, and so by duality $\mathbf{\Sigma}^4_{3,4} = \mathbf{S}_+^{3^2\times 3^2}$. Moreover, in a recent working paper Ahmadi et al. [1] showed that every tri-variate convex quartic polynomial is sos-convex, implying $\mathbf{\Sigma}^2_{\nabla^2_{3,4}} = \left(\mathbf{\Sigma}^2_{3,4}\bigcap \mathbf{S}^{3^4}_{\mathrm{cvx}}\right)$. Thus we have at most three distinctive cones of quartic forms. Example 4.1 in $\mathbf{\Sigma}^2_{2,4}\setminus\mathbf{S}^{2^4}_{\mathrm{cvx}}$ and Example 4.2 in $\mathbf{\Sigma}^2_{\nabla^2_{3,4}}\setminus\mathbf{S}_+^{3^2\times 3^2}$ show that there are in fact three distinctive cones.

Proposition 5.6 *For the cone of tri-variate quartic forms, it holds that*

$$\mathbf{\Sigma}^4_{3,4} = \mathbf{S}_+^{3^2\times 3^2} \subsetneq \mathbf{\Sigma}^2_{\nabla^2_{3,4}} = \left(\mathbf{\Sigma}^2_{3,4}\bigcap \mathbf{S}^{3^4}_{\mathrm{cvx}}\right) \subsetneq \mathbf{\Sigma}^2_{3,4} = \mathbf{S}^{3^4}_+.$$

5.3 Interiors of the Cones

Unlike the cone of nonnegative quadratic forms, where its interior is completely decided by the positive definiteness, the interior of quartic forms is much more complicated. Here we study two particular simple quartic forms: $\left(x^{\mathrm{T}}x\right)^2$ whose corresponding tensor is $\mathrm{sym}\,(I \otimes I)$, and $\sum_{i=1}^n {x_i}^4$ whose corresponding tensor is denoted by $\mathcal{I}$. Even for these two simple forms, to decide if they belong to the interior of certain quartic forms is already nontrivial. The results in this subsection can be found in Reznick [45], in the context of polynomials, while the framework here is the space of fourth order super-symmetric tensors.

First, it is easy to see that both $\mathrm{sym}\,(I \otimes I)$ and $\mathcal{I}$ are in the interior of $\mathbf{S}_+^{n^4}$. This is because the inner product between $\mathcal{I}$ and any nonzero form in $\mathbf{\Sigma}_{n,4}^4$ (the dual cone of $\mathbf{S}_+^{n^4}$) is positive. The same situation holds for $\mathrm{sym}\,(I \otimes I)$. Besides, they are both in $\mathbf{\Sigma}_{n,4}^4$ according to Theorem 3.3. Then one may want to know whether they are both in the interior of $\mathbf{\Sigma}_{n,4}^4$. At a first glance, one may think that $\mathcal{I}$ is in the interior of $\mathbf{\Sigma}_{n,4}^4$ as it is analogous to the identity matrix in the space of symmetric matrices. However, this is not the case. In fact, it was shown in [45] (Theorem 3.14) that any quartic form in the interior of $\mathbf{\Sigma}_{n,4}^4$ has to be written as a sum of at least $\binom{n+1}{2}$ fourth powers of linear forms in the shortest possible representation, which clearly rules $\mathcal{I}$ out as an element in the interior.

Proposition 5.7 *It holds that* $\mathrm{sym}\,(I \otimes I) \in \mathrm{Int}\,(\mathbf{S}_+^{n^2 \times n^2})$ *and* $\mathcal{I} \notin \mathrm{Int}\,(\mathbf{S}_+^{n^2 \times n^2})$.

Before providing the proof, let us first discuss the definition of $\mathrm{Int}\,(\mathbf{S}_+^{n^2 \times n^2})$. Following Definition 2.3, one may define a quartic form $\mathcal{F} \in \mathrm{Int}\,(\mathbf{S}_+^{n^2 \times n^2})$ if

$$\mathcal{F}(X, X) > 0 \quad \forall\, X \in \mathbf{R}^{n^2} \setminus O. \tag{15}$$

However, this condition is sufficient but not necessary. Since for any $\mathcal{F} \in \mathbf{S}^{n^4}$ and any skewness matrix Y, we have $\mathcal{F}(Y, Y) = 0$ according to the proof of Lemma 3.4, which leads to empty interior for $\mathbf{S}_+^{n^2 \times n^2}$ if we strictly follow (15). Noticing that $\mathbf{S}_+^{n^2 \times n^2} = \left\{\mathcal{F} \in \mathbf{S}^{n^4} \,|\, \mathcal{F}(X, X) \geq 0 \;\forall\; X \in \mathbf{S}^{n^2}\right\}$ by Theorem 3.3, the interior of $\mathbf{S}_+^{n^2 \times n^2}$ shall be correctly defined as follows, which is easy to verify by checking the standard definition of the cone interior.

Definition 5.1 *A quartic form* $\mathcal{F} \in \mathrm{Int}\,(\mathbf{S}_+^{n^2 \times n^2})$ *if and only if* $\mathcal{F}(X, X) > 0$ *for any* $X \in \mathbf{S}^{n^2} \setminus O$.

Proof of Proposition 5.7. For any $X \in \mathbf{S}^{n^2} \setminus O$, we observe that $\mathrm{sym}\,(I \otimes I)(X, X) = 2(\mathrm{tr}\,(X))^2 + 4\,\mathrm{tr}\,(XX^{\mathrm{T}}) > 0$, implying that $\mathrm{sym}\,(I \otimes I) \in \mathrm{Int}\,(\mathbf{S}_+^{n^2 \times n^2})$.

To prove the second part, we let $Y \in \mathbf{S}^{n^2} \setminus O$ with $\mathrm{diag}\,(Y) = 0$. Then we have $\mathcal{I}(Y, Y) = \sum_{i=1}^n Y_{ii}^2 = 0$, implying that $\mathcal{I} \notin \mathrm{Int}\,(\mathbf{S}_+^{n^2 \times n^2})$. □

The following theorems help to position $\mathcal{I}$ and $\mathrm{sym}\,(I \otimes I)$ in the interior of a particular cone in the hierarchy (10), respectively.

