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ON CONFORMAL CHANGES OF RIEMANNIAN METRICS

Dedicated to Professor Yûsaku Komatu on his sixtieth birthday

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§ 1. Introduction.

Let M be an n-dimensional connected differentiable manifold and g a Riemannian metric tensor field on M. We denote by (M, g) a Riemannian manifold with the metric tensor field g. Let there be given two metric tensor fields g and g^* on M. If g^* is conformal to g, that is, if there exists a function ρ on M such that $g^*=e^{2\rho}g$, then we call such a change of metric tensor field $g\to g^*$ a conformal change of metric. In particular, if ρ=constant then the conformal change of metric is said to be homothetic and if $\rho=0$ then the conformal change of metric is said to be isometric.

Let (M,g) and (M',g') be two Riemannian manifolds and $f: M\to M'$ a diffeomorphism. Then $g^*=f^*g'$ is a Riemannian metric tensor field on M. When g^* is conformal to g, that is, when there exists a function ρ on M such that $g^*=e^{2\rho}g$, we call $f:(M, g)\to(M', g')$ a conformal transformation. In particular, if ρ=constant then f is called a homothetic transformation or a homothety and if $\rho=0$ then f is called an isometric transformation or an isometry.

If a vector field v on M satisfies

$$L_v g=2\phi g, \tag{1.1}$$

where L_v denotes the Lie derivation with respect to v and ϕ a function on M, then v is called an infinitesimal conformal transformation. The v is said to be homothetic or isometric according as ϕ is a constant or zero.

Given a Riemannian manifold (M, g), we denote by g_{ji}, $\left\{{h \atop j\ i}\right\}$, ∇_i, $K_{kji}{}^h$, K_{ji} and K, respectively, the components of the metric tensor field g, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\left\{{h \atop j\ i}\right\}$, the components of the curvature tensor field, the components of the Ricci tensor field and the scalar curvature of (M,g), where and in the sequel, indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$. Hereafter we assume that functions under consideration are always differentiable.

When we consider a conformal change of metric

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$$g^*=e^{2\rho}g,$$

if Ω is a quantity formed with g then we denote by Ω^* the quantity formed with g^* by the same rule as that Ω is formed with g.

Recently, Goldberg and Yano [2] studied non-homothetic conformal changes of metrics and obtained the following

THEOREM A. *Let (M, g) be a compact orientable Riemannian manifold of dimension $n>3$ with constant scalar curvature K and admitting a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that $K^*=K$. Then if*

$$\int_M u^{-n+1}G_{ji}u^ju^idV\geqq 0, \tag{1.2}$$

where

$$G_{ji}=K_{ji}-\frac{K}{n}g_{ji} \tag{1.3}$$

and $u=e^{-\rho}$, $u_i=\nabla_i u$, $u^h=u_ig^{ih}$ and dV denotes the volume element of (M, g), then (M, g) is isometric to a sphere.

Yano and Obata [13] proved following theorems.

THEOREM B. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M(\Delta u)KdV=0, \qquad G^*_{ji}G^{*ji}=u^4G_{ji}G^{ji},$$

where $\Delta u=g^{ji}\nabla_j\nabla_i u$, then (M, g) is conformal to a sphere.

THEOREM C. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ and with K=constant admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$G^*_{ji}G^{*ji}=u^4G_{ji}G^{ji},$$

then (M, g) is isometric to a sphere.

THEOREM D. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M(\Delta u)KdV=0, \qquad Z^*_{kjih}Z^{*kjih}=u^4Z_{kjih}Z^{kjih},$$

where

$$Z_{kji}{}^h=K_{kji}{}^h-\frac{K}{n(n-1)}(\delta_k^h g_{ji}-\delta_j^h g_{ki}), \tag{1.4}$$

then (M, g) is conformal to a sphere.

THEOREM E. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ and with K=constant admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$Z^*_{kjih}Z^{*kjih}=u^4Z_{kjih}Z^{kjih},$$

then (M, g) is isometric to a sphere.

THEOREM F. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M (\Delta u) K dV = 0 ,$$

$$W^*_{kjih} W^{*kjih} = u^4 W_{kjih} W^{kjih} , \qquad a+(n-2)b \neq 0 ,$$

where

$$W_{kji}{}^h = a Z_{kji}{}^h + b(\delta_k^h G_{ji} - \delta_j^h G_{ki} + G_k{}^h g_{ji} - G_j{}^h g_{ki}) , \tag{1.5}$$

a and b being constants, then (M, g) is conformal to a sphere.

THEOREM G. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ and with K=constant admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$W^*_{kjih} W^{*kjih} = u^4 W_{kjih} W^{kjih} , \qquad a+(n-2)b \neq 0 ,$$

then (M, g) is isometric to a sphere.

THEOREM H. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$K^* = K , \qquad L_{du} K = 0 , \qquad \int_M u^{-n+1} G_{ji} u^j u^i dV \geqq 0 ,$$

where L_{du} denotes the Lie derivation with respect to a vector field $u^h = g^{ih} \nabla_i u$, then (M, g) is isometric to a sphere.

THEOREM I. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$K^* = K , \qquad L_{du} K = 0 , \qquad G^*_{ji} G^{*ji} = G_{ji} G^{ji} ,$$

then (M, g) is isometric to a sphere.

(See also Barbance [1].)

THEOREM J. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$K^* = K , \qquad L_{du} K = 0 , \qquad Z^*_{kjih} Z^{*kjih} = Z_{kjih} Z^{kjih} ,$$

then (M, g) is isometric to a sphere.

(See also Hsiung and Liu [3].)

THEOREM K. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$K^* = K , \qquad L_{du} K = 0 ,$$

$$W^*_{kjih} W^{*kjih} = W_{kjih} W^{kjih} , \qquad a+(n-2)b \neq 0 ,$$

then (M, g) is isometric to a sphere.

(See also Hsiung and Liu [3].)

