J. Math. Soc. Japan
Vol. 23, No. 1, 1971

On conformal diffeomorphisms of complete Riemannian manifolds with parallel Ricci tensor

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(Received Oct. 6, 1969)
(Revised July 8, 1970)

We shall assume, throughout this paper, that Riemannian manifolds are connected and of dimension >2, their metrics are positive-definite, and manifolds and diffeomorphisms are of differentiability class C^{∞}.

Let M and M^* be Riemannian manifolds with metric tensor g and g^* respectively, and f a diffeomorphism of M onto M^*. If the induced metric f^*g^* of g^* by f is related to g by

$$f^*g^* = \rho^{-2}g,$$

then f is called a *conformal diffeomorphism* of M onto M^*, where ρ is a positively valued scalar field on M. If the scalar field ρ satisfies the equation

$$\nabla\nabla\rho = \phi g,$$

∇ indicating covariant differentiation and ϕ being a scalar field, then the diffeomorphism f is called a *concircular* one, which carries Riemannian circles of M to those of M^*. If ρ is a constant or equal to 1, then f is said to be *homothetic* or *isometric*, respectively.

THEOREM 1. *Let M and M^* be complete Riemannian manifolds with parallel Ricci tensor. If there is a non-isometric conformal diffeomorphism f of M onto M^*, then f is homothetic or both the manifolds M and M^* are isometric to the sphere.*

THEOREM 2. *Let M be a complete Riemannian manifold with parallel Ricci tensor. If M admits a non-isometric conformal transformation f, then occurs one of the following two cases:*

(1) *f is homothetic and M is a Euclidean space, or*

(2) *M is isometric to the sphere.*

The first and principal part of Theorem 1 was proved by N. Tanaka [6] by a group-theoretic method under some additional conditions on Ricci tensors of M and M^*. Then T. Nagano [5] dealt with the cases excepted by Tanaka, showed that f is properly conformal only in the case where both M and M^* are Einstein manifolds of positive curvature, and completed Theorem

1 as stated above by use of a famous theorem due to K. Yano and T. Nagano [11]:

THEOREM A. *If a complete Einstein manifold M admits a global one-parameter group of non-homothetic conformal transformations, then M is isometric to the sphere.*

A generalization of Theorem A to a conformal diffeomorphism is stated as follows:

THEOREM B. *Let M and M^* be complete Einstein manifolds. If there is a non-homothetic conformal diffeomorphism of M onto M^*, then both M and M^* are isometric to the sphere.*

While T. Nagano stated this fact on the way of the reduction of Theorem 1 to Theorem A, an immediate proof was given by S. Ishihara and Y. Tashiro [2] and Y. Tashiro [7] since a conformal diffeomorphism between Einstein manifolds is concircular, cf. [8, 9].

As a generalization of Theorem 1 or A, the present authors [10] have recently proved the following.

THEOREM C. *If a complete and reducible Riemannian manifold admits a global one-parameter group of non-isometric conformal transformations, then the group is homothetic and the manifold is a Euclidean space.*

It has remained open, however, to give an alternative proof of Theorem 1 by tensor calculus. The purpose of the present paper is to prove Theorem 1 by an elementary method similar to that employed in [10].

§1. Let f be a conformal diffeomorphism of M into M^*, and denote the metric tensors g and f^*g^* by components $g_{\mu\lambda}$ and $g^*_{\mu\lambda}$. Then they are related by the equation

$$g^*_{\mu\lambda}=\rho^{-2}g_{\mu\lambda}\,. \tag{1.1}$$

We denote the Christoffel symbol, the curvature tensor, the Ricci tensor and the scalar curvature of g by $\left\{ {\kappa \atop \mu\lambda} \right\}$, $K_{\nu\mu\lambda}{}^{\kappa}$, $K_{\mu\lambda}$ and κ respectively, where the scalar curvature κ is defined by

$$\kappa=\frac{1}{n(n-1)}K_{\mu\lambda}g^{\mu\lambda}\,, \tag{1.2}$$

and indicate the corresponding quantities of g^* by asterisking. Then we obtain the transformation formulas