Theorem 5.8 *It holds that* $\mathcal{I} \notin \mathrm{Int}\,(\mathbf{S}_{\mathrm{cvx}}^{n^4})$ *and* $\mathcal{I} \in \mathrm{Int}\,(\mathbf{\Sigma}_{n,4}^2)$.

Proof. To prove the first part, we denote quartic form $\mathcal{F}_\epsilon$ to be $\mathcal{F}_\epsilon(x,x,x,x) = \sum_{i=1}^n x_i^4 - \epsilon x_1^2 x_2^2$, which is perturbed from $\mathcal{I}$. By Proposition 4.2, $\mathcal{F}_\epsilon \in \mathbf{S}^{n^4}_{\text{cvx}}$ if and only if

$$\mathcal{F}_\epsilon(x,x,y,y) = \sum_{i=1}^n x_i^2 y_i^2 - \frac{\epsilon}{6}\left(x_1^2 y_2^2 + x_2^2 y_1^2 + 4x_1x_2y_1y_2\right) \geq 0 \quad \forall\, x,y \in \mathbf{R}^n.$$

However, choosing $\hat{x} = (1,0,0,\dots,0)$ and $\hat{y} = (0,1,0,\dots,0)$ leads to $\mathcal{F}_\epsilon(\hat{x},\hat{x},\hat{y},\hat{y}) = -\frac{\epsilon}{6} < 0$ for any $\epsilon > 0$. Therefore $\mathcal{F}_\epsilon \notin \mathbf{S}^{n^4}_{\text{cvx}}$, implying that $\mathcal{I} \notin \operatorname{Int}(\mathbf{S}^{n^4}_{\text{cvx}})$.

For the second part, recall that the dual cone of $\mathbf{\Sigma}^2_{n,4}$ is $\mathbf{S}^{n^2\times n^2}_+$. It suffices to show that $\mathcal{I}\cdot\mathcal{F} > 0$ for any $\mathcal{F} \in \mathbf{S}^{n^2\times n^2}_+ \setminus \mathcal{O}$, or equivalently $\mathcal{I}\cdot\mathcal{F} = 0$ for $\mathcal{F} \in \mathbf{S}^{n^2\times n^2}_+$ implies $\mathcal{F} = \mathcal{O}$. Now rewrite $\mathcal{F}$ as an $n^2 \times n^2$ symmetric matrix $M_\mathcal{F}$. Clearly, $\mathcal{F} \in \mathbf{S}^{n^2\times n^2}_+$ implies $M_\mathcal{F} \succeq 0$, with its diagonal components $\mathcal{F}_{ijij} \geq 0$ for any i,j, in particular $\mathcal{F}_{iiii} \geq 0$ for any i. Combing this fact and the assumption that $\mathcal{I}\cdot\mathcal{F} = \sum_{i=1}^n \mathcal{F}_{iiii} = 0$ yeilds $\mathcal{F}_{iiii} = 0$ for any i. Next, we noticed that for any $i \neq j$, the matrix $\left[\begin{smallmatrix} \mathcal{F}_{iiii} & \mathcal{F}_{iijj} \\ \mathcal{F}_{jjii} & \mathcal{F}_{jjjj} \end{smallmatrix}\right]$ is a principle minor of the positive semidefinite matrix $M_\mathcal{F}$; as a result $\mathcal{F}_{iijj} = 0$ for any $i \neq j$. Since $\mathcal{F}$ is super-symmetric, we further have $\mathcal{F}_{ijij} = \mathcal{F}_{iijj} = 0$. Therefore $\operatorname{diag}(M_\mathcal{F}) = 0$, which combining $M_\mathcal{F} \succeq 0$ leads to $M_\mathcal{F} = O$. Hence $\mathcal{F} = \mathcal{O}$ and the conclusion follows. □

Remark that $\mathcal{I} \notin \operatorname{Int}(\mathbf{S}^{n^4}_{\text{cvx}})$ can also be observed from the fact that the Hessian matrix of the polynomial $\sum_{i=1}^n {x_i}^4$ is not everywhere positive definite.

Theorem 5.9 *It holds that* $\operatorname{sym}(I \otimes I) \in \operatorname{Int}(\mathbf{\Sigma}^4_{n,4})$.

Proof. By the duality relationship between $\mathbf{\Sigma}^4_{n,4}$ and $\mathbf{S}^{n^4}_+$, it suffices to show that any $\mathcal{F} \in \mathbf{S}^{n^4}_+$ with $\operatorname{sym}(I\otimes I)\cdot\mathcal{F} = 0$ implies $\mathcal{F} = \mathcal{O}$. For this qualified $\mathcal{F}$, we have $\mathcal{F}(x,x,x,x) \geq 0$ for any $x \in \mathbf{R}^n$. For any given i, let $x_i = 1$ and other entries be zeros, and it leads to

$$\mathcal{F}_{iiii} \geq 0 \quad \forall\, i. \tag{16}$$

Next, let $\xi \in \mathbf{R}^n$ whose entries are i.i.d. symmetric Bernoulli random variables, i.e., $\text{Prob}\{\xi_i = 1\} = \text{Prob}\{\xi_i = -1\} = \frac{1}{2}$ for all i. Then it is easy to compute

$$\mathsf{E}[\mathcal{F}(\xi,\xi,\xi,\xi)] = \sum_{i=1}^n \mathcal{F}_{iiii} + 6\sum_{1\leq i<j\leq n} \mathcal{F}_{iijj} \geq 0. \tag{17}$$

Besides, for any given $i \neq j$, let $\eta \in \mathbf{R}^n$ where η_i and η_j are independent symmetric Bernoulli random variables and other entries are zeros. Then

$$\mathsf{E}[\mathcal{F}(\eta,\eta,\eta,\eta)] = \mathcal{F}_{iiii} + \mathcal{F}_{jjjj} + 6\mathcal{F}_{iijj} \geq 0 \quad \forall\, i \neq j. \tag{18}$$

