ON CONFORMAL KILLING SYMMETRIC TENSOR FIELDS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. A vector field on a Riemannian manifold is called *conformal Killing* if it generates one-parameter group of conformal transformation. The class of conformal Killing symmetric tensor fields of an arbitrary rank is a natural generalization of the class of conformal Killing vector fields, and appears in different geometric and physical problems. We prove the statement: A trace-free conformal Killing tensor field is identically zero if it vanishes on some hypersurface. This statement is a basis of the theorem on decomposition of a symmetric tensor field on a compact manifold with boundary to a sum of three fields of special types. We also establish triviality of the space of trace-free conformal Killing tensor fields on some closed manifolds.

1. INTRODUCTION

Conformal transformation of a Riemannian manifold (M, g) is a diffeomorphism $\varphi : M \to M$ such that $\varphi^* g = \lambda g$ for some positive function λ on M. A vector field u on M is called *conformal Killing* if it generates one parameter transformation group of conformal mappings. In local coordinates, a conformal Killing vector field satisfies the equation

$$\frac{1}{2}(\nabla_i u_j + \nabla_j u_i) = v g_{ij} \tag{1.1}$$

for some scalar function v (depending on u). Here $\nabla_i u_j$ denote the covariant derivatives of the field u.

The notion of conformal Killing tensor fields is a generalization of the notion of conformal Killing vector fields to the case of higher rank tensors, and the equation that defines the first class of fields generalizes equation (1.1).

Given a Riemannian manifold (M, g), let $C^\infty(S^m\tau'_M)$ be the space of smooth symmetric covariant tensor field of rank m on M. The first order differential operator

$$d = \sigma\nabla : C^\infty(S^{m-1}\tau'_M) \to C^\infty(S^m\tau'_M),$$

is called the *inner derivative.* Here ∇ denotes the covariant derivative, and σ is the symmetrization. The *divergence*

$$\delta : C^\infty(S^m\tau'_M) \to C^\infty(S^{m-1}\tau'_M)$$

is defined in local coordinates by $(\delta u)_{i_1\dots i_{m-1}} = g^{jk}\nabla_j u_{ki_1\dots i_{m-1}}$. The operators d and $-\delta$ are dual to each other with respect to the natural L^2-product on the space of symmetric tensor fields (see Section 3). We denote by

$$i : C^\infty(S^m\tau'_M) \to C^\infty(S^{m+2}\tau'_M)$$

the following algebraic operator of symmetric multiplication by the metric tensor: $iu = \sigma(g \otimes u)$, and the adjoint of i is denoted by $j : C^\infty(S^{m+2}\tau'_M) \to C^\infty(S^m\tau'_M)$. In local coordinates, $(ju)_{i_1\dots i_m} = g^{jk}u_{jki_1\dots i_m}$. The tensor field ju is called the *trace* of the field u. A symmetric tensor field u is called *trace-free* if its trace is identically equal to zero:

$$ju = 0. \tag{1.2}$$

A *Killing* tensor field is a symmetric tensor field u satisfying $du = 0$. A *conformal Killing* tensor field is a symmetric tensor field u satisfying the equation

$$du = iv \tag{1.3}$$

for some v. Equation (1.3) is a natural generalization of (1.1).

Conformal Killing vector (covector) fields are the classic object of the Riemannian geometry. Conformal Killing symmetric tensor fields of higher rank naturally appear in various problems of physics and geometry (see [21, 7, 8, 13, 20, 6, 1]). There are also many papers where antisymmetric conformal Killing tensor fields are studied (conformal Killing forms) because of their role in Gravitation Theory and in the Maxwell equations (see [18, 10] and the references there).

It should be observed that (in some sense) there are "too many" conformal Killing fields of rank $m \geq 2$. Indeed, if v is any field of rank $m-2$ then the field $u = iv$ satisfies the equation $du = i(dv)$ because the operators i and d commute. For this reason, it makes sense to study the *trace-free conformal Killing fields*, i.e., conformal Killing fields u satisfying equation (1.2). In some papers (for example, see [6, 1]), equation (1.2) is included into the definition of a conformal Killing tensor field, but we prefer to speak on the *trace-free conformal Killing fields* in this case.

Eliminating v from equation (1.3), we get $pdu = 0$, where p is an algebraic operator defined in Section 3 below. As is shown in Theorem 5.1 below, the operator δpd is elliptic on the bundle of trace-free tensor fields. Therefore, the equation $\delta pdu = 0$ implies the following statement: *a trace-free conformal Killing field is smooth.* Hereafter in the paper, the term "smooth" always means "C^∞-smooth".

The definition of a (trace-free) conformal Killing field is invariant with respect to a conformal change of the metric in the following sense. If $u \in C^\infty(S^m\tau'_M)$ is a (trace-free) conformal Killing field with respect to a Riemannian metric g then $\lambda^m u$ is a (trace-free) conformal Killing field with respect to the metric λg for any smooth positive function λ on M.

As well known, the space of conformal Killing vector fields on M has a finite dimension if $n = \dim M \geq 3$. In the two-dimensional case, the space can be of infinite dimension. Nevertheless, in every dimension, a conformal Killing vector field is uniquely determined by its C^∞-jet at any point. The following theorem is a generalization of the latter fact.