Yano and Sawaki [14] proved following theorems.

THEOREM L. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$L_{du}K=0, \qquad L_{du}K^*=0,$$

$$u^p G_{ji}G^{ji}=\{(u-1)\varphi+1\}G^*_{ji}G^{*ji},$$

where p is a real number such that $p\leqq 4$ and φ a non-negative function on M, then (M, g) is isometric to a sphere.

THEOREM M. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$L_{du}K=0, \qquad L_{du}K^*=0,$$

$$u^p Z_{kjih}Z^{kjih}=\{(u-1)\varphi+1\}Z^*_{kjih}Z^{*kjih},$$

where p and φ are the same as in Theorem L, then (M, g) is isometric to a sphere.

THEOREM N. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$L_{du}K=0, \qquad L_{du}K^*=0,$$

$$u^p W_{kjih}W^{kjih}=\{(u-1)\varphi+1\}W^*_{kjih}W^{*kjih}, \qquad a+(n-2)b\neq 0,$$

where p and φ are the same as in Theorem L, then (M, g) is isometric to a sphere.

The purpose of the present paper is to prove generalizations of Theorems A$\sim$N.

In the sequel, we need the following two theorems.

THEOREM O (Tashiro [8]). *If a compact Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-constant function u on M such that*

$$\nabla_j\nabla_i u-\frac{1}{n}\Delta u g_{ji}=0,$$

then (M, g) is conformal to a sphere in an $(n+1)$-dimensional Euclidean space.

(See also Ishihara [4], Ishihara and Tashiro [5].)

THEOREM P (Yano and Obata [13]). *If a complete Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-constant function u on M such that*

$$\nabla_j\nabla_i u-\frac{1}{n}\Delta u g_{ji}=0, \qquad L_{du}K=0,$$

then (M, g) is isometric to a sphere in an $(n+1)$-dimensional Euclidean space.

§ 2. Preliminaries.

We consider a conformal change of metric

$$g^*_{ji}=e^{2\rho}g_{ji}\,. \tag{2.1}$$

First of all, we have

$$\left\{ {h \atop j\ i} \right\}^* = \left\{ {h \atop j\ i} \right\} + \delta_j^h \rho_i + \delta_i^h \rho_j - g_{ji}\rho^h\,, \tag{2.2}$$

where

$$\rho_i=\nabla_i\rho\,, \qquad \rho^h=\rho_i g^{ih}\,,$$

from which

$$K^*_{kji}{}^h=K_{kji}{}^h-\delta_k^h\rho_{ji}+\delta_j^h\rho_{ki}-\rho_k{}^h g_{ji}+\rho_j{}^h g_{ki}\,, \tag{2.3}$$

where

$$\rho_{ji}=\nabla_j\rho_i-\rho_j\rho_i+\frac{1}{2}\rho_t\rho^t g_{ji}\,, \qquad \rho_j{}^h=\rho_{ji}g^{ih}\,,$$

and consequently

$$K^*_{ji}=K_{ji}-(n-2)\rho_{ji}-\rho_t{}^t g_{ji} \tag{2.4}$$

and

$$e^{2\rho}K^*=K-2(n-1)\rho_t{}^t\,, \tag{2.5}$$

where

$$\rho_t{}^t=\Delta\rho+\frac{n-2}{2}\rho_t\rho^t\,, \qquad \Delta\rho=g^{ji}\nabla_j\rho_i\,.$$

From (2.3), (2.4) and (2.5) and the definitions of G_{ji}, $Z_{kji}{}^h$ and $W_{kji}{}^h$, we have

$$G^*_{ji}=G_{ji}-(n-2)(\nabla_j\rho_i-\rho_j\rho_i)+\frac{n-2}{n}(\Delta\rho-\rho_t\rho^t)g_{ji}\,, \tag{2.6}$$

$$\begin{aligned} Z^*_{kji}{}^h=Z_{kji}{}^h&-\delta_k^h(\nabla_j\rho_i-\rho_j\rho_i)+\delta_j^h(\nabla_k\rho_i-\rho_k\rho_i)\\ &-(\nabla_k\rho^h-\rho_k\rho^h)g_{ji}+(\nabla_j\rho^h-\rho_j\rho^h)g_{ki}\\ &+\frac{2}{n}(\Delta\rho-\rho_t\rho^t)(\delta_k^h g_{ji}-\delta_j^h g_{ki}) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} W^*_{kji}{}^h=W_{kji}{}^h+\{a+(n-2)b\}\Big\{&-\delta_k^h(\nabla_j\rho_i-\rho_j\rho_i)+\delta_j^h(\nabla_k\rho_i-\rho_k\rho_i)\\ &-(\nabla_k\rho^h-\rho_k\rho^h)g_{ji}+(\nabla_j\rho^h-\rho_j\rho^h)g_{ki}\\ &+\frac{2}{n}(\Delta\rho-\rho_t\rho^t)(\delta_k^h g_{ji}-\delta_j^h g_{ki})\Big\}\,. \end{aligned} \tag{2.8}$$

If we put

$$u=e^{-\rho}, \qquad u_i=\nabla_i u, \tag{2.9}$$

then we have

$$\nabla_j u_i=-u(\nabla_j \rho_i-\rho_j\rho_i) \tag{2.10}$$

and

$$\Delta u=-u(\Delta\rho-\rho_t\rho^t), \tag{2.11}$$

and consequently

$$K^*=u^2K+2(n-1)u\Delta u-n(n-1)u_tu^t, \tag{2.12}$$

$$G^*_{ji}=G_{ji}+(n-2)P_{ji}, \tag{2.13}$$

$$Z^*_{kji}{}^h=Z_{kji}{}^h+Q_{kji}{}^h \tag{2.14}$$

and

$$W^*_{kji}{}^h=W_{kji}{}^h+\{a+(n-2)b\}Q_{kji}{}^h, \tag{2.15}$$

where

$$P_{ji}=u^{-1}\left(\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}\right), \tag{2.16}$$

$$Q_{kji}{}^h=\delta_k^h P_{ji}-\delta_j^h P_{ki}+P_k{}^h g_{ji}-P_j{}^h g_{ki} \tag{2.17}$$

and

$$P_j{}^h=P_{ji}g^{ih}.$$

From (2.16) and (2.17), we obtain

$$P_{ji}P^{ji}=u^{-2}\left\{(\nabla^j u^i)(\nabla_j u_i)-\frac{1}{n}(\Delta u)^2\right\} \tag{2.18}$$

and

$$Q_{kjih}Q^{kjih}=4(n-2)P_{ji}P^{ji} \tag{2.19}$$

respectively.