Since we assume $\operatorname{sym}(I\otimes I)\cdot\mathcal{F} = 0$, it follows that

$$\sum_{i=1}^n \mathcal{F}_{iiii} + 2\sum_{1\leq i<j\leq n} \mathcal{F}_{iijj} = \frac{1}{3}\left(\sum_{i=1}^n \mathcal{F}_{iiii} + 6\sum_{1\leq i<j\leq n} \mathcal{F}_{iijj}\right) + \frac{2}{3}\sum_{i=1}^n \mathcal{F}_{iiii} = 0. \tag{19}$$

Combining (16), (17) and (19), we get

$$\mathcal{F}_{iiii} = 0 \quad \forall\, i. \tag{20}$$

It further leads to $\mathcal{F}_{iijj} \geq 0$ for any $i \neq j$ by (18). Combining this result again with (19) and (20), we get

$$\mathcal{F}_{iijj} = 0 \quad \forall\, i \neq j. \tag{21}$$

Now it suffices to prove $\mathcal{F}_{iiij} = 0$ for all $i \neq j$, $\mathcal{F}_{iijk} = 0$ for all distinctive i, j, k, and $\mathcal{F}_{ijk\ell} = 0$ for all distinctive i, j, k, ℓ. To this end, for any given $i \neq j$, let $x \in \mathbf{R}^n$ where $x_i = t^2$ and $x_j = \frac{1}{t}$ and other entries are zeros. By (20) and (21), it follows that

$$\mathcal{F}(x, x, x, x) = 4\mathcal{F}_{iiij}\, x_i^3 x_j + 4\mathcal{F}_{ijjj}\, x_i x_j^3 = 4\mathcal{F}_{iiij}\, t^5 + 4\mathcal{F}_{ijjj}/t \geq 0 \quad \forall\, i \neq j.$$

Letting $t \to \pm\infty$, we get

$$\mathcal{F}_{iiij} = 0 \quad \forall\, i \neq j. \tag{22}$$

For any given distinctive i, j, k, let $x \in \mathbf{R}^n$ whose only nonzero entries are x_i, x_j and x_k, and we have

$$\mathcal{F}(x, x, x, x) = 12\mathcal{F}_{iijk}\, x_i^2 x_j x_k + 12\mathcal{F}_{jjik}\, x_j^2 x_i x_k + 12\mathcal{F}_{kkij}\, x_k^2 x_i x_j \geq 0 \quad \forall\, \text{distinctive } i, j, k.$$

Taking $x_j = 1, x_k = \pm 1$ in the above leads to $\pm(\mathcal{F}_{iijk}\, x_i^2 + \mathcal{F}_{jjik}\, x_i) + \mathcal{F}_{kkij}\, x_i \geq 0$ for any $x_i \in \mathbf{R}$, and we get

$$\mathcal{F}_{iijk} = 0 \quad \forall\, \text{distinctive } i, j, k. \tag{23}$$

Finally, for any given distinctive i, j, k, ℓ, let $x \in \mathbf{R}^n$ whose only nonzero entries are x_i, x_j, x_k and x_ℓ, and we have

$$\mathcal{F}(x, x, x, x) = 24\mathcal{F}_{ijk\ell}\, x_i x_j x_k x_\ell \geq 0 \quad \forall\, \text{distinctive } i, j, k, \ell.$$

Taking $x_i = x_j = x_k = 1$ and $x_\ell = \pm 1$ leads to

$$\mathcal{F}_{ijk\ell} = 0 \quad \forall\, \text{distinctive } i, j, k, \ell. \tag{24}$$

Combining equations (20), (21), (22), (23) and (24) yields $\mathcal{F} = \mathcal{O}$. $\square$

We remark that a generalization of Theorem 5.9 (to any even degree) can be found in Theorem 8.15 of [45].

6 Quartic Conic Programming

The study of quartic forms in the previous sections gives rise some new modeling opportunities. In this section we shall discuss quartic conic programming, i.e., optimizing a linear function over the intersection of an affine subspace and a cone of quartic forms. In particular, we shall investigate the following quartic conic programming model:

$$\begin{array}{rll} (QCP) & \max & \mathcal{C} \bullet \mathcal{X} \\ & \text{s.t.} & \mathcal{A}^i \bullet \mathcal{X} = b_i,\ i = 1, \ldots, m \\ & & \mathcal{X} \in \mathbf{\Sigma}^4_{n,4}, \end{array}$$

where $\mathcal{C}, \mathcal{A}^i \in \mathbf{S}^{n^4}$ and $b_i \in \mathbf{R}$ for $i = 1, \ldots, m$. As we will see later, a large class of non-convex quartic polynomial optimization models can be formulated as a special class of (QCP). In fact we will study a few concrete examples to show the modeling power of the quartic forms that we introduced.

6.1 Quartic Polynomial Optimization

Quartic polynomial optimization received much attention in the recent years; see e.g. [36, 34, 19, 20, 49, 31]. Essentially, all the models studied involve optimization of a quartic polynomial function subject to some linear and/or homogenous quadratic constraints, including spherical constraints, binary constraints, the intersection of co-centered ellipsoids, and so on. Below we consider a very general quartic polynomial optimization model:

$$
\begin{array}{ll}
(P) \max & p(x) \\
\quad \text{s.t.} & (a^i)^{\mathrm{T}} x = b_i,\ i = 1, \dots, m \\
& x^{\mathrm{T}} A^j x = c_j,\ j = 1, \dots, l \\
& x \in \mathbf{R}^n,
\end{array}
$$

where $p(x)$ is a general inhomogeneous quartic polynomial function.