$$\left\{ {\kappa \atop \mu\lambda} \right\}^*=\left\{ {\kappa \atop \mu\lambda} \right\}-\frac{1}{\rho}(\delta^{\kappa}_{\mu}\rho_{\lambda}+\delta^{\kappa}_{\lambda}\rho_{\mu}-g_{\mu\lambda}\rho^{\kappa})\,, \tag{1.3}$$

$$K^{*}_{\nu\mu\lambda}{}^{\kappa}=K_{\nu\mu\lambda}{}^{\kappa}+\frac{1}{\rho}(\delta^{\kappa}_{\nu}\nabla_{\mu}\rho_{\lambda}-\delta^{\kappa}_{\mu}\nabla_{\nu}\rho_{\lambda}+g_{\mu\lambda}\nabla_{\nu}\rho^{\kappa}-g_{\nu\lambda}\nabla_{\mu}\rho^{\kappa})$$
$$-\frac{1}{\rho^{2}}\rho_{\omega}\rho^{\omega}(\delta^{\kappa}_{\nu}g_{\mu\lambda}-\delta^{\kappa}_{\mu}g_{\nu\lambda}), \tag{1.4}$$

$$K^{*}_{\mu\lambda}=K_{\mu\lambda}+\frac{1}{\rho}(n-2)\nabla_{\mu}\rho_{\lambda}+\frac{1}{\rho}g_{\mu\lambda}\nabla_{\kappa}\rho^{\kappa}-\frac{1}{\rho^{2}}(n-1)\rho_{\kappa}\rho^{\kappa}g_{\mu\lambda}, \tag{1.5}$$

$$\kappa^{*}=\rho^{2}\kappa+\frac{2}{n}\rho\nabla_{\kappa}\rho^{\kappa}-\rho_{\kappa}\rho^{\kappa}, \tag{1.6}$$

where we have put $\rho_{\lambda}=\nabla_{\lambda}\rho$.

We have from (1.6)

$$\nabla_{\kappa}\rho^{\kappa}=\frac{n}{2\rho}(\kappa^{*}-\rho^{2}\kappa+\rho_{\kappa}\rho^{\kappa}).$$

Substituting this into (1.5) and putting

$$L_{\mu\lambda}=K_{\mu\lambda}-\frac{n}{2}\kappa g_{\mu\lambda}\quad\text{and}\quad L^{*}_{\mu\lambda}=K^{*}_{\mu\lambda}-\frac{n}{2}\kappa^{*}g^{*}_{\mu\lambda}, \tag{1.7}$$

we obtain the equation

$$L^{*}_{\mu\lambda}=L_{\mu\lambda}+\frac{1}{\rho}(n-2)\nabla_{\mu}\rho_{\lambda}-\frac{1}{2\rho^{2}}(n-2)g_{\mu\lambda}\rho_{\kappa}\rho^{\kappa}. \tag{1.8}$$

Differentiating covariantly this equation with respect to $\left\{{\kappa \atop \mu\lambda}\right\}^{*}$ and substituting (1.3), we obtain the equation

$$\nabla^{*}_{\nu}L^{*}_{\mu\lambda}=\nabla_{\nu}L_{\mu\lambda}+\frac{1}{2\rho^{2}}(n-2)[\nabla_{\nu}\nabla_{\mu}\nabla_{\lambda}\rho^{2}-g_{\mu\lambda}\nabla_{\nu}(\rho_{\kappa}\rho^{\kappa})$$
$$-g_{\nu\lambda}\nabla_{\mu}(\rho_{\kappa}\rho^{\kappa})-g_{\nu\mu}\nabla_{\lambda}(\rho_{\kappa}\rho^{\kappa})]+\frac{1}{2\rho^{2}}[2L_{\mu\lambda}\nabla_{\nu}\rho^{2}$$
$$+L_{\nu\mu}\nabla_{\lambda}\rho^{2}+L_{\nu\lambda}\nabla_{\mu}\rho^{2}-(g_{\nu\lambda}L_{\mu\kappa}+g_{\nu\mu}L_{\lambda\kappa})\nabla^{\kappa}\rho^{2}]. \tag{1.9}$$