We also have, from (2.13), (2.14) and (2.15),

$$G^*_{ji}G^{*ji}=u^4\{G_{ji}G^{ji}+2(n-2)G_{ji}P^{ji}+(n-2)^2P_{ji}P^{ji}\}, \tag{2.20}$$

$$Z^*_{kjih}Z^{*kjih}=u^4\{Z_{kjih}Z^{kjih}+8G_{ji}P^{ji}+4(n-2)P_{ji}P^{ji}\} \tag{2.21}$$

and

$$W^*_{kjih}W^{*kjih}=u^4[W_{kjih}W^{kjih}+8\{a+(n-2)b\}^2G_{ji}P^{ji} +4(n-2)\{a+(n-2)b\}^2P_{ji}P^{ji}] \tag{2.22}$$

respectively. For the expression $G_{ji}P^{ji}$ in (2.20), (2.21) and (2.22), we have, from (2.16),

$$G_{ji}P^{ji}=u^{-1}G_{ji}\nabla^{j}u^{i}\,, \tag{2.23}$$

where $\nabla^{j}=g^{ji}\nabla_{i}$.

§ 3. Lemmas.

LEMMA 1 (Lichnerowicz [6], Satō [7], Yano [9, 11]). *For a vector field v^h on a compact orientable Riemannian manifold (M, g), we have*

$$\begin{aligned}\int_M\Big(g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{n-2}{n}\nabla^h\nabla_t v^t\Big)v_h dV \\ +\frac{1}{2}\int_M\Big(\nabla^j v^i+\nabla^i v^j-\frac{2}{n}\nabla_t v^t g^{ji}\Big) \\ \times\Big(\nabla_j v_i+\nabla_i v_j-\frac{2}{n}\nabla_s v^s g_{ji}\Big)dV=0\,.\end{aligned} \tag{3.1}$$

Proof. By a straightforward computation, we have

$$\begin{aligned}\nabla_i\Big\{\Big(\nabla^i v^h+\nabla^h v^i-\frac{2}{n}\nabla_t v^t g^{ih}\Big)v_h\Big\} \\ =\Big(g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{n-2}{n}\nabla^h\nabla_i v^i\Big)v_h \\ +\frac{1}{2}\Big(\nabla^j v^i+\nabla^i v^j-\frac{2}{n}\nabla_t v^t g^{ji}\Big) \\ \times\Big(\nabla_j v_i+\nabla_i v_j-\frac{2}{n}\nabla_s v^s g_{ji}\Big)\,,\end{aligned}$$

and consequently, integrating over M, we have (3.1).

REMARK. If a vector field v^h defines an infinitesimal conformal transformation, then we have

$$L_v g_{ji}=2\rho g_{ji}\,,$$

that is,

$$\nabla_j v_i+\nabla_i v_j-\frac{2}{n}\nabla_t v^t g_{ji}=0\,.$$

From this, we can deduce

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{n-2}{n}\nabla^h\nabla_t v^t=0\,. \tag{3.2}$$

Formula (3.1) shows that this is not only necessary but also sufficient in order that the vector field v^h defines an infinitesimal conformal transformation in a compact orientable Riemannian manifold.

LEMMA 2 (Yano [10]). *For a function u on a compact orientable Riemannian manifold (M, g), we have*

(3.3)
$$\int_M \Big(g^{ji}\nabla_j\nabla_i u^h + K_i{}^h u^i + \frac{n-2}{n}\nabla^h \Delta u\Big) u_h dV$$
$$+2\int_M \Big(\nabla^j u^i - \frac{1}{n}\Delta u g^{ji}\Big)\Big(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\Big) dV = 0$$

and

(3.4)
$$\int_M \Big\{(g^{ji}\nabla_j\nabla_i u^h + K_i{}^h u^i) u_h - \frac{n-2}{n}(\Delta u)^2\Big\} dV$$
$$+2\int_M \Big(\nabla^j u^i - \frac{1}{n}\Delta u g^{ji}\Big)\Big(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\Big) dV = 0,$$

where $u_i = \nabla_i u$, $u^h = u_i g^{ih}$ *and* $\Delta u = g^{ji}\nabla_j\nabla_i u$.

Proof. Putting $v^h = u^h$ in (3.1) and using $\nabla^j u^i = \nabla^i u^j$, we obtain (3.3). (3.4) follows from (3.3) because of

$$\int_M (\nabla^h \Delta u) u_h dV = -\int_M (\Delta u)^2 dV.$$

LEMMA 3 (Yano [10]). *For a function u on a Riemannian manifold (M, g), we have*

(3.5)
$$\nabla^h \Delta u = g^{ji}\nabla_j\nabla_i u^h - K_i{}^h u^i,$$

that is,

(3.6)
$$g^{ji}\nabla_j\nabla_i u^h = \nabla^h \Delta u + K_i{}^h u^i.$$

Proof. We have

$$\begin{aligned}\nabla_h(\Delta u) &= \nabla_h(g^{ji}\nabla_j u_i) = g^{ji}\nabla_h\nabla_j u_i \\ &= g^{ji}(\nabla_j\nabla_h u_i - K_{hji}{}^t u_t) \\ &= g^{ji}\nabla_j\nabla_i u_h - K_h{}^t u_t,\end{aligned}$$

from which (3.5) follows.