Theorem 1.1. *Let (M, g) be a connected n-dimensional Riemannian manifold, with $n \geq 3$. If a trace-free conformal Killing symmetric field u of rank $m \geq 0$ satisfies the conditions*

$$u(x_0) = 0, \quad \nabla u(x_0) = 0, \quad \dots, \quad \nabla^l u(x_0) = 0 \tag{1.4}$$

at some point $x_0 \in M$, where $l = l(m) \leq 6m$ depends only on m, then $u \equiv 0$. In particular, the dimension of the space of trace-free conformal Killing fields of rank m is finite. If $n = 2$ then the first statement is true if (1.4) *is replaced by the following condition: All the derivatives of the field u vanish at the point x_0.*

The theorem was first proved in [14], with $l(m) = 6m$. The results of [6, 1] provide the exact value $l(m) = 2m$ (in the case of $m = 2$, see also [21]). Arguments of [6, 1] are based on delicate facts of representation theory and the theory of overdetermined systems. At the same time, the proof given in [14] is quite elementary although it does not allow to obtain the exact value of $l(m)$. For reader's convenience, we reproduce the latter proof in Sections 9 and 11. For $n \geq 3$, the scheme of the proof is as follows. Being written in local coordinates, (1.2) and (1.3) constitute a linear homogeneous system of equations in components of the fields u and v, and their first order partial derivatives.

We differentiate these equations l times and show that the resulting system can be solved with respect to all partial derivatives of the highest order $l+1$. This means that the components of the tensors $u(t)=u\big(x(t)\big)$ and $v(t)=v\big(x(t)\big)$ satisfy a homogeneous linear system of ordinary differential equations of order $l+1$ along every smooth curve $x=x(t)$. Together with homogeneous initial conditions (1.4), this yields the required result. In the two-dimensional case, the proof consists of reducing system (1.2)–(1.3) to the Cauchy-Riemann equations.

Corollary 1.2. *Let (M,g) be a connected Riemannian manifold of dimension $n\ge 3$. If tensor fields $u\in C^\infty(S^m\tau'_M)$ and $v\in C^\infty(S^{m-1}\tau'_M)$ $(m\ge 0)$ satisfy equation* (1.3) *and initial conditions* (1.4) *with the same $l=l(m)$ as in Theorem* 1.1 *then there exists a field $w\in C^\infty(S^{m-2}\tau'_M)$ such that $u=iw$ and $v=dw$. For $n=2$, the statement is true if* (1.4) *is replaced by the following condition: All the derivatives of the field u vanish at the point x_0.*

Theorem 1.1 implies Corollary 1.2 by an algebraic trick presented in Section 3. Theorem 1.1 is also used in the proof of the following proposition.

Theorem 1.3. *Let (M,g) be a connected Riemannian manifold of dimension at least* 2, *and let $\varnothing\ne\Gamma\subset M$ be a smooth hypersurface. In particular, Γ may be a relatively open subset of the boundary ∂M. If a trace-free conformal Killing field u vanishes on Γ then $u\equiv 0$.*

In the case of $m=1$ and $\Gamma=\partial M$, Theorem 1.3 follows from [12, Proposition 3.3].

The authors are indebted to the anonymous referee for verification of the following fact: Theorem 1.3 is not valid if, in the condition $u\big|_\Gamma=0$, the hypersurface Γ is replaced by a submanifold of dimension less than $\dim M-1$. We quote: "The dimension of the space of trace-free conformal Killing fields of rank 2 on $\mathbb{R}^3$ which vanish on a straight line, equals 10, i.e., in some sense, Theorem 1.3 is the best possible result."

Corollary 1.4. *Let (M,g) and Γ satisfy the hypotheses of Theorem* 1.3. *If tensor fields $u\in C^\infty(S^m\tau'_M)$ and $v\in C^\infty(S^{m-1}\tau'_M)$ satisfy the equation $du=iv$ and the condition $u\big|_\Gamma=0$ then there exists a field $w\in C^\infty(S^{m-2}\tau'_M)$ such that $u=iw$, $v=dw$, and $w\big|_\Gamma=0$.*

The following definition was introduced in [16, § 3]. A Riemannian manifold with boundary is called *conformally rigid* if there is no nonzero conformal Killing vector field that vanishes on the boundary. Theorem 1.3 implies conformal rigidity of an arbitrary connected Riemannian manifold with nonempty boundary. For such a compact manifold (M,g), Theorem 3.3 of [16] can be formulated as follows: *Every rank* 2 *symmetric tensor field f on M can be uniquely represented in the form*

$$f_{ij}=\frac{1}{2}(\nabla_iv_j+\nabla_jv_i)+\lambda g_{ij}+\tilde f_{ij},\quad v|_{\partial M}=0,\quad \operatorname{tr}\tilde f=0,\quad \delta\tilde f=0.$$

We generalize this result to higher rank tensor fields. Given a compact M, let $H^k(S^m\tau'_M)$ denote the Hilbert space of symmetric tensor fields of rank m whose components are locally square integrable together with their partial derivatives up to order k in an arbitrary coordinate system.

Theorem 1.5. *Let (M,g) be a compact connected Riemannian manifold with nonempty boundary. Every symmetric tensor field $f\in H^k(S^m\tau'_M