We first homogenize $p(x)$ by introducing a new homogenizing variable, say x_{n+1}, which is set to one, and get a homogeneous quartic form

$$
p(x) = \mathcal{F}(\bar{x}, \bar{x}, \bar{x}, \bar{x}) = \mathcal{F} \bullet (\bar{x} \otimes \bar{x} \otimes \bar{x} \otimes \bar{x}),
$$

where $\mathcal{F} \in \mathbf{S}^{(n+1)^4}$, $\bar{x} = \binom{x}{x_{n+1}}$ and $x_{n+1} = 1$. By adding some redundant constraints, we have an equivalent formulation of (P):

$$
\begin{array}{ll}
\max & \mathcal{F}(\bar{x}, \bar{x}, \bar{x}, \bar{x}) \\
\text{s.t.} & (a^i)^{\mathrm{T}} x = b_i,\ \left((a^i)^{\mathrm{T}} x\right)^2 = {b_i}^2,\ \left((a^i)^{\mathrm{T}} x\right)^4 = {b_i}^4,\ i = 1, \dots, m \\
& x^{\mathrm{T}} A^j x = c_j,\ \left(x^{\mathrm{T}} A^j x\right)^2 = {c_j}^2,\ j = 1, \dots, l \\
& \bar{x} = \binom{x}{1} \in \mathbf{R}^{n+1}.
\end{array}
$$

The objective function of the above problem can be taken as a linear function of $\bar{x} \otimes \bar{x} \otimes \bar{x} \otimes \bar{x}$, and we introduce new variables of a super-symmetric fourth order tensor $\bar{\mathcal{X}} \in \mathbf{S}^{(n+1)^4}$. The notations x, X, and $\mathcal{X}$ extract part of the entries of $\bar{\mathcal{X}}$, which are defined as:

$$
\begin{array}{ll}
x \in \mathbf{R}^n,\ x_i = \bar{\mathcal{X}}_{i,n+1,n+1,n+1} & \forall\, 1 \le i \le n, \\
X \in \mathbf{S}^{n^2},\ X_{i,j} = \bar{\mathcal{X}}_{i,j,n+1,n+1} & \forall\, 1 \le i, j \le n, \\
\mathcal{X} \in \mathbf{S}^{n^4},\ \mathcal{X}_{i,j,k,\ell} = \bar{\mathcal{X}}_{i,j,k,\ell} & \forall\, 1 \le i, j, k, \ell \le n.
\end{array}
$$

Essentially they can be treated as linear constraints on $\bar{\mathcal{X}}$. Now by taking $\bar{\mathcal{X}} = \bar{x} \otimes \bar{x} \otimes \bar{x} \otimes \bar{x}$, $\mathcal{X} = x \otimes x \otimes x \otimes x$, and $X = x \otimes x$, we may equivalently represent the above problem as a quartic conic programming model with a rank-one constraint:

$$
\begin{array}{ll}
(Q) \max & \mathcal{F} \bullet \bar{\mathcal{X}} \\
\quad \text{s.t.} & (a^i)^{\mathrm{T}} x = b_i,\ (a^i \otimes a^i) \bullet X = {b_i}^2,\ (a^i \otimes a^i \otimes a^i \otimes a^i) \bullet \mathcal{X} = {b_i}^4,\ i = 1, \dots, m \\
& A^j \bullet X = c_j,\ (A^j \otimes A^j) \bullet \mathcal{X} = {c_j}^2,\ j = 1, \dots, l \\
& \bar{\mathcal{X}}_{n+1,n+1,n+1,n+1} = 1,\ \bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4},\ \operatorname{rank}(\bar{\mathcal{X}}) = 1.
\end{array}
$$

Dropping the rank-one constraint, we obtain a relaxation problem, which is exactly in the form of quartic conic program (QCP):

$$
\begin{array}{ll}
(RQ) \max & \mathcal{F} \bullet \bar{\mathcal{X}} \\
\quad \text{s.t.} & (a^i)^{\mathrm{T}} x = b_i,\ (a^i \otimes a^i) \bullet X = {b_i}^2,\ (a^i \otimes a^i \otimes a^i \otimes a^i) \bullet \mathcal{X} = {b_i}^4,\ i = 1, \dots, m \\
& A^j \bullet X = c_j,\ (A^j \otimes A^j) \bullet \mathcal{X} = {c_j}^2,\ j = 1, \dots, l \\
& \bar{\mathcal{X}}_{n+1,n+1,n+1,n+1} = 1,\ \bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4}.
\end{array}
$$

Interestingly, the relaxation from (Q) to (RQ) is not lossy; or, to put it differently, (RQ) is a tight relaxation of (Q), under some mild conditions.

Theorem 6.1 *If $A^j \in \mathbf{S}_+^{n^2}$ for all $1 \le j \le l$ in the model (P), then (RQ) is equivalent to (P) in the sense that: 1. they have the same optimal value; 2. if $\bar{\mathcal{X}}$ is optimal to (RQ), then x is in the convex hull of the optimal solution of (P). Moreover, the minimization counterpart of (P) is also equivalent to the minimization counterpart of (RQ).*

This result shows that (P) is in fact a conic quartic program (QCP) when the matrices A^j's in (P) are positive semidefinite. Notice that the model (P) actually includes quadratic inequality constraints $x^{\mathrm{T}}A^jx \le c_j$ as its subclasses, for one can always add a slack variable $y_j \in \mathbf{R}$ with $x^{\mathrm{T}}A^jx + {y_j}^2 = c_j$, while reserving the new data matrix $\left[\begin{smallmatrix} A^j & 0 \\ 0 & 1 \end{smallmatrix}\right]$ in the quadratic term still being positive semidefinite. The proof of Theorem 6.1 is dedicated to Appendix A.

As mentioned before, Burer [8] established the equivalence between a large class of mixed-binary quadratic programs and copositive programs. Theorem 6.1 may be regarded as a quartic extension of Burer's result. The virtue of this equivalence is to alleviate the highly non-convex objective and/or constraints of (QCP) and retain the problem in convex form, although the difficulty is all absorbed into the dealing of the quartic cone, which is nonetheless a convex one. Note that this is characteristically a property for polynomial of degree higher than 2: the SDP relaxation for similar quadratic models can never be tight.