LEMMA 4. *For a function u on a compact orientable Riemannian manifold (M, g), we have*

(3.7)
$$\int_M \Big(K_{ji}u^j u^i + \frac{n-1}{n} u^h \nabla_h \Delta u\Big) dV$$
$$+\int_M \Big(\nabla^j u^i - \frac{1}{n}\Delta u g^{ji}\Big)\Big(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\Big) dV = 0$$

and

(3.8)
$$\int_M \Big\{K_{ji}u^j u^i - \frac{n-1}{n}(\Delta u)^2\Big\} dV$$
$$+\int_M \Big(\nabla^j u^i - \frac{1}{n}\Delta u g^{ji}\Big)\Big(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\Big) dV = 0.$$

Proof. Substituting (3.6) into (3.3), we have (3.7), and substituting (3.6) into (3.4), we have (3.8).

LEMMA 5. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a conformal change of metric $g^* = e^{2\rho} g$, then, for any real number p, we have*

$$(3.9)\qquad \int_M u^{p-1} G_{ji} u^j u^i dV + (p+n-2)\int_M u^{p-2}(\nabla_j u_i) u^j u^i dV + \frac{1}{2n}\int_M (u^{p-2} L_{du} K^* - u^p L_{du} K) dV - \frac{p+n-2}{2}\int_M u^{p-3}(u_t u^t)^2 dV - \frac{p+n-2}{2n(n-1)}\int_M u^{p-3} u_t u^t K^* dV + \frac{p+n-2}{2n(n-1)}\int_M u^{p-1} u_t u^t K dV + \int_M u^{p+1} P_{ji} P^{ji} dV = 0,$$

$$(3.10)\qquad \int_M u^{p-1} K_{ji} u^j u^i dV - \frac{p+n-2}{n}\int_M u^{p-1}(\Delta u)^2 dV - \frac{(p-1)(p+n-2)}{n}\int_M u^{p-2} u_t u^t \Delta u\, dV + \frac{p-1}{n(n-1)}\int_M u^{p-1} u_t u^t K dV - \frac{p-1}{2n(n-1)}\int_M (u^{p-2} L_{du} K^* - u^p L_{du} K) dV + \int_M u^{p+1} P_{ji} P^{ji} dV = 0$$

and

$$(3.11)\qquad \int_M u^{p-1} K_{ji} u^j u^i dV + \frac{p+n-2}{n}\int_M u^{p-1} u^i \nabla_i(\Delta u) dV + \frac{p-1}{n(n-1)}\int_M u^{p-1} u_t u^t K dV - \frac{p-1}{2n(n-1)}\int_M (u^{p-2} L_{du} K^* - u^p L_{du} K) dV + \int_M u^{p+1} P_{ji} P^{ji} dV = 0.$$

In particular, if $p = -n+2$ then

$$(3.12)\qquad \int_M u^{-n+1} G_{ji} u^j u^i dV + \frac{1}{2n}\int_M (u^{-n} L_{du} K^* - u^{-n+2} L_{du} K) dV + \int_M u^{-n+3} P_{ji} P^{ji} dV = 0,$$

and if $p = 1$ then

$$(3.13)\qquad \int_M K_{ji} u^j u^i dV - \frac{n-1}{n}\int_M (\Delta u)^2 dV + \int_M u^2 P_{ji} P^{ji} dV = 0$$

and

$$(3.14)\qquad \int_M K_{ji} u^j u^i dV + \frac{n-1}{n}\int_M u^i \nabla_i(\Delta u) dV + \int_M u^2 P_{ji} P^{ji} dV = 0.$$

Proof. We first have

$$\begin{aligned}\nabla_j(u^{p-1}u_i\nabla^j u^i)&=(p-1)u^{p-2}(\nabla^j u^i)u_j u_i+u^{p-1}(\nabla_j u_i)(\nabla^j u^i)+u^{p-1}u_i\nabla_j\nabla^j u^i\\&=(p-1)u^{p-2}(\nabla^j u^i)u_j u_i+u^{p-1}(\nabla_j u_i)(\nabla^j u^i)\\&\quad+u^{p-1}K_{ji}u^j u^i+u^{p-1}u_i\nabla^i(\Delta u)\,,\end{aligned}$$

where we have used (3.6), that is,

$$\nabla_j\nabla^j u^i=K_t{}^i u^t+\nabla^i\Delta u\,,$$

and consequently, integrating over M, we have

$$\begin{aligned}\int_M u^{p-1}(\nabla_j u_i)(\nabla^j u^i)dV+(p-1)\int_M u^{p-2}(\nabla^j u^i)u_j u_i dV&\\+\int_M u^{p-1}K_{ji}u^j u^i dV&\\+\int_M u^{p-1}u_i\nabla^i(\Delta u)dV=0\,.&\end{aligned}\tag{3.15}$$

Similarly, computing $\nabla_i(u^{p-1}u^i\Delta u)$ and integrating over M, we have

$$\begin{aligned}(p-1)\int_M u^{p-2}u_t u^t\Delta u\,dV+\int_M u^{p-1}(\Delta u)^2 dV&\\+\int_M u^{p-1}u^i\nabla_i(\Delta u)dV=0\,.&\end{aligned}\tag{3.16}$$

By using (2.18), (3.15) and (3.16), we get

$$\begin{aligned}\int_M u^{p+1}P_{ji}P^{ji}dV&=\int_M u^{p-1}(\nabla_j u_i)(\nabla^j u^i)dV-\frac{1}{n}\int_M u^{p-1}(\Delta u)^2 dV\\&=-(p-1)\int_M u^{p-2}(\nabla^j u^i)u_j u_i dV-\int_M u^{p-1}K_{ji}u^j u^i dV\\&\quad+\frac{p-1}{n}\int_M u^{p-2}u_t u^t\Delta u\,dV-\frac{n-1}{n}\int_M u^{p-1}u^i\nabla_i(\Delta u)dV\,.\end{aligned}\tag{3.17}$$

On the other hand, from (2.12), we have

$$\Delta u=\frac{1}{2(n-1)}(u^{-1}K^*-uK)+\frac{n}{2}u^{-1}u_t u^t\,,\tag{3.18}$$

from which

$$\begin{aligned}\nabla_i(\Delta u)&=-\frac{1}{2(n-1)}(u^{-2}u_i K^*+u_i K)+\frac{1}{2(n-1)}(u^{-1}\nabla_i K^*-u\nabla_i K)\\&\quad-\frac{n}{2}u^{-2}u_i u_t u^t+nu^{-1}(\nabla_i u_t)u^t\,.\end{aligned}\tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17) and using

$$K_{ji}=G_{ji}+\frac{K}{n}g_{ji}\,,$$

we have (3.9).

