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ON CONFIDENCE LIMITS FOR THE RELIABILITY OF SYSTEMS

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0. Summary. The asymptotic chi-square distribution of the log-likelihood ratio is used to obtain approximate confidence intervals for the reliability of any system which may be represented by a monotone function of Bernoulli variates. This generalizes the results of A. Madansky, *Technometrics*, November 1965, for series, parallel and series-parallel systems. The method used is to parameterize the log-likelihood equation so as to find the interval of parameter values which keeps the log-likelihood less than or equal to the specified quantile of the chi-square distribution. This is done by introducing an operator depending upon the parameter, a fixed point of which is the solution of the likelihood ratio equation, and by showing the operator is a contractive map and hence has a unique fixed point depending continuously on the parameter. The solution can be found simply by iteration.

1. Introduction. Several approaches have been made to the problem of determining confidence limits for system reliability. Buehler [2] has given upper confidence limits for the product of two binomial parameters using Poisson approximations with an accuracy believed to be adequate whenever both sample sizes exceed forty. Buehler's method has been extended by Lipow [5] and Steck [8], for certain cases and short tabulations have been made. Madansky [6] has made use of the log-likelihood ratio's asymptotic chi-square distribution to obtain confidence intervals for series, parallel and series-parallel systems. It is the extension of this last result to a much wider class of systems that we present here.

The reason for this effort is the acknowledged inadequacy in many practical instances of the only known method of finding confidence bounds, from data on the performance of the components, for the reliability of general systems. This known method is the use of the asymptotic normality of the maximum likelihood estimates. A comparison, for certain cases of practical importance, of the method presented here with this alternative method is carried out in [7]. There the authors have shown that the use of the asymptotic chi-square method to determine confidence limits yields results which are substantially better for small sample sizes and equivalent for large ones for the cases considered.

In order to be specific about the class of systems which we consider here, it is necessary to introduce some notation. For the ith component among m, let the Bernoulli random variable Y_i indicate performance by taking the value one for success and zero for failure. The state of the components then is the vector $Y = (Y_1, \cdots, Y_m)$. By a coherent (monotone) structure we follow [1] to mean there exists a non-decreasing Boolean function ϕ of the state of the components, which

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is the indicator of the state of the structure. If $EY_i = p_i$ is the reliability of the ith component for $i = 1, \cdots, m$, then the vector $p = (p_1, \cdots, p_m)$ is the reliability of the components and $E\phi(Y) = h(p)$ is the reliability of the structure.

Our data consists of the vectors $x = (x_1, \cdots, x_m)$ and $n = (n_1, \cdots, n_m)$ where we have observed x_i successes in n_i trials of the ith component. The number of successes x_i has a binomial distribution given by

$$b(j:n_i, p_i) = \binom{n_i}{j} p_i^j (1 - p_i)^{n_i - j} \quad \text{for} \quad j = 0, \cdots, n_i.$$

The problem is to obtain confidence limits for $h(p)$ without further testing of the system.

Let $L(r)$ be the logarithm of the likelihood ratio

$$L(r) = \sup_{\{p:h(p)=r\}} \left[\sum_1^m \ln b(x_i : n_i, p_i)\right] - \sup_{p \in \mathcal{H}} \left[\sum_1^m \ln b(x_i : n_i, p_i)\right]$$

where $\mathcal{H}$ is the unit hypercube of dimension m.

Following the well-known theorem of Wilks [9] on the asymptotic behavior of the logarithm of the likelihood ratio, a confidence set of level γ for the system reliability is approximately

$$\{r: -2L(r) \leqq \chi^2_{1-\gamma}(1)\}$$

where $\chi^2_{1-\gamma}(1)$ is the γth quantile of the chi-square distribution with one degree of freedom.