6.2 Biquadratic Assignment Problems

The biquadratic assignment problem $(BQAP)$ is a generalization of the quadratic assignment problem (QAP), which is to minimize a quartic polynomial of an assignment matrix:

$$
\begin{array}{lll}
(BQAP) & \min & \sum_{1\le i,j,k,\ell,s,t,u,v\le n} \mathcal{A}_{ijk\ell}\mathcal{B}_{stuv}X_{is}X_{jt}X_{ku}X_{\ell v} \\
& \text{s.t.} & \sum_{i=1}^n X_{ij} = 1,\ j = 1,\dots,n \\
& & \sum_{j=1}^n X_{ij} = 1,\ i = 1,\dots,n \\
& & X_{ij} \in \{0,1\},\ i,j = 1,\dots,n \\
& & X \in \mathbf{R}^{n^2},
\end{array}
$$

where $\mathcal{A}, \mathcal{B} \in \mathbf{R}^{n^4}$. This problem was first considered by Burkard et al. [11] and was shown to have applications in the VLSI synthesis problem. After that, several heuristics for $(BQAP)$ were developed by Burkard and Cela [10], and Mavridou et al. [37].

In this subsection we shall show that $(BQAP)$ can be formulated as a quartic conic program (QCP). First notice that the objective function of $(BQAP)$ is a fourth order polynomial function with respect to the variables X_{ij}'s, where X is taken as an n^2-dimensional vector. The assignment constraints $\sum_{i=1}^n X_{ij} = 1$ and $\sum_{j=1}^n X_{ij} = 1$ are clearly linear equality constraints. Finally by imposing a new variable $x_0 \in \mathbf{R}$, and the binary constraints $X_{ij} \in \{0,1\}$ is equivalent to

$$
\begin{pmatrix} X_{ij} \\ x_0 \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} X_{ij} \\ x_0 \end{pmatrix} = \frac{1}{4} \quad \text{and} \quad x_0 = \frac{1}{2},
$$

where the coefficient matrix in the quadratic term is indeed positive semidefinite. Applying Theorem 6.1 we have the following result:

Corollary 6.2 *The biquadratic assignment problem* $(BQAP)$ *can be formulated as a quartic conic program* (QCP).

6.3 Eigenvalues of Fourth Order Super-Symmetric Tensor

The notion of eigenvalue for matrices has been extended to tensors, proposed by Lim [32] and Qi [41] independently; see also [33]. Versatile extensions turned out to be possible, among which the most popular one is called *Z-eigenvalue* (in the notion by Qi [41]). Restricting to the space of fourth order super-symmetric tensors $\mathbf{S}^{n^4}$, $\lambda \in \mathbf{R}$ is called a Z-eigenvalue of the super-symmetric tensor $\mathcal{F} \in \mathbf{S}^{n^4}$, if the following system holds

$$\begin{cases} \mathcal{F}(x,x,x,\cdot) = \lambda\, x, \\ x^{\mathrm{T}} x = 1, \end{cases}$$

where $x \in \mathbf{R}^n$ is the corresponding eigenvector with respect to λ. Notice that the Z-eigenvalues are the usual eigenvalues for a symmetric matrix, when restricting to the space of symmetric matrices $\mathbf{S}^{n^2}$. We refer interested readers to [32,41] for various other definitions of tensor eigenvalues and [42] for their applications in polynomial optimizations.

Observe that x is a Z-eigenvector of the fourth order tenor $\mathcal{F}$ if and only if x is a KKT point to following polynomial optimization problem:

$$\begin{array}{ll} (E) & \max\ \mathcal{F}(x,x,x,x) \\ & \text{s.t.}\quad x^{\mathrm{T}} x = 1. \end{array}$$

Furthermore, x is the Z-eigenvector with respect to the largest (respective smallest) Z-eigenvalue of $\mathcal{F}$ if and only if x is optimal to (E) (respective the minimization counterpart of (E)). As the quadratic constraint $x^{\mathrm{T}}x = 1$ satisfies the condition in Theorem 6.1, we reach the following conclusion:

Corollary 6.3 *The problem of finding a Z-eigenvector with respect to the largest or smallest Z-eigenvalue of a fourth order super-symmetric tensor* $\mathcal{F}$ *can be formulated as a quartic conic program* (QCP).

To conclude this section, as well as the whole paper, we remark here that quartic conic problems have many potential applications, alongside their many intriguing theoretical properties. The hierarchical structure of the quartic cones that we presented in the previous sections paves a way for possible relaxation methods to be viable. For instance, according to the hierarchy relationship (10), by relaxing the cone $\mathbf{\Sigma}^4_{n,4}$ to an easy cone $\mathbf{S}_+^{n^2\times n^2}$ lends a hand to solve the quartic conic optimization problem approximately. Such relaxations are different from the existing ones (e.g. [36,31]) for approximation algorithms for polynomial optimization models. The quality of such new solution methods and possible enhancements remain to be a topic for future research.

Acknowledgements We would like to thank three anonymous referees for their insightful comments, which helped significantly improve this paper from its original version.

This work was partially supported by National Science Foundation of China (Grants 11401364 and 11371242) and the US National Science Foundation (Grant CMMI-1161242).

A Proof of Theorem 6.1

Here we only prove the equivalent relation for the maximization problems since the proof for their minimization counterparts is exactly the same. That is, we shall prove the equivalence between (Q) and (RQ).

To start with, let us first investigate the feasible regions of these two problems, to be denoted by $\mathrm{feas}\,(Q)$ and $\mathrm{feas}\,(RQ)$, respectively. The relationship between $\mathrm{feas}\,(Q)$ and $\mathrm{feas}\,(RQ)$ is revealed by the following lemma.

Lemma A.1 *It holds that* $\mathrm{conv}\,(\mathrm{feas}\,(Q)) \subseteq \mathrm{feas}\,(RQ) = \mathrm{conv}\,(\mathrm{feas}\,(Q)) + \mathbf{P}$, *where*

$$\mathbf{P} := \mathrm{cone}\left\{ \binom{x}{0} \otimes \binom{x}{0} \otimes \binom{x}{0} \otimes \binom{x}{0} \,\middle|\, \begin{array}{l} (a^i)^{\mathrm{T}} x = 0 \ \forall\, 1 \le i \le m, \\ x^{\mathrm{T}} A^j x = 0 \ \forall\, 1 \le j \le l \end{array} \right\} \subseteq \mathbf{\Sigma}^4_{n+1,4}.$$