Substituting

$$u^{p-3}u_t u^t K^* = 2(n-1)u^{p-2}u_t u^t \Delta u - n(n-1)u^{p-3}(u_t u^t)^2 + u^{p-1}u_t u^t K$$

which can be obtained from (3.18) into

$$\begin{aligned}\int_M u^{p-1}u^i \nabla_i(\Delta u)dV &= n\int_M u^{p-2}(\nabla_j u_i)u^j u^i dV \\ &-\frac{1}{2(n-1)}\int_M u^{p-3}u_t u^t K^* dV - \frac{1}{2(n-1)}\int_M u^{p-1}u_t u^t K dV \\ &+\frac{1}{2(n-1)}\int_M (u^{p-2}L_{du}K^* - u^p L_{du}K)dV - \frac{n}{2}\int_M u^{p-3}(u_t u^t)^2 dV\end{aligned}$$

which follows from (3.19), we have

$$\begin{aligned}&\int_M u^{p-1}u^i \nabla_i(\Delta u)dV \\ &\quad = n\int_M u^{p-2}(\nabla_j u_i)u^j u^i dV - \int_M u^{p-2}u_t u^t \Delta u\, dV \\ &\quad -\frac{1}{n-1}\int_M u^{p-1}u_t u^t K dV + \frac{1}{2(n-1)}\int_M (u^{p-2}L_{du}K^* - u^p L_{du}K)dV,\end{aligned}$$

and consequently, by using

$$\int_M u^{p-1}u^i \nabla_i(\Delta u)dV = -(p-1)\int_M u^{p-2}u_t u^t \Delta u\, dV - \int_M u^{p-1}(\Delta u)^2 dV \tag{3.20}$$

which is equivalent to (3.16), we obtain

$$\begin{aligned}\int_M u^{p-2}(\nabla_j u_i)u^j u^i dV &= -\frac{p-2}{n}\int_M u^{p-2}u_t u^t \Delta u\, dV \\ &-\frac{1}{n}\int_M u^{p-1}(\Delta u)^2 dV + \frac{1}{n(n-1)}\int_M u^{p-1}u_t u^t K dV \\ &-\frac{1}{2n(n-1)}\int_M (u^{p-2}L_{du}K^* - u^p L_{du}K)dV.\end{aligned} \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.17), we get (3.10). From (3.16) and (3.20), we have (3.11) immediately.

LEMMA 6. *If a compact orientable Riemannian manifold (M, g) admits a conformal change of metric $g^* = e^{2\rho}g$, then, for any real number p,*

$$\begin{aligned}&\int_M (u^{p-3}G^*_{ji}G^{*ji} - u^{p+1}G_{ji}G^{ji})dV \\ &\quad +2(n-2)p\int_M u^{p-1}G_{ji}u^j u^i dV + \frac{(n-2)^2}{n}\int_M u^p L_{du}K dV \\ &\quad -(n-2)^2\int_M u^{p+1}P_{ji}P^{ji}dV = 0.\end{aligned} \tag{3.22}$$

In particular, if $p=-n+2$ then

(3.23)
$$\int_M (u^{-n-1}G^*_{ji}G^{*ji}-u^{-n+3}G_{ji}G^{ji})dV$$
$$-2(n-2)^2\int_M u^{-n+1}G_{ji}u^ju^idV+\frac{(n-2)^2}{n}\int_M u^{-n+2}L_{du}KdV$$
$$-(n-2)^2\int_M u^{-n+3}P_{ji}P^{ji}dV=0\,,$$

and if $p=0$ then

(3.24)
$$\int_M (u^{-3}G^*_{ji}G^{*ji}-uG_{ji}G^{ji})dV$$
$$+\frac{(n-2)^2}{n}\int_M L_{du}KdV-(n-2)^2\int_M uP_{ji}P^{ji}dV=0\,.$$

Proof. Using (2.20) and (2.23), we have

(3.25)
$$\int_M (u^{p-3}G^*_{ji}G^{*ji}-u^{p+1}G_{ji}G^{ji})dV$$
$$=2(n-2)\int_M u^pG_{ji}\nabla^ju^idV+(n-2)^2\int_M u^{p+1}P_{ji}P^{ji}dV\,.$$

On the other hand, calculating $\nabla^j(u^pG_{ji}u^i)$ and using

$$\nabla^jG_{ji}=\frac{n-2}{2n}\nabla_iK\,,$$

we have

$$\nabla^j(u^pG_{ji}u^i)=pu^{p-1}G_{ji}u^ju^i+\frac{n-2}{2n}u^pu^i\nabla_iK+u^pG_{ji}\nabla^ju^i\,,$$

and consequently, integrating over M, we have

(3.26)
$$\int_M u^pG_{ji}\nabla^ju^idV=-p\int_M u^{p-1}G_{ji}u^ju^idV-\frac{n-2}{2n}\int_M u^pu^i\nabla_iKdV\,.$$

Substituting this into (3.25), we have (3.22) to be proved.