Now we assume that the Ricci tensors $K_{\mu\lambda}$ and $K^{*}_{\mu\lambda}$ are parallel, $\nabla_{\nu}K_{\mu\lambda}=\nabla^{*}_{\nu}K^{*}_{\mu\lambda}=0$. Then the scalar curvatures κ and κ^{*} are constants and the tensors $L_{\mu\lambda}$ and $L^{*}_{\mu\lambda}$ are also parallel. Hence the scalar field ρ satisfies the differential equation

$$\nabla_{\nu}\nabla_{\mu}\nabla_{\lambda}\rho^{2}=g_{\mu\lambda}\nabla_{\nu}(\rho_{\kappa}\rho^{\kappa})+g_{\nu\lambda}\nabla_{\mu}(\rho_{\kappa}\rho^{\kappa})+g_{\nu\mu}\nabla_{\lambda}(\rho_{\kappa}\rho^{\kappa})$$
$$-\frac{1}{n-2}[2L_{\mu\lambda}\nabla_{\nu}\rho^{2}+L_{\nu\lambda}\nabla_{\mu}\rho^{2}+L_{\nu\mu}\nabla_{\lambda}\rho^{2}$$
$$-(g_{\nu\lambda}L_{\mu\kappa}+g_{\nu\mu}L_{\lambda\kappa})\nabla^{\kappa}\rho^{2}]. \tag{1.10}$$

Applying Ricci's formula to (1.10) itself and its covariant derivative, we have the equations

$$K_{\nu\mu\lambda}{}^{\kappa}\rho_{\kappa}=\frac{1}{n-2}[L_{\mu\lambda}\rho_{\nu}-L_{\nu\lambda}\rho_{\mu}+(g_{\mu\lambda}L_{\nu\kappa}-g_{\nu\lambda}L_{\mu\kappa})\rho^{\kappa}], \tag{1.11}$$

$$K_{\omega\nu\mu}{}^{\kappa}\nabla_{\lambda}\nabla_{\kappa}\rho^2+K_{\omega\nu\lambda}{}^{\kappa}\nabla_{\mu}\nabla_{\kappa}\rho^2=g_{\omega\lambda}\nabla_{\nu}\nabla_{\mu}(\rho_{\kappa}\rho^{\kappa})-g_{\nu\lambda}\nabla_{\omega}\nabla_{\mu}(\rho_{\kappa}\rho^{\kappa})$$
$$+g_{\omega\mu}\nabla_{\nu}\nabla_{\lambda}(\rho_{\kappa}\rho^{\kappa})-g_{\nu\mu}\nabla_{\omega}\nabla_{\lambda}(\rho_{\kappa}\rho^{\kappa})$$
$$(1.12)\qquad -\frac{1}{n-2}[L_{\omega\lambda}\nabla_{\nu}\nabla_{\mu}\rho^2-L_{\nu\lambda}\nabla_{\omega}\nabla_{\mu}\rho^2+L_{\omega\mu}\nabla_{\nu}\nabla_{\lambda}\rho^2$$
$$-L_{\nu\mu}\nabla_{\omega}\nabla_{\lambda}\rho^2+(g_{\nu\lambda}L_{\mu\kappa}+g_{\nu\mu}L_{\lambda\kappa})\nabla_{\omega}\nabla^{\kappa}\rho^2$$
$$-(g_{\omega\lambda}L_{\mu\kappa}+g_{\omega\mu}L_{\lambda\kappa})\nabla_{\nu}\nabla^{\kappa}\rho^2]\,.$$

We first prove the following.

LEMMA 1. *Let M and M^* be Riemannian manifolds with parallel Ricci tensor and f be a non-homothetic conformal diffeomorphism of M to M^*. If, in addition, M is simply connected, complete and reducible, then the manifold M is the Pythagorean product $M_1\times M_2$ of two Einstein manifolds M_1 and M_2, the scalar curvatures k_1 and k_2 of which have one of the following properties:*

1) *$k_1+k_2=0$ and $k_1\neq 0$, and consequently $k_2\neq 0$,*
2) *one of the manifolds, say M_2, is one-dimensional and $k_1\neq 0$,*
3) *$k_1=k_2=0$, and M itself is an Einstein manifold of zero curvature.*

PROOF. By virtue of de Rham's well known decomposition theorem, the manifold M is a Pythagorean product

$$M=M_1\times M_2\times\cdots\times M_r\,,$$

where $M_1, M_2, \cdots, M_r$ are irreducible complete Riemannian manifolds and called *parts* of M. Suppose that the dimension of each part M_s is n_s $(s=1, 2, \cdots, r)$. Let $(x^{i_1}, x^{i_2}, \cdots, x^{i_r})$ be a local coordinate system in a neighborhood of a point such that (x^{i_s}) is a local coordinate system of each part M_s, and $g_{j_s i_s}$ components of the metric tensor of M_s. In such a separate coordinate system, the matrix $(g_{\mu\lambda})$ of components of the metric tensor g is expressed as the direct sum of the matrices $(g_{j_s i_s})$ of components of the metrics of the parts, and the Christoffel symbol, the curvature tensor and the Ricci tensor have pure components only.

Since the Ricci tensor is parallel, we have

$$(1.13)\qquad \nabla_{k_s}K_{j_s i_s}=0 \qquad (s=1, 2, \cdots, r)$$

on each part M_s. The irreducibility of each part implies that each part M_s is an Einstein manifold, that is,

$$(1.14)\qquad K_{j_s i_s}=(n_s-1)k_s g_{j_s i_s} \qquad (s=1, 2, \cdots, r)\,,$$

where k_s is the constant scalar curvature of M_s. By our definition, the scalar curvatures κ and k_s satisfy the relation

$$(1.15)\qquad n_1(n_1-1)k_1+\cdots+n_r(n_r-1)k_r=n(n-1)\kappa\,.$$

The components of $L_{\mu\lambda}$ on M_s are equal to

$$L_{j_s i_s}=\left[(n_s-1)k_s-\frac{n}{2}\kappa\right]g_{j_s i_s}\,. \tag{1.16}$$

Putting $\lambda=i_s$, $\mu=j_s$ and $\nu=h_t$ $(s\neq t)$ in (1.11), we have

$$L_{j_s i_s}\rho_{h_t}+g_{j_s i_s}L_{h_t k_t}\rho^{k_t}=0 \tag{1.17}$$

and, substituting (1.16),

$$[(n_s-1)k_s+(n_t-1)k_t-n\kappa]g_{j_s i_s}\rho_{h_t}=0\,. \tag{1.18}$$

Since ρ is supposed not to be a constant, there exists a point P where the gradient vector ρ_λ does not vanish and so does not the restriction of ρ_λ on some part, say ρ_{h_1} on $M_1(P)$, through P. Therefore we have the relations

$$(n_1-1)k_1+(n_s-1)k_s=n\kappa \qquad (s=2,\cdots,r)\,, \tag{1.19}$$

from which it follows that

$$(n_2-1)k_2=\cdots=(n_r-1)k_r\,. \tag{1.20}$$

Hence the product $M_2\times\cdots\times M_r$ is an Einstein manifold.

Putting together the product $M_2\times\cdots\times M_r$ into a manifold, we may assume from the biginning that M is the Pythagorean product $M_1\times M_2$ of two Einstein manifolds M_1 and M_2. Then, similarly to (1.15) and (1.19), the scalar curvatures k_1 and k_2 satisfy the relations

$$n_1(n_1-1)k_1+n_2(n_2-1)k_2=n(n-1)\kappa \tag{1.21}$$

and

$$(n_1-1)k_1+(n_2-1)k_2=n\kappa\,. \tag{1.22}$$

It follows from these equations that

$$(n_1-1)(n_2-1)(k_1+k_2)=0\,. \tag{1.23}$$

Therefore, if neither M_1 nor M_2 are one-dimensional, we have

$$k_1+k_2=0\,. \tag{1.24}$$

If $k_1=0$ and consequently $k_2=0$, or if $k_1=0$ and M_2 is one-dimensional, then the manifold M itself is an Einstein one of zero scalar curvature. Q.E.D.

§ 2. We shall derive the differential equations which the scalar field ρ satisfies in the cases 1) and 2) of Lemma 1. For brevity, let (x^i, x^p) be a separate local coordinate system of $M=M_1\times M_2$ $(i,j,k=1,\cdots,n_1;\ p,q=n_1+1,\cdots,n)$ and write k for the non-zero constant scalar curvature k_1 of M_1.

Eliminating k_2 from (1.21) and (1.22), we have the relation

$$nk = (2n_1 - n)k\,. \tag{2.1}$$

The components L_{ji} of $L_{\mu\lambda}$ belonging to M_1 given by (1.16) are then equal to

$$L_{ji} = \frac{n-2}{2} k g_{ji}\,. \tag{2.2}$$

Since

$$\begin{aligned}(n_2-1)k_2 &= n\kappa - (n_1-1)k \\ &= [(2n_1-n)-(n_1-1)]k \\ &= (n_1-n+1)k\end{aligned}$$

by (1.22) and (2.1), the components L_{qp} belonging to M_2 are equal to

$$L_{qp} = -\frac{n-2}{2} k g_{qp} \tag{2.3}$$

including the case $n_2 = 1$.

Putting the indices $\lambda = i$, $\mu = j$, $\nu = k$ and $\omega = p$ in the equation (1.12), we have

$$\nabla_p \nabla_i (\rho_\kappa \rho^\kappa) = 0\,, \tag{2.4}$$

and, putting $\lambda = i$, $\mu = p$, $\nu = j$ and $\omega = q$,

$$g_{ji} \nabla_q \nabla_p (\rho_\kappa \rho^\kappa - k\rho^2) = g_{qp} \nabla_j \nabla_i (\rho_\kappa \rho^\kappa + k\rho^2)\,. \tag{2.5}$$

From the equation (2.4), we may put

$$\rho_\kappa \rho^\kappa = \phi_1 + \phi_2\,, \tag{2.6}$$

where the functions ϕ_1 and ϕ_2 respectively depend on (x^i) and (x^p) only. From the equation (2.5), we may put

$$\begin{aligned}\nabla_j \nabla_i (\rho_\kappa \rho^\kappa + k\rho^2) &= \psi g_{ji}\,, \\ \nabla_q \nabla_p (\rho_\kappa \rho^\kappa - k\rho^2) &= \psi g_{qp}\,,\end{aligned} \tag{2.7}$$

where ψ is a function determined as follows. Differentiating covariantly the first equation of (2.7) with respect to x^p and taking account of (2.1), we have

$$k\nabla_p \nabla_j \nabla_i \rho^2 = (\nabla_p \psi) g_{ji}\,.$$

Comparing this equation with (1.10) for $\lambda = i$, $\mu = j$ and $\nu = p$, we obtain

$$\nabla_p \psi = k\nabla_p (\phi_2 - k\rho^2)\,,$$

and similarly from (1.10) and the second of (2.7)

$$\nabla_i \psi = -k\nabla_i (\phi_1 + k\rho^2)\,.$$

Hence ψ is given in the form

$$\psi = -k(\phi_1 - \phi_2 + k\rho^2 + b)\,,$$

b being a constant.

Therefore the equations (2.7) turn to

$$\nabla_j\nabla_i(\phi_1+k\rho^2)=-k(\phi_1-\phi_2+k\rho^2+b)g_{ji}\,,$$
$$\nabla_q\nabla_p(\phi_2-k\rho^2)=-k(\phi_1-\phi_2+k\rho^2+b)g_{qp}\,. \tag{2.8}$$

These equations show that the function $\phi_1-\phi_2+k\rho^2$ is a special concircular scalar field on each part. Subtracting the equation (1.10) referred to M_1 and M_2 from the covariant derivatives of the equations (2.8), we see that the functions ϕ_1 and ϕ_2 satisfy the equations

$$\nabla_k\nabla_j\nabla_i\phi_1=-k(2g_{ji}\nabla_k\phi_1+g_{kj}\nabla_i\phi_1+g_{ki}\nabla_j\phi_1)\,,$$
$$\nabla_r\nabla_q\nabla_p\phi_2=k(2g_{qp}\nabla_r\phi_2+g_{rq}\nabla_p\phi_2+g_{rp}\nabla_q\phi_2)\,. \tag{2.9}$$

§ **3.** We shall prove the following simple lemmas, the proof of which we have not seen.

LEMMA 2. *Let g and g^* be complete Riemannian metrics on a manifold M, and C a differentiable curve defined on the interval* [0, 1) *in M. If the length of the curve C is bounded with respect to the metric g, then so is it with respect to the metric g^*.*

PROOF. Denote by $|X|$ and $|X|^*$ the lengths of a vector X with respect to g and g^*, respectively. Since the metric g is complete and the curve C is bounded with respect to g, the curve C lies in a compact subset of M. If we make X vary over unit vectors at points in the subset, the length $|X|^*$ is bounded. Hence, for any vector X at any point in the subset, we have the inequality

$$|X|^*\leqq K|X|,$$

K being a constant.

On the other hand, the curve C tends to a point $C(1)$ as t tends to 1. Adding the point $C(1)$ to C, we get the curve $\bar{C}$ which is continuous on the closed interval [0, 1] and differentiable on [0, 1). If the tangent vector field of C is denoted by $\dot{C}(t)$, then the inequality

$$|\dot{C}(t)|^*\leqq K|\dot{C}(t)|$$

is valid in [0, 1). The integral of the right hand side from 0 to 1 is convergent, and so is the integral of the left hand side. Hence the length of $\bar{C}$ is finite with respect to g^*, and the length of the curve C is bounded with respect to g^*.

LEMMA 3. *Let g be a complete metric on a manifold M and f be a diffeomorphism of M. Then the induced metric f^*g is also complete.*

PROOF. Let C be a geodesic with respect to the Riemannian connection of the induced metric f^*g. Then the curve $f(C)$ is a geodesic with respect to the Riemannian connection of the metric g, and has the same arc length as C. Since g is complete, $f(C)$ is extendable for any large value of the arc length and so is C. Hence the induced metric f^*g is complete. Q. E. D.

§ **4.** We are now going to give a

PROOF OF THEOREM 1. If, in addition to the assumptions of the theorem, both M and M^* are irreducible, then they are Einstein manifolds. If there is a non-homothetic conformal diffeomorphism of M onto M^*, then M and M^* are isometric to the sphere by virtue of Theorem B. Hence it suffices for us to show that a conformal diffeomorphism f is homothetic if M or M^* is reducible.

We may assume that the manifolds M and M^* are simply connected, by taking the universal covering manifolds if necessary, and M is reducible. If f is supposed not to be homothetic, then it follows from Lemma 1 that M is the Pythagorean product $M_1 \times M_2$ of Einstein manifolds M_1 and M_2 and occurs one of the following cases:

(I) $k_1 = k = -c^2$ and $k_2 = -k = c^2$,

(II) $k_1 = k = -c^2$ and $n_2 = 1$,

(III) $n_1 = 1$ and $k_2 = -k = c^2$,

(IV) M itself is an Einstein manifold of zero curvature,

where c is a positive constant.

CASE (I). We first suppose that the restriction of the gradient vector field ρ_λ on $M_1(P)$ through a point P does not identically vanish. Then there is a geodesic curve C in $M_1(P)$, along which ρ is not constant. Denote by s the arc length of the curve C. Since $M_1(P)$ is complete, C is infinitely extendable.

The first equations of (2.8) and (2.9) reduce along C to the ordinary differential equations

$$\frac{d^2}{ds^2}(\phi_1 - c^2\rho^2) = c^2(\phi_1 - c^2\rho^2 + A) \tag{4.1}$$

$$\frac{d^3\phi_1}{ds^3} = 4c^2\frac{d\phi_1}{ds}, \tag{4.2}$$

where we have put $-\phi_2(P) + b = A$. The general solution of (4.2) is written in the form

$$\phi_1 = A_1 e^{2cs} + B_1 e^{-2cs} + C_1 \tag{4.3}$$

and consequently the general solution of (4.1) is in the form

$$\rho^2 = \frac{1}{c^2}A_1 e^{2cs} + \frac{1}{c^2}B_1 e^{-2cs} + A_2 e^{cs} + B_2 e^{-cs} + C_2, \tag{4.4}$$

where the coefficients A_1, B_1 and so on are integral constants.

By our assumption, at least one of A_1, B_1, A_2 and B_2 is different from zero. If, for example, $A_1 \neq 0$, then A_1 should be positive because ρ is positively valued. Put $A_1 = a^2$ and take a value s_0 so large that the inequality

$$\rho > \frac{a}{2c} e^{cs}$$

is valid for $s > s_0$. Let C^* be the image of C under the conformal diffeomorphism, and s^* the arc length of C^*. Then s^* is related to s by the differential equation

$$\frac{ds^*}{ds} = \frac{1}{\rho},$$

from which we have the inequality

$$s^* - s_0^* = \int_{s_0}^{s} \frac{1}{\rho} ds < \frac{2}{a} e^{-cs_0},$$

s_0^* being the value corresponding to s_0. Hence the length of C^* is bounded. This contradicts to Lemma 2 because the length of C is unbounded.

We next assume that the components ρ_i identically vanish on $M_1(P)$ through any point P and consequently all over M. Then ρ is independent of points of M_1, and $\phi_1 = 0$. We see from the first equation of (2.8) that

$$\phi_2 + c^2\rho^2 = b. \tag{4.5}$$

Since M_2 is a complete Einstein manifold of positive scalar curvature, it is compact by Myers' theorem [4]. Hence ρ takes the minimal value m at a point of M_2, where $\phi_2 = \rho_p \rho^p$ vanishes. Thus we see $b = c^2 m^2$ and

$$\phi_2 + c^2\rho^2 = c^2 m^2. \tag{4.6}$$

Since $\phi_2 \geqq 0$, it follows from (4.6) that ϕ_2 identically vanishes. Thus ρ should be a constant, and the diffeomorphism f is homothetic.

CASE (II). In the same way as that in Case (I), ρ is independent of points of M_1. We take the arc length s as coordinate of M_2. Then we have

$$\phi_2 = \rho_p \rho^p = \left(\frac{d\rho}{ds}\right)^2,$$

and the equation (4.5) reduces to

$$\left(\frac{d\rho}{ds}\right)^2 + c^2\rho^2 = b.$$

The general solution of this equation is written in the form

$$\rho = A \cos(cs + B),$$

A and B being constants. Since ρ is negative for some value of s, this is a contradiction.

CASE (III). As is seen from the discussion in § 2, the equation (2.2) is valid with $k=-k_2=c^2$ and so are the other equations in this case. Therefore, by taking M_1 itself for the geodesic curve C, the argument in Case (I) is also applicable to this case. Q. E. D.

CASE (IV). Applying the above discussions to the inverse diffeomorphism f^{-1} of M^* onto M, we may consider only the case where M and M^* are Einstein manifolds and M is of zero curvature. Then the diffeomorphism f is homothetic by virtue of Theorem B. Q. E. D.

PROOF OF THEOREM 2. By means of Lemma 3, the theorem follows immediately from Theorem 1 and the fact that a complete Riemannian manifold admitting a non-isometric homothetic transformation is isometric to the Euclidean space [**12**], see also [**1, 3**]. Q. E. D.

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