Proof. First, it is obvious that $\mathrm{conv}\,(\mathrm{feas}\,(Q)) \subseteq \mathrm{feas}\,(RQ)$ as (RQ) is a relaxation of (Q) and $\mathrm{feas}\,(RQ)$ is convex. Next, we notice that the recession cone of $\mathrm{feas}\,(RQ)$ is equal to

$$\left\{ \bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4} \,\middle|\, \begin{array}{ll} \mathcal{X}_{n+1,n+1,n+1,n+1} = 0, & \\ (a^i)^{\mathrm{T}} x = 0,\ (a^i \otimes a^i) \bullet X = 0,\ (a^i \otimes a^i \otimes a^i \otimes a^i) \bullet \mathcal{X} = 0 & \forall\, 1 \le i \le m, \\ A^j \bullet X = 0,\ (A^j \otimes A^j) \bullet \mathcal{X} = 0 & \forall\, 1 \le j \le l \end{array} \right\}.$$

Observing that $\bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4}$ and $\mathcal{X}_{n+1,n+1,n+1,n+1} = 0$, it is easy to see that $x = 0$ and $X = 0$. Thus the recession cone of $\mathrm{feas}\,(RQ)$ is reduced to

$$\left\{ \bar{\mathcal{X}} \in \mathbf{\Sigma}^4_{n+1,4} \,\middle|\, \begin{array}{ll} \mathcal{X}_{n+1,n+1,n+1,n+1} = 0,\ x = 0,\ X = 0, & \\ (a^i \otimes a^i \otimes a^i \otimes a^i) \bullet \mathcal{X} = 0 & \forall\, 1 \le i \le m, \\ (A^j \otimes A^j) \bullet \mathcal{X} = 0 & \forall\, 1 \le j \le l \end{array} \right\} \supseteq \mathbf{P},$$

which proves $\mathrm{feas}\,(RQ) \supseteq \mathrm{conv}\,(\mathrm{feas}\,(Q)) + \mathbf{P}$.

Finally, we shall show the inverse inclusion, i.e., $\mathrm{feas}\,(RQ) \subseteq \mathrm{conv}\,(\mathrm{feas}\,(Q)) + \mathbf{P}$. Suppose $\bar{\mathcal{X}} \in \mathrm{feas}\,(RQ)$, and it can be decomposed as

$$\bar{\mathcal{X}} = \sum_{k \in K} \binom{y^k}{\alpha_k} \otimes \binom{y^k}{\alpha_k} \otimes \binom{y^k}{\alpha_k} \otimes \binom{y^k}{\alpha_k}, \tag{25}$$

where $\alpha_k \in \mathbf{R}$, $y^k \in \mathbf{R}^n$ for all $k \in K$. Immediately we have

$$\sum_{k \in K} {\alpha_k}^4 = \mathcal{X}_{n+1,n+1,n+1,n+1} = 1. \tag{26}$$

Now divide the index set K into two parts $K_0 := \{k \in K \mid \alpha_k = 0\}$ and $K_1 := \{k \in K \mid \alpha_k \neq 0\}$, and let $z^k = y^k/\alpha_k$ for all $k \in K_1$. The decomposition (25) is then equivalent to

$$\bar{\mathcal{X}} = \sum_{k \in K_1} {\alpha_k}^4 \binom{z^k}{1} \otimes \binom{z^k}{1} \otimes \binom{z^k}{1} \otimes \binom{z^k}{1} + \sum_{k \in K_0} \binom{y^k}{0} \otimes \binom{y^k}{0} \otimes \binom{y^k}{0} \otimes \binom{y^k}{0}.$$

If we can prove that

$$\binom{z^k}{1} \otimes \binom{z^k}{1} \otimes \binom{z^k}{1} \otimes \binom{z^k}{1} \in \mathrm{feas}\,(Q) \quad \forall\, k \in K_1 \tag{27}$$

$$\binom{y^k}{0} \otimes \binom{y^k}{0} \otimes \binom{y^k}{0} \otimes \binom{y^k}{0} \in \mathbf{P} \qquad \forall\, k \in K_0 \tag{28}$$

then by (26), we shall have $\bar{\mathcal{X}} \in \mathrm{conv}\,(\mathrm{feas}\,(Q)) + \mathbf{P}$, proving the inverse inclusion.

In the following we shall prove (27) and (28). Since $\bar{\mathcal{X}} \in \mathrm{feas}\,(RQ)$, together with $x = \sum_{k \in K} {\alpha_k}^3 y^k$, $X = \sum_{k \in K} {\alpha_k}^2 y^k \otimes y^k$ and $\mathcal{X} = \sum_{k \in K} y^k \otimes y^k \otimes y^k \otimes y^k$, we obtain the following equalities:

$$\begin{array}{ll} \sum_{k \in K} {\alpha_k}^3 (a^i)^{\mathrm{T}} y^k = b_i,\ \sum_{k \in K} {\alpha_k}^2 \left((a^i)^{\mathrm{T}} y^k\right)^2 = {b_i}^2,\ \sum_{k \in K} \left((a^i)^{\mathrm{T}} y^k\right)^4 = {b_i}^4 & \forall\, 1 \le i \le m \\ \sum_{k \in K} {\alpha_k}^2 (y^k)^{\mathrm{T}} A^j y^k = c_j,\ \sum_{k \in K} \left((y^k)^{\mathrm{T}} A^j y^k\right)^2 = {c_j}^2 & \forall\, 1 \le j \le l. \end{array}$$

As a direct consequence of the above equalities and (26), we have

$$\begin{array}{ll}
\left(\sum_{k\in K} {\alpha_k}^2\cdot\alpha_k(a^i)^{\mathrm{T}}y^k\right)^2 = {b_i}^2 = \left(\sum_{k\in K}{\alpha_k}^4\right)\left(\sum_{k\in K}{\alpha_k}^2\left((a^i)^{\mathrm{T}}y^k\right)^2\right) & \forall\, 1\le i\le m\\
\left(\sum_{k\in K} {\alpha_k}^2\left((a^i)^{\mathrm{T}}y^k\right)^2\right)^2 = {b_i}^4 = \left(\sum_{k\in K}{\alpha_k}^4\right)\left(\sum_{k\in K}\left((a^i)^{\mathrm{T}}y^k\right)^4\right) & \forall\, 1\le i\le m\\
\left(\sum_{k\in K} {\alpha_k}^2(y^k)^{\mathrm{T}}A^jy^k\right)^2 = {c_j}^2 = \left(\sum_{k\in K}{\alpha_k}^4\right)\left(\sum_{k\in K}\left((y^k)^{\mathrm{T}}A^jy^k\right)^2\right) & \forall\, 1\le j\le l.
\end{array}$$