LEMMA 7. *If a compact orientable Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$, then*

(3.27)
$$\int_M (u^{-n-1}G^*_{ji}G^{*ji}-u^{-n+3}G_{ji}G^{ji})dV$$
$$+\frac{(n-2)^2}{n}\int_M u^{-n}L_{du}K^*dV+(n-2)^2\int_M u^{-n+3}P_{ji}P^{ji}dV=0\,.$$

Proof. Adding (3.12) $\times 2(n-2)^2$ and (3.23), we have (3.27).

LEMMA 8. *If a compact orientable Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$, then, for any real number p,*

(3.28)
$$\int_M (u^{p-3}Z^*{}_{kjih}Z^{*kjih} - u^{p+1}Z_{kjih}Z^{kjih})dV$$
$$+8p\int_M u^{p-1}G_{ji}u^j u^i dV + \frac{4(n-2)}{n}\int_M u^p L_{du}K dV$$
$$-4(n-2)\int_M u^{p+1}P_{ji}P^{ji}dV = 0 .$$

In particular, if $p=-n+2$ then

(3.29)
$$\int_M (u^{-n-1}Z^*{}_{kjih}Z^{*kjih} - u^{-n+3}Z_{kjih}Z^{kjih})dV$$
$$-8(n-2)\int_M u^{-n+1}G_{ji}u^j u^i dV + \frac{4(n-2)}{n}\int_M u^{-n+2}L_{du}K dV$$
$$-4(n-2)\int_M u^{-n+3}P_{ji}P^{ji}dV = 0 ,$$

and if $p=0$ then

(3.30)
$$\int_M (u^{-3}Z^*{}_{kjih}Z^{*kjih} - uZ_{kjih}Z^{kjih})dV$$
$$+\frac{4(n-2)}{n}\int_M L_{du}K dV - 4(n-2)\int_M uP_{ji}P^{ji}dV = 0 .$$

Proof. Using (2.21) and (2.23), we have

(3.31)
$$\int_M (u^{p-3}Z^*{}_{kjih}Z^{*kjih} - u^{p+1}Z_{kjih}Z^{kjih})dV$$
$$-8\int_M u^p G_{ji}\nabla^j u^i dV - 4(n-2)\int_M u^{p+1}P_{ji}P^{ji}dV = 0 .$$

Substituting (3.26) into (3.31), we have (3.28).

LEMMA 9. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$, then*

(3.32)
$$\int_M (u^{-n-1}Z^*{}_{kjih}Z^{*kjih} - u^{-n+3}Z_{kjih}Z^{kjih})dV$$
$$+\frac{4(n-2)}{n}\int_M u^{-n}L_{du}K^* dV + 4(n-2)\int_M u^{-n+3}P_{ji}P^{ji}dV = 0 .$$

Proof. (3.32) follows from (3.12) and (3.29).

LEMMA 10. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$, then, for any real number p,*

(3.33)
$$\int_M (u^{p-3}W^*{}_{kjih}W^{*kjih} - u^{p+1}W_{kjih}W^{kjih})dV$$
$$+8\{a+(n-2)b\}^2 p\int_M u^{p-1}G_{ji}u^j u^i dV$$

$$+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M u^p L_{du} K dV$$

$$-4(n-2)\{a+(n-2)b\}^2\int_M u^{p+1} P_{ji} P^{ji} dV=0 .$$

In particular, if $p=-n+2$ then

(3.34)
$$\int_M (u^{-n-1} W^*{}_{kjih} W^{*kjih} - u^{-n+3} W_{kjih} W^{kjih}) dV$$
$$-8(n-2)\{a+(n-2)b\}^2\int_M u^{-n+1} G_{ji} u^j u^i dV$$
$$+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M u^{-n+2} L_{du} K dV$$
$$-4(n-2)\{a+(n-2)b\}^2\int_M u^{-n+1} P_{ji} P^{ji} dV=0 ,$$

and if $p=0$ then

(3.35)
$$\int_M (u^{-3} W^*{}_{kjih} W^{*kjih} - u W_{kjih} W^{kjih}) dV$$
$$+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M L_{du} K dV$$
$$-4(n-2)\{a+(n-2)b\}^2\int_M u P_{ji} P^{ji} dV=0 .$$

Proof. Using (2.22) and (2.23), we have

(3.36)
$$\int_M (u^{p-3} W^*{}_{kjih} W^{*kjih} - u^{p+1} W_{kjih} W^{kjih}) dV$$
$$-8\{a+(n-2)b\}^2\int_M u^p G_{ji} \nabla^j u^i dV$$
$$-4(n-2)\{a+(n-2)b\}^2\int_M u^{p+1} P_{ji} P^{ji} dV=0 .$$

Substituting (3.26) into (3.36), we have (3.33).

LEMMA 11. *If a compact orientable Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$, then*

(3.37)
$$\int_M (u^{-n-1} W^*{}_{kjih} W^{*kjih} - u^{-n+3} W_{kjih} W^{kjih}) dV$$
$$+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M u^{-n} L_{du} K^* dV$$
$$+4(n-2)\{a+(n-2)b\}^2\int_M u^{-n+3} P_{ji} P^{ji} dV=0 .$$

Proof. (3.37) follows from (3.12) and (3.34).

LEMMA 12. *Suppose that a Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$ and f and f^* are non-negative functions on M such that*

$$u^p f=\{u^q+(u^r-1)\varphi\}f^*, \tag{3.38}$$

where p is a real number such that $p\leqq 4$, q and r non-negative numbers and φ a non-negative function on M. Then

$$(u^{-n-1}f^*-u^{-n+3}f)-(u^{-3}f^*-uf)\geqq 0. \tag{3.39}$$

Proof. We have

$$\begin{aligned}&(u^{-n-1}f^*-u^{-n+3}f)-(u^{-3}f^*-uf)\\ &\quad=u^{-n-1}(1-u^{n-2})(f^*-u^4 f)\\ &\quad=u^{-n-1}(1-u^{n-2})(f^*-u^q f^*-u^p f+u^q f^*+u^p f-u^4 f)\\ &\quad=u^{-n-1}(1-u^{n-2})(1-u^q)f^*-u^{-n-1+p}(1-u^{n-2})(1-u^{4-p})f\\ &\qquad+u^{-n-1}(1-u^{n-2})(1-u^r)\varphi f^*.\end{aligned}$$

We can easily prove that

$$(1-u^{n-2})(1-u^q)\geqq 0, \qquad (1-u^{n-2})(1-u^{4-p})\geqq 0, \qquad (1-u^{n-2})(1-u^r)\geqq 0,$$

and consequently that (3.39) holds.