To maximize the joint density with respect to p subject to the restraint $h(p) = r$ we use the method of Lagrange and examine the system of equations for $j = 1, \cdots, m$

$$(\partial/\partial p_j)\left[\sum_1^m \ln b(x_i : n_i, p_i) - \delta h(p)\right] = 0 \tag{1.0}$$

with

$$h(p) = r, \tag{1.1}$$

where δ is a Lagrange multiplier. The system (1.0) becomes for $j = 1, \cdots, m$

$$x_j p_j^{-1} - (n_j - x_j)(1 - p_j)^{-1} = \delta \partial_j h(p) \tag{1.2}$$

where ∂_j represents the partial derivative with respect to the jth variable. This partial derivative always exists as can be easily shown from the definition of $h(p)$, see [1]. For given δ, call $p(\delta) = (p_1(\delta), \cdots, p_m(\delta))$ the vector solution of (1.2), assuming it exists and is unique within $\mathcal{H}$.

If for given (x, n) we define N on $\mathcal{H}$ by

$$N(p) = \sum_{i=1}^m n_i[\tilde{p}_i \ln p_i + (1 - \tilde{p}_i) \ln (1 - p_i)] \tag{1.3}$$

where $\tilde{p}_i = x_i/n_i$, we see that

$$L(r) = Np(\delta_r) - N(\tilde{p})$$

where we make the convention here and throughout that juxtaposition of functions indicates composition, and where $p(\delta_r)$ is the solution of (1.2) with δ_r chosen so that (1.1) is satisfied.

Let us define

(1.4) $$\Lambda(\delta) = Np(\delta) - N(\hat{p})$$

for those values of δ for which $p(\delta)$ exists. Presuming for the moment that the existence of $p_i(\delta)$ implies that $p_i'(\delta)$ exists, which we prove in Section 2, we have

$$\Lambda'(\delta) = \textstyle\sum_1^m [x_i(p_i(\delta))^{-1} - (n_i - x_i)(1 - p_i(\delta))^{-1}]p_i'(\delta)$$

but from (1.2) we see

(1.5) $$\Lambda'(\delta) = \textstyle\sum_1^m \delta\partial_i h(p(\delta))p_i'(\delta) = \delta(d/d\delta)hp(\delta).$$

Thus from equation (1.5) we have

(1.6) LEMMA. *At each point δ for which $p(\delta)$ exists, $hp(\delta)$ is monotone decreasing iff $\Lambda(\delta)$ is monotone decreasing for $\delta > 0$ and monotone increasing for $\delta < 0$.*

From this follows the

(1.7) THEOREM. *If $hp(\delta)$ is monotone decreasing as a function of δ across an interval $[\delta^-, \delta^+]$ where $\delta^- < 0 < \delta^+$ are two values of δ for which*

$$\Lambda(\delta) = -\chi^2_{1-\gamma}(1)/2$$

then

$$\{r{:}hp(\delta^-) > r > hp(\delta^+)\} = \{r{:}2L(r) \leqq \chi^2_{1-\gamma}(1)\}.$$

PROOF. Since $hp(\delta)$ is a 1-1 transformation the adjustment of δ to satisfy (1.1) is superfluous and we can replace the parameterization of L by r with that of Λ by δ. ∎

The above discussion gives the general idea of the procedure that we have in mind to obtain an interval estimate.

For a given coherent structure h with failure data (x, n) on the components we seek to find, for a given confidence level γ, the two values of the Lagrange multiplier parameter, δ, one value being positive and one negative, such that the log-likelihood evaluated at each of the two parameter points is equal to minus one half of the γth quantile of the chi-square distribution with one degree of freedom. Moreover, we must assure ourselves that the reliability function, $hp(\delta)$, is monotone decreasing across the interval of parameter values. The confidence interval is the interval of values included between the reliability function evaluated at the end points of the interval of parameter values.

Since the problem of determining confidence intervals for the reliability of a component when it has experienced either no successes or no failures must be separately considered, unless one has some prior information which can be utilized with Bayesian techniques, it is not surprising that such cases should also require special attention when they occur in the components of a structure. For example, when a series structure has at least one component with no successes the lower confidence limit is fixed at zero.

It may be that for some structures and data we may be able only to find two values $\delta_1 > \delta_2$ across which $hp(\delta)$ is decreasing and then obtain the approximate confidence level of the interval estimate $(hp(\delta_1), hp(\delta_2))$ from $\frac{1}{2}[H_1(-2\Lambda(\delta_1)) + H_1(-2\Lambda(\delta_2))]$, where H_1 is the distribution of the chi-square random variable with one degree of freedom.