Noticing that the equalities hold for the above Cauchy-Schwarz inequalities, it follows that for every $1\le i\le m$ and every $1\le j\le l$, there exist $\delta_i,\epsilon_i,\theta_j\in\mathbf{R}$, such that

$$\delta_i{\alpha_k}^2=\alpha_k(a^i)^{\mathrm{T}}y^k,\ \epsilon_i{\alpha_k}^2=\left((a^i)^{\mathrm{T}}y^k\right)^2 \text{ and } \theta_j{\alpha_k}^2=(y^k)^{\mathrm{T}}A^jy^k \quad \forall\, k\in K. \tag{29}$$

If $\alpha_k=0$, then $(a^i)^{\mathrm{T}}y^k=0$ and $(y^k)^{\mathrm{T}}A^jy^k=0$, which implies (28). Moreover, due to (29) and (26),

$$\delta_i=\delta_i\left(\sum_{k\in K}{\alpha_k}^4\right)=\sum_{k\in K}\delta_i{\alpha_k}^2\cdot{\alpha_k}^2=\sum_{k\in K}\alpha_k(a^i)^{\mathrm{T}}y^k\cdot{\alpha_k}^2=b_i \quad \forall\, 1\le i\le m.$$

Similarly, we have $\theta_j=c_j$ for all $1\le j\le l$. If $\alpha_k\neq 0$, noticing $z^k=y^k/\alpha_k$, it follows from (29) that

$$\begin{array}{ll}
(a^i)^{\mathrm{T}}z^k=(a^i)^{\mathrm{T}}y^k/\alpha_k=\delta_i=b_i & \forall\, 1\le i\le m\\
(z^k)^{\mathrm{T}}A^jz^k=(y^k)^{\mathrm{T}}A^jy^k/{\alpha_k}^2=\theta_j=c_j & \forall\, 1\le j\le l,
\end{array}$$

which implies (27). $\square$

To prove Theorem 6.1, we notice that if A^j is positive semidefinite, then

$$x^{\mathrm{T}}A^jx=0\Longleftrightarrow A^jx=0.$$

Therefore, $\binom{x}{0}\otimes\binom{x}{0}\otimes\binom{x}{0}\otimes\binom{x}{0}\in\mathbf{P}$ implies that x is a recession direction of the feasible region for (P). Applying this property and using a similar argument of Theorem 2.6 in [8], Theorem 6.1 follows immediately.

References

1. A.A. Ahmadi, G. Blekherman, P.A. Parrilo, *Convex Ternary Quartics Are SOS-Convex*, Preprint, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge (2011)
2. A.A. Ahmadi, A. Olshevsky, P.A. Parrilo, J.N. Tsitsiklis, *NP-Hardness of Deciding Convexity of Quartic Polynomials and Related Problems*, Mathematical Programming, Series A, 137, 453–476 (2013)
3. A.A. Ahmadi, P.A. Parrilo, *A Convex Polynomial That Is Not SOS-Convex*, Mathematical Programming, Series A, 135, 275–292 (2012)
4. A.A. Ahmadi, P.A. Parrilo, *A Complete Characterization of the Gap between Convexity and SOS-Convexity*, SIAM Journal on Optimization, 23, 811–833 (2013)
5. A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, Volume 54, American Mathematical Society, Providence (2002)
6. P. Biswas, T.-C. Liang, T.-C. Wang, Y. Ye, *Semidefinite Programming Based Algorithms for Sensor Network Localization*, ACM Transactions on Sensor Networks, 2, 188–220 (2006)
7. G. Blekherman, *Convex Forms That Are Not Sums of Squares*, Preprint (2009) `http://arxiv.org/abs/0910.0656`