§ 4. Propositions.

PROPOSITION 1. *If a compact Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-constant function u on M, then*

$$(\nabla_j u_i)(\nabla^j u^i)\geqq \frac{1}{n}(\Delta u)^2, \tag{4.1}$$

equality holding if and only if (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then the equality holds if and only if (M, g) is isometric to a sphere.

Proof. (4.1) is equivalent to

$$\left(\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}\right)\left(\nabla^j u^i-\frac{1}{n}\Delta u g^{ji}\right)\geqq 0,$$

and consequently equality in (4.1) holds if and only if

$$\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}=0,$$

that is, by Theorem O, if and only if (M, g) is conformal to a sphere. The latter part of this proposition follows from Theorem P.

PROPOSITION 2. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a non-constant function u on M such that*

$$K_i{}^h u^i + \frac{n-1}{n} \nabla^h \Delta u = 0, \tag{4.2}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. From (3.5), we have

$$g^{ji}\nabla_j \nabla_i u^h - K_i{}^h u^i - \nabla^h \Delta u = 0 .$$

Adding (4.2)×2 and this relation, we have

$$g^{ji}\nabla_j \nabla_i u^h + K_i{}^h u^i + \frac{n-2}{n} \nabla^h \Delta u = 0 .$$

Thus, by the Remark to Lemma 1, we see that the vector field u^h on M defines an infinitesimal conformal transformation and consequently that

$$\nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0 .$$

Thus, by Theorem O, (M, g) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

PROPOSITION 3. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a non-homothetic conformal change of metric $g^* = e^{2\rho} g$ such that*

$$K_i{}^h u^i + \frac{n-1}{n} \nabla^h \Delta u = 0,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. This is an immediate consequence of Proposition 2. But, an another proof is as follows. From (3.14) and (4.2), we have $P_{ji}=0$, that is,

$$\nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

PROPOSITION 4. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a non-constant function u on M such that*

$$\int_M K_{ji} u^j u^i dV \geqq \frac{n-1}{n} \int_M (\Delta u)^2 dV, \tag{4.3}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. From (3.8) and (4.3), we have

$$\nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0 ,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

PROPOSITION 5. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a non-homothetic conformal change of metric $g^* = e^{2\rho} g$ such that*

$$\int_M K_{ji} u^j u^i dV \geqq \frac{n-1}{n} \int_M (\Delta u)^2 dV ,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K = 0$ or $K =$ constant, then (M, g) is isometric to a sphere.

Proof. This is an immediate consequence of Proposition above. But, we can give an another proof. From (3.13) and the above relation, we find $P_{ji} = 0$, that is,

$$\nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0 ,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

(For Propositions 2~5, see Yano and Hiramatu [12].)

PROPOSITION 6. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a non-homothetic conformal change of metric $g^* = e^{2\rho} g$ such that*

$$\text{(4.4)} \qquad \int_M u^{-n+1} G_{ji} u^j u^i dV + \frac{1}{2n} \int_M (u^{-n} L_{du} K^* - u^{-n+2} L_{du} K) dV \geqq 0 ,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K = 0$ or $K =$ constant, then (M, g) is isometric to a sphere.

Proof. By using (3.12) and (4.4), we have $P_{ji} = 0$, that is,

$$\nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0 ,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. We have the latter part of the proposition by Theorem P.

The latter part of the proposition above is a generalization of Theorems A and H.

PROPOSITION 7. *If a compact orientable Riemannian manifold (M, g) of dimension $n > 2$ admits a non-homothetic conformal change of metric $g^* = e^{2\rho} g$ such that*

$$\int_M (u^{-3}G^*{}_{ji}G^{*ji} - uG_{ji}G^{ji})dV + \frac{(n-2)^2}{n}\int_M L_{du}KdV \leqq 0\,, \tag{4.5}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. By using (3.24) and (4.5), we have $P_{ji}=0$, that is,

$$\nabla_j u_i - \frac{1}{n}\Delta u g_{ji} = 0\,,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. Using Theoroem P, we can prove the latter part of the proposition.

The first part of Proposition 7 is a generalization of Theorem B because of

$$\int_M (\Delta u)KdV = -\int_M L_{du}KdV\,,$$

and the latter part a generalization of Theorem C.

PROPOSITION 8. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M (u^{-n-1}G^*{}_{ji}G^{*ji} - u^{-n+3}G_{ji}G^{ji})dV + \frac{(n-2)^2}{n}\int_M u^{-n}L_{du}K^*dV \geqq 0\,, \tag{4.6}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. This follows from (3.27) and Theorems O and P.

PROPOSITION 9. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M (u^{-3}Z^*{}_{kjih}Z^{*kjih} - uZ_{kjih}Z^{kjih})dV + \frac{4(n-2)}{n}\int_M L_{du}KdV \leqq 0, \tag{4.7}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. This follows from (3.30) and Theorems O and P.

The first part of this proposition is a generalization of Theorem D and the latter part is a generalization of Theorem E.

PROPOSITION 10. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M (u^{-n-1}Z^*{}_{kjih}Z^{*kjih} - u^{-n+3}Z_{kjih}Z^{kjih})dV + \frac{4(n-2)}{n}\int_M u^{-n}L_{du}K^*dV \geqq 0\,, \tag{4.8}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. This follows from (3.32) and Theorems O and P.