In order to limit our discussion in what follows to two-sided confidence intervals (the necessary modifications are easily made for one-sided) we shall treat the case

$$0 < x_i < n_i \quad \text{for} \quad i = 1, \cdots, m. \tag{1.8}$$

Since as a practical matter we are usually interested in components of high reliability, the values x_j should be nearer n_j. Moreover, because this procedure is based on an asymptotic approximation and its validity is in question if the sample sizes are too small, we are examining the most frequent and the most important case.

Unfortunately, to solve the equations in general, we still find it necessary to use an iteration procedure which requires the use of a computing machine at least as sophisticated as the IBM 1620.

However, since the advent of contracts in certain industries which call for vendors to demonstrate a reliability goal with at least a specified confidence, economic importance in such applications easily justifies the use of computing machines.

2. Iteration procedure for $p(\delta)$. We now turn to the practical problem of finding a method to obtain the solution $p(\delta)$ of equation (1.2), as a function of δ. We can then use it to obtain the graphs of $hp(\delta)$ and $-\Lambda(\delta)$ as functions of δ from which we can obtain an approximate confidence interval across any interval in which $hp(\delta)$ is decreasing.

One of the best methods, in that it lends itself readily to both machine computation and to theoretical study, is to define an operator such that the solution sought is a fixed point.

From (1.2) we have

$$x_j p_j^{-1} - (n_j - x_j)(1 - p_j)^{-1} = \delta p_j^{-1} a_j(p) \quad \text{for} \quad j = 1, \cdots, m,$$

where $a_j(p) = p_j \partial_j h(p)$. Now we introduce the transformation A where

$$A_j(p, \delta) = (x_j - \delta a_j(p))(n_j - \delta a_j(p))^{-1}, \qquad j = 1, \cdots, m, \tag{2.1}$$

which is suggested by solving the equations above as if $a_j(p)$ were constant.

Note that if, contrary to (1.8), we had $x_j = 0$ only values of $\delta < 0$ could be used otherwise $A_j(p, \delta) < 0$. In such a case only an upper confidence limit for the system reliability could be found. On the other hand, if some $x_j = n_j$ then A_j equals one. In effect, that component's reliability is taken as unity and thus the order of the structure reduced by one, essentially removing that component from consideration. Thus by suitably modifying the structure our data would be reduced to (1.8).

This lack of symmetry in behavior is caused by our definition of the transformation A and our choice of the form of the restraint. This discussion is taken up again in Section 4.

It is easily checked that for each fixed $\delta < \min(x_1, \cdots, x_m)$ the operator $A(\cdot, \delta)$ is a continuous transformation on $\mathcal{H}$ into itself.

We know immediately that a solution exists from Schauder's principle, (see e.g. Kantorovich and Akilov, p. 640, [3]), that a continuous operator mapping a convex closed compact subset of a Banach space (in this case $\mathcal{H}$) into itself has a fixed point. However, we desire more, namely, that the solution be unique and a continuous function of δ. For such results additional information is generally required either in the form of monotonicity of the operator or some uniformity condition like the one of Lipschitz.

We will now prove that the transformation A is a contractive map of a complete metric space $\mathcal{H}$ into itself and hence has a unique fixed point.

Define for a given coherent structure ϕ the constants for $j = 1, \cdots, m$

$$c_j = \sum \phi(1_j, y) - \phi(0_j, y) \tag{2.2}$$

where we introduce the notation $(1_j, y) = (y_1 y, \cdots, y_{j-1}, 1, y_{j+1}, \cdots, y_m)$, with the obvious meaning for $(0_j, y)$, and the summation extends over all 2^{m-1} values with $y_i = 0, 1$ for $i = j$. (Note that for the class of structures called the quorum or "k out of m" structures we have $c_j = \binom{m-1}{k-1}$ which gives some idea of the magnitude of these constants.)

We now make the obvious

REMARK. If $S, t \in (0, 1)$ and s, T are real, then

$$\begin{aligned} |sS - tT| &= |S(s - t) + t(S - T)| \\ &\leqq S|s - t| + t|S - T| \leqq |s - t| + |S - T|. \end{aligned} \tag{2.3}$$

(2.4) THEOREM. *For each given coherent structure ϕ (or correspondingly h) we have for all δ such that*

$$\min(x_1, \cdots, x_m) > \delta > 0 \quad \textit{and} \quad \sum_{j=1}^m (n_j - x_j)\delta(n_j - \delta)^{-2} c_j < 1 \tag{2.4.1}$$

or

$$\delta < 0 \quad \textit{and} \quad \sum_{j=1}^m (n_j - x_j) n_j^{-2} |\delta| c_j < 1, \tag{2.4.2}$$

the transformation $A(\cdot, \delta)$ is a contractive map of the metric space $(\mathcal{H}, d)$ into itself where d is the metric on the unit hypercube defined by

$$d(p, q) = m^{-1} \sum_1^m |p_i - q_i|. \tag{2.4.3}$$

PROOF. Let δ be fixed satisfying (2.4.1) or (2.4.2). We now must show that there exists an $r < 1$ such that

$$d(Ap, Aq) \leqq r d(p, q).$$

Of course both A and r depend upon δ but we omit its mention in the remainder of this argument. Take $p \neq q \in \mathcal{H}$, then from (2.1)

$$A_j(p) - A_j(q) = (n_j - x_j)\delta[a_j(q) - a_j(p)][(n_j - \delta a_j(p))(n_j - \delta a_j(q))]^{-1}$$

and

$$d(Ap, Aq) = m^{-1}\sum_j |A_jp - A_jq| . \tag{2.4.4}$$

By applying the remark above, we see since $0 < \partial_j hp < 1$

$$|a_j(p) - a_j(q)| = |p_j\partial_j hp - q_j\partial_j hq| \leqq |p_j - q_j| + |\partial_j hp - \partial_j hq| .$$

Now using the notation just introduced for coherent structures we have

$$\partial_j hp = h(1_j, p) - h(0_j, p)$$
$$= \sum [\phi(1_j, y) - \phi(0_j, y)]\prod_{i\neq j} [y_ip_i + (1 - y_i)(1 - p_i)]$$

and hence

$$\partial_j hp - \partial_j hq = \sum [\phi(1_j, y) - \phi(0_j, y)]$$
$$\cdot\{\prod_{i\neq j} [y_ip_i + (1 - y_i)(1 - p_i)] - \prod_{i\neq j} [y_iq_i + (1 - y_i)(1 - q_i)]\}.$$

By applying the remark repeatedly, we obtain

$$|\partial_j hp - \partial_j hq| \leqq \sum [\phi(1_j, y) - \phi(0_j, y)]\sum_{i\neq j} |p_i - q_i| .$$

Therefore

$$|a_jp - a_jq| \leqq |p_j - q_j| + c_j\sum_{i\neq j} |p_i - q_i| .$$

Since without loss of generality in coherent structures each element is essential, p. 64, [1], we must have $c_j \geqq 1$. Hence $|a_jp - a_jq| \leqq mc_jd(p, q)$ and

$$d(Ap, Aq) \leqq \sum_j (n_j - x_j)|\delta|\, c_j[(n_j - \delta a_jp)(n_j - \delta a_jq)]^{-1} d(p, q).$$

By taking into account that $1 \geqq a_j(p) \geqq 0$ and the sign of δ, we have the theorem proved. □

We define for given $p^0 \varepsilon \mathcal{H}$ the sequence

$$p^k(\delta) = A(p^{k-1}(\delta), \delta) \quad \text{for} \quad k = 1, 2, \cdots .$$

(2.5) THEOREM. *For every δ in the neighborhood of zero defined by the inequalities* (2.4.1), (2.4.2), *the solution of the system of equations* (1.2) *exists, call it $p(\delta)$, is unique and can be found for any initial $p^0 \varepsilon \mathcal{H}$ as*

$$\lim p^k(\delta) = p(\delta).$$

Moreover, $p(\delta)$ is a continuously differentiable function of δ in that neighborhood.

PROOF. That $A(\cdot, \delta)$, for δ in a neighborhood of zero, has a unique fixed point which is the limit of an iteration from any initial point follows from the known behavior of contraction maps, p. 32, [4]. The fixed point $p(\delta)$ is the solution of (1.2) and is given by

$$p(\delta) = \lim p^k(\delta) = A(\lim p^{k-1}(\delta), \delta) = A(p(\delta), \delta).$$

Let F be the vector valued function defined by

$$F_i(p, \delta) = \delta\partial_i h(p) - x_i p_i^{-1} + (n_i - x_i)(1 - p_i)^{-1}.$$

We have just argued that to the equation $F(p, \delta) = 0$, a unique solution, say p, exists for certain fixed δ. We assert that the Jacobian is not zero, i.e.

$$\det \partial_j F_i(p, \delta) \neq 0,$$

for if this determinant were zero, there would either be infinitely many solutions or none which is a contradiction. Now we can make use of the implicit function theorem to assert that the continuous differentiability of the function F is inherited by $p(\delta)$. Thus we have that if $p(\delta)$ exists, it is continuously differentiable. □

We now show that any structure h and any failure data (x, n) there always exists an approximate confidence interval of some level about the maximum likelihood estimate of the structure reliability $h(\tilde{p})$.

In view of Theorem (2.5), we have merely to show

(2.6) THEOREM. *For any reliability structure h, and any failure data there exists a neighborhood of zero in δ across which $hp(\delta)$ is decreasing in δ.*

PROOF. Letting $p_i'(\delta) = (d/d\delta)p_i(\delta)$,

$$(d/d\delta)hp(\delta) = \textstyle\sum_{i=1}^m \partial_i hp(\delta)\cdot p_i'(\delta). \tag{2.6.1}$$

From (1.2) by taking the derivative of both sizes with respect to δ and setting $s_j = x_j p_j^{-2} + (n_j - x_j)(1 - p_j)^{-2}$, where s_j, like p_j, is a function of δ, we obtain

$$-s_j p_j' - \delta\textstyle\sum_{i=1}^m \partial_{ij}h(p)\cdot p_i' = \partial_j h(p). \tag{2.6.2}$$

Multiplying by p_j' and summing, we obtain

$$-(d/d\delta)hp(\delta) = \delta\textstyle\sum_{i,j=1}^m \partial_{ij}h(p)p_i'p_j' + \sum_{j=1}^m s_j(p_j')^2.$$

Now clearly for $\delta = 0$, we obtain

$$s_j(0) = x_i(\tilde{p}_j)^{-2} + (n_j - x_j)(1 - \tilde{p}_j)^{-2} > 0,$$

and therefore $-dhp(0)/d\delta > 0$. □

REMARK. It may not be that an approximate confidence interval of high level, such as .99, can be obtained for every structure and every set of failure data but we can always obtain an interval estimate of some confidence.

3. The effect of large sample size. What we wish to examine in this section is the behavior of the approximate confidence limits as a function of sample size. It is not surprising that these limits for structural reliability as computed using $p(\delta)$ are consistent. To be precise we state

(3.0) THEOREM. *If at a given level γ, using data (x, n) the equation*

$$\Lambda(\delta : x, n) = -\chi^2_{1-\gamma}(1)/2$$

has two solutions $\delta^+_{x,n}(\gamma)$, $\delta^-_{x,n}(\gamma)$, then both these points tend stochastically to a *common value as n increases. With probability one we have*

$$\{r: hp(\delta^-) > r > hp(\delta^+)\} \rightarrow \{h(\pi)\} \quad as \quad \textstyle\sum_1^m n_i^{-1} \rightarrow 0$$

where π is the vector of true component reliability.

PROOF. Suppose that one has values δ, $p(\delta)$ which satisfy (1.2). We omit the argument of p in what follows:

By rewriting (1.2) in the form $\tilde{p}_j - p_j = (\delta/n_j)p_j(1 - p_j)\partial_j h(p)$, substituting into (1.4) and expanding the logarithms, it follows after some tedious manipulation that for $\sum n_i^{-1}$ sufficiently close to zero,

$$\Lambda(\delta) = -\{\delta^2 \textstyle\sum_{i=1}^m \beta_2(i)n_i^{-1} + \delta^3 \sum_{i=1}^m \beta_3(i)n_i^{-2} + \cdots\}$$

where

$$(3.1) \quad \beta_2(i) = [\partial_i h(p)]^2[p_i(1 - p_i) - \tfrac{1}{2}\tilde{p}_i(1 - p_i)^2 - \tfrac{1}{2}(1 - \tilde{p}_i)p_i^2].$$

For $j \geqq 3$ one can show $|\beta_j(i)| \leqq 1$. Thus by equating $\Lambda(\delta)$ to $\frac{1}{2}\chi^2_{1-\gamma}(1)$ and neglecting terms of order $\sum n_i^{-2}$ we can obtain an asymptotic solution, namely,

$$(3.2) \quad \delta^{\pm} \cong \pm(\chi^2_{1-\gamma}(1)/2\textstyle\sum_1^m \beta_2(i)n_i^{-1})^{\frac{1}{2}}.$$

It is sufficient to show that as $\sum n_i^{-1} \rightarrow 0$ that $|hp(\delta^+) - h(\tilde{p})| + |h(\tilde{p}) - hp(\delta^-)| \rightarrow 0$. Following the same methods that were used in Section (2.4) we see that h satisfies a Lipschitz condition and since $|\tilde{p}_j - p_j| \leqq |\delta|/(4n_j)$ we have

$$d(p(\delta^+), \tilde{p}) \leqq \tfrac{1}{4}|\delta^+| \textstyle\sum_1^m n_i^{-1} = O((\sum n_i^{-1})^{\frac{1}{2}}).$$

By the strong law of large numbers we have that $\tilde{p} \rightarrow \pi$ with probability one as $\sum n_i^{-1} \rightarrow 0$ and hence that $p(\delta^+) \rightarrow \pi$ with probability one, also. This completes the proof. □

From the equation (3.2) we can obtain an estimate for δ^+, by merely substituting $\tilde{p}_j$ for p_j in (3.1) and using this estimate in (3.2).

For some numerical comparisons of the consistency of this procedure, see [7].

4. An alternative formulation. Suppose that in maximizing the joint density to obtain the first term of $L(r)$ we impose the restraint in the form $\ln h(p) = \ln r$, then we would have, instead of (1.2), the system

$$(4.1) \quad x_j p_j^{-1} - (n_j - x_j)(1 - p_j)^{-1} = \lambda(h(p))^{-1}\partial_j h(p)$$

where λ is a Lagrange multiplier. We call the solution of (4.1), $\hat{p}(\lambda)$, when it exists uniquely within $\mathcal{H}$ in order to distinguish it from $p^*(\delta)$, the solution of (1.2). Of course the use of the restraint in a different form serves only to reparameterize the solutions of (1.2).

We can obtain in a manner similar to that of (1.7) the

(4.2) THEOREM. *If $h\hat{p}(\lambda)$ is monotone decreasing as a function of λ for $\lambda \varepsilon [\lambda^-, \lambda^+]$ where $\lambda^- < 0 < \lambda^+$ are two values of λ for which*

$$N\hat{p}(\lambda) - N(\tilde{p}) = -\chi^2_{1-\gamma}(1)/2$$

then an approximate confidence interval of level γ *is*

$$\{r: h\hat{p}(\lambda^-) > r > h\hat{p}(\lambda^+)\}.$$

REMARK. In the case of a series system $h(p) = \prod_1^m p_i$, and clearly $\partial_j h(p)/h(p) = p_j^{-1}$, so the system (4.1) has the explicit solution obtained by Madansky in [6], namely,

$$\hat{p}_j(\lambda) = (x_j - \lambda)(n_j - \lambda)^{-1}. \tag{4.3}$$

Clearly, $\hat{p}_j(\lambda)$ is a strictly decreasing function of λ since it has a derivative of $-(n_j - x_j)/(n_j - \lambda)^2$. We restrict λ to the range which keeps $0 \leqq p_j(\lambda) \leqq 1$ for $j = 1, \cdots, m$. From the monotone behavior of $h(p)$ in the series case, we see the hypothesis of Theorem (4.2) is not vacuous. This is the method presented by Madansky in the series case in [6].

Notice that for each δ such that $p^*(\delta)$ exists it provides a solution to (4.1) for the value $\lambda = \delta hp^*(\delta)$. To see this, merely multiply numerator and denominator of the right-hand side of (1.2) by $hp^*(\delta)$ and relabel. But, moreover, for each λ such that $\hat{p}(\lambda)$ exists for (4.1) there is a solution to (1.2) for the value $\delta = \lambda/h\hat{p}(\lambda)$. These transformations from one parameter to another are from the real line onto the real line but they are not always necessarily one-to-one. To see this consider $\lambda/h\hat{p}(\lambda)$, where h is a series reliability function. This transformation is increasing if $\lambda > 0$ but one can show that for any $\lambda < 0$, the derivative is negative for the order m sufficiently large.

We now mention an advantage of p^*. For every coherent structure ϕ, there exists a dual structure ϕ_D which is related by the condition

$$\phi_D(y_1, \cdots, y_m) = 1 - \phi(1 - y_1, \cdots, 1 - y_m)$$

see pp. 58–59, [1]. Hence if $h(p)$ is a reliability function of a coherent structure, then the dual structure has a reliability h_D defined for $p \,\varepsilon\, \mathfrak{K}$, letting $\nu^0 = (1, 1, \cdots, 1) \,\varepsilon\, \mathfrak{K}$ by $h_D(p) = 1 - h(\nu^0 - p)$.

We now state

(4.4) THEOREM. *If* $p^*(\delta)$ *is the solution of* (1.2) *which exists for all* δ *over an interval* $[\delta^-, \delta^+]$ *where* $\delta^- < 0 < \delta^+$ *and* $hp^*(\delta)$ *is decreasing in* δ *across the same interval then* $h_D p^{\#}(\delta)$ *is decreasing in* δ *over the reversed interval* $[-\delta^+, -\delta^-]$, *where* $p^{\#}(\delta)$ *is the vector solution of the dual system of equations in* p

$$z_j p_j^{-1} - (n_j - z_j)(1 - p_j)^{-1} = \delta \partial_j h_D(p) \tag{4.4.1}$$

with z_j *the number of successful trials of the* n_j *made for the* jth *component.*

PROOF. Let $q_i = 1 - p_i$, $y_i = n_i - z_i$, then (4.4.1) becomes

$$y_j q_j^{-1} - (n_j - y_j)(1 - q_j)^{-1} = -\delta \partial_j h(q). \tag{4.4.2}$$

Now by comparison of (4.4.2) with (1.2) it follows by hypothesis that $hq^{\#}(-\delta)$ is decreasing in δ over the reversed interval where $q^{\#}(-\delta)$ is the solution of (4.4.2). Therefore, we have $1 - hq^{\#}(-\delta)$ is increasing and consequently, the function $h_D p^{\#}(\delta) = 1 - hq^{\#}(\delta)$ is decreasing in δ which is the conclusion sought. □

The transformation is thus

$$p_j \to 1 - p_j, \qquad x_j \to n_j - x_j, \qquad \delta \to -\delta. \tag{4.5}$$

The first two we recognize as being characteristic of duality.

Consider h as a series structure reliability. Applying the transformation (4.5) to (4.3), which we know holds for $p^*(\delta)$, we obtain formally an explicit solution for the parallel case, namely,

$$p_j^{\#}(\lambda) = x_j(n_j + \lambda)^{-1}. \tag{4.6}$$

This is a decreasing function of λ and yields $p^{\#} \varepsilon \mathcal{H}$ whenever $-\min(n_j - x_j) < \lambda < \infty$.

We now derive this parameterized solution. From (1.2) we write

$$x_j p_j^{-1} - (n_j - x_j)(1 - p_j)^{-1} = \delta(1 - p_j)^{-1}(1 - p_j)\partial_j h(p)$$
$$= \delta(1 - p_j)^{-1}[h(1_j, p) - h(p)],$$

where as before $h(1_j, p) = h(p_1, \cdots, p_{j-1}, 1, p_{j+1}, \cdots, p_m)$. But if $h(p) = 1 - \prod_1^m (1 - p_i)$, let $\tau = \delta(1 - h(p))$ and the above becomes

$$x_j p_j^{-1} - (n_j - x_j)(1 - p_j)^{-1} = \tau(1 - p_j)^{-1},$$

the solution of which is (4.6) with λ replaced by τ. What we have done is equivalent with imposing the restriction in the form $1 - h(p) = 1 - r$ and then replacing the Lagrange multiplier by its negative.

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