8. S. Burer, *On the Copositive Representation of Binary and Continuous Nonconvex Quadratic Programs*, Mathematical Programming, Series A, 120, 479–495 (2009)
9. S. Burer, H. Dong, *Representing Quadratically Constrained Quadratic Programs as Generalized Copositive Programs*, Operations Research Letters, 40, 203–206 (2012)
10. R.E. Burkard, E. Cela, *Heuristis for Biqudratic Assignment Problems and Their Computational Comparison*, European Journal of Operational Research, 83, 283–300 (1995)
11. R.E. Burkard, E. Cela, B. Klinz, *On the Biquadratic Assignment Problem*, Quadratic Assignment and Related Problems (P.M. Pardalos, H. Wolkowicz, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 16, 117–146, American Mathematical Society, Providence (1994)
12. C. Chen, P.P. Vaidyanathan, *MIMO Radar Waveform Optimization with Prior Information of the Extended Target and Clutter*, IEEE Transactions on Signal Processing, 57, 3533–3544 (2009)
13. E.C. Chi, T.G. Kolda, *On Tensors, Sparsity and Nonnegative Factorizations*, SIAM Journal on Matrix Analysis and Applications, 33, 1272–1299 (2012)
14. M.D. Choi, T.Y. Lam, *An Old Question of Hilbert*, Queen's Papers in Pure and Applied Mathematics, 46, 385–405 (1977)
15. M.D. Choi, T.Y. Lam, *Extremal Positive Semidefinite Forms*, Mathematische Annalen, 231, 1–18 (1977)
16. G. Dahl, J.M. Leinaas, J. Myrheim, E. Ovrum, *A Tensor Product Matrix Approximation Problem in Quantum Physics*, Linear Algebra and its Applications, 420, 711–725 (2007)
17. P.J.C. Dickinson, L. Gijben, *On the Computational Complexity of Membership Problems for the Completely Positive Cone and Its Dual*, Computational Optimization and Applications, 57, 403–415 (2014)
18. V.I. Dmitriev, *The Structure of a Cone in a Five-Dimensional Space (Russian)*, Vorone. Gos. Univ. Trudy Naun.-Issled. Inst. Mat. VGU Vyp., 7, 13–22 (1973)
19. S. He, Z. Li, S. Zhang, *Approximation Algorithms for Homogeneous Polynomial Optimization with Quadratic Constraints*, Mathematical Programming, Series B, 125, 353–383 (2010)
20. S. He, Z. Li, S. Zhang, *Inhomogeneous Polynomial Optimization Over a Convex Set: An Approximation Approach*, Mathematics of Computation, 84, 715–741 (2015)
21. J.W. Helton, J. Nie, *Semidefinite Representation of Convex Sets*, Mathematical Programming, Series A, 122, 21–64 (2010)
22. D. Hilbert, *Über die Darstellung Definiter Formen als Summe von Formenquadraten*, Mathematische Annalen, 32, 342–350 (1888)
23. C.J. Hillar, L.-H. Lim, *Most Tensor Problems Are NP-Hard*, Journal of the ACM, 60, Article 45 (2013)
24. R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge (1995)
25. B. Jiang, S. He, Z. Li, S. Zhang, *Moments Tensors, Hilbert's Identity, and k-wise Uncorrelated Random Variables*, Mathematics of Operations Research, 39, 775–788 (2014)
26. B. Jiang, S. Ma, S. Zhang, *Tensor Principal Component Analysis via Convex Optimization*, Mathematical Programming, Series A, 150, 423–457 (2015)
27. P.M. Kleniati, P. Parpas, B. Rustem, *Partitioning Procedure for Polynomial Optimization: Application to Portfolio Decisions with Higher Order Moments*, COMISEF Working Papers Series, WPS-023 (2009)
28. J.B. Lasserre, *Global Optimization with Polynomials and the Problem of Moments*, SIAM Journal on Optimization, 11, 769–817 (2001)
29. M. Laurent, *Sums of Squares, Moment Matrices and Optimization over Polynomials*, Emerging Applications of Algebraic Geometry (M. Putinar, S. Sullivant, eds.), The IMA Volumes in Mathematics and Its Applications, Volumn 149, Springer, Berlin (2009)
30. B. Lenore, F. Cucker, M. Shub, S. Smale, *Complexity and Real Computation*, Springer, New York (1997)
31. Z. Li, S. He, S. Zhang, *Approximation Methods for Polynomial Optimization: Models, Algorithms, and Applications*, SpringerBriefs in Optimization, Springer, New York (2012)
32. L.-H. Lim, *Singular Values and Eigenvalues of Tensors: A Variational Approach*, Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, 1, 129–132 (2005)
33. L.-H. Lim, M. Ng, L. Qi, eds., *The Spectral Theory of Tensors and Its Applications*, Numerical Linear Algebra with Applications, 20, Issue 6, (2013)
34. C. Ling, J. Nie, L. Qi, Y. Ye, *Biquadratic Optimization Over Unit Spheres and Semidefinite Programming Relaxations*, SIAM Journal on Optimization, 20, 1286–1310 (2009)

35. Z.-Q. Luo, J.F. Sturm, S. Zhang, *Multivariate Nonnegative Quadratic Mappings*, SIAM Journal on Optimization, 14, 1140–1162 (2004)
36. Z.-Q. Luo, S. Zhang, *A Semidefinite Relaxation Scheme for Multivariate Quartic Polynomial Optimization With Quadratic Constraints*, SIAM Journal on Optimization, 20, 1716–1736 (2010)
37. T. Mavridou, P.M. Pardalos, L.S. Pitsoulis, M.G.C. Resende, *A GRASP for the Biquadratic Assignment Problem*, European Journal of Operational Research, 105, 613–621 (1998)
38. J. Nie, *The $\mathcal{A}$-Truncated K-Moment Problem*, Foundations of Computational Mathematics, 14, 1243–1276 (2014)
39. P.M. Pardalos, S.A. Vavasis, *Open Questions in Complexity Theory for Numerical Optimization*, Mathematical Programming, 57, 337–339 (1992)
40. P.A. Parrilo, *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, Ph.D. Dissertation, California Institute of Technology, Pasadena (2000)
41. L. Qi, *Eigenvalues of a Real Supersymmetric Tensor*, Journal of Symbolic Computation, 40, 1302–1324 (2005)
42. L. Qi, F. Wang, Y. Wang, *Z-eigenvalue Methods for a Global Polynomial Optimization Problem*, Mathematical Programming, Series A, 118, 301–316 (2009)
43. S. Ragnarsson, C.F. Van Loan, *Block Tensors and Symmetric Embeddings*, Linear Algebra and its Applications, 438, 853–874 (2013)
44. B. Reznick, *Banach Spaces with Polynomial Norms*, Pacific Journal of Mathematics, 82, 223–235 (1979)
45. B. Reznick, *Sums of Even Powers of Real Linear Forms*, Memoirs of the American Mathematical Society, Volume 96, Number 463, American Mathematical Society, Providence (1992)
46. B. Reznick, *Some Concrete Aspects of Hilbert's 17th Problem*, Real Algebraic Geometry and Ordered Structures, Contemporary Mathematics, Volume 253, American Mathematical Society, Providence (2000)
47. B. Reznick, *Blenders*, Notions of Positivity and the Geometry of Polynomials (P. Brändén, M. Passare, M. Putinar, eds.), Trends in Mathematics, 345–373, Springer, Basel (2011)
48. R.M. Robinson, *Some Definite Polynomials Which Are Not Sums of Squares of Real Polynomials (Russian)*, Selected Questions of Algebra and Logic, 264–282, Academy of Sciences USSR, Novosibirsk (1973)
49. A.M.-C. So, *Deterministic Approximation Algorithms for Sphere Constrained Homogeneous Polynomial Optimization Problems*, Mathematical Programming, Series B, 129, 357–382 (2011)
50. J.F. Sturm, S. Zhang, *On Cones of Nonnegative Quadratic Functions*, Mathematics of Operations Research, 28, 246–267 (2003)
51. S. Zhang, Z. Li, S. Ma, eds., *Special Issue on Polynomial and Tensor Optimization*, Pacific Journal of Optimization, Volume 11, Number 2, (2015)