PROPOSITION 11. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M (u^{-3}W^*_{kjih}W^{*kjih}-uW_{kjih}W^{kjih})dV + \frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M L_{du}KdV \leqq 0, \quad a+(n-2)b\neq 0, \tag{4.9}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. This follows from (3.35) and Theorems O and P.

The first part of Proposition 11 generalizes Theorem F and the latter part generalizes Theorem G.

PROPOSITION 12. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M (u^{-n-1}W^*_{kjih}W^{*kjih}-u^{-n+3}W_{kjih}W^{kjih})dV + \frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M u^{-n}L_{du}K^*dV \geqq 0, \quad a+(n-2)b\neq 0, \tag{4.10}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. This follows from (3.37) and Theorems O and P.

PROPOSITION 13. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$u^pG_{ji}G^{ji}=\{u^q+(u^r-1)\varphi\}G^*_{ji}G^{*ji} \tag{4.11}$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0, \tag{4.12}$$

where p is a real number such that $p\leqq 4$, q and r non-negative numbers and φ a *non-negative function on M, then (M, g) is conformal to a sphere. If moreover* $L_{du}K=0$ *or* $K=constant$, *then (M, g) is isometric to a sphere.*

Proof. Subtracting (3.24) from (3.27), we obtain

$$\int_M \{(u^{-n-1}G^*_{ji}G^{*ji}-u^{-n+3}G_{ji}G^{ji})-(u^{-3}G^*_{ji}G^{*ji}-uG_{ji}G^{ji})\}dV \tag{4.13}$$
$$+\frac{(n-2)^2}{n}\int_M (u^{-n}L_{du}K^*-L_{du}K)dV$$
$$+(n-2)^2\int_M (u^{-n+3}+u)P_{ji}P^{ji}dV=0\,.$$

By Lemma 12, from (4.11), (4.12) and (4.13), we have $P_{ji}=0$, that is,

$$\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}=0$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. By using Theorem P, we can prove the latter part of this proposition.

The latter part of Proposition 13 is a generalization of Theorem L.

COROLLARY 1. *If a compact orientable Riemannian manifold (M, g) of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$G_{ji}G^{ji}=G^*_{ji}G^{*ji} \tag{4.14}$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0\,,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ *or* $K=constant$, *then (M, g) is isometric to a sphere.*

Proof. Putting $p=q=r=0$ in (4.11), we have (4.14), and consequently this corollary follows immediately from Proposition 13.

The latter part of this corollary is a generalization of Theorem I.

PROPOSITION 14. *If a compact orientable Riemannian manifold (M, g) of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$u^pZ_{kjih}Z^{kjih}=\{u^q+(u^r-1)\varphi\}Z^*_{kjih}Z^{*kjih} \tag{4.15}$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0\,,$$

where p, q, r *and* φ *are the same as in Proposition 13, then (M, g) is conformal to a sphere. If moreover* $L_{du}K=0$ *or* $K=constant$, *then (M, g) is isometric to a* *sphere.*

Proof. Subtracting (3.30) from (3.32), we have

$$\begin{aligned}(4.16)\qquad &\int_M \{(u^{-n-1}Z^*_{kjih}Z^{*kjih}-u^{-n+3}Z_{kjih}Z^{kjih})\\ &\qquad -(u^{-3}Z^*_{kjih}Z^{*kjih}-uZ_{kjih}Z^{kjih})\}dV\\ &+\frac{4(n-2)}{n}\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\\ &+4(n-2)\int_M (u^{-n+3}+u)P_{ji}P^{ji}dV=0\,.\end{aligned}$$

Using Lemma 12, (4.12), (4.15) and (4.16), we have $P_{ji}=0$, that is,

$$\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}=0\,,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. By using Theorem P, we can prove the latter part of the proposition.

The latter part of Proposition 14 is a generalization of Theorem M.

COROLLARY 2. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$(4.17)\qquad Z^*_{kjih}Z^{*kjih}=Z_{kjih}Z^{kjih}$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0\,,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=constant$, then (M, g) is isometric to a sphere.

Proof. Putting $p=q=r=0$ in (4.15), we get (4.17), and consequently Corollary 2 follows immediately from Proposition 14.

The latter part of Corollary 2 generalizes Theorem J.

PROPOSITION 15. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$(4.18)\qquad u^p W_{kjih}W^{kjih}=\{u^q+(u^r-1)\varphi\}W^*_{kjih}W^{*kjih}\,,$$

$$a+(n-2)b\neq 0$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0\,,$$

where p, q, r and φ are the same as in Proposition 13, then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. Subtracting (3.35) from (3.37), we have

$$\int_M \{(u^{-n-1}W^*{}_{kjih}W^{*kjih} - u^{-n+3}W_{kjih}W^{kjih}) \\ -(u^{-3}W^*{}_{kjih}W^{*kjih} - uW_{kjih}W^{kjih})\}dV \\ +\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M (u^{-n}L_{du}K^* - L_{du}K)dV \\ +4(n-2)\{a+(n-2)b\}^2\int_M (u^{-n+3}+u)P_{ji}P^{ji}dV=0\,. \tag{4.19}$$

By using Lemma 12, from (4.12), (4.18) and (4.19), we have $P_{ji}=0$, that is,

$$\nabla_j u_i - \frac{1}{n}\Delta u g_{ji} = 0\,,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. By using Theorem P, we can prove the latter part of Proposition 15.

The latter part of Proposition 15 is a generalization of Theorem N.

COROLLARY 3. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$W^*{}_{kjih}W^{*kjih} = W_{kjih}W^{kjih}\,, \qquad a+(n-2)b \neq 0 \tag{4.20}$$

and

$$\int_M (u^{-n}L_{du}K^* - L_{du}K)dV \geqq 0\,,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. Putting $p=q=r=0$ in (4.18), we get (4.20), and consequently Corollary 3 follows immediately from Proposition 15.

The latter part of Corollary 3 is a generalization of Theorem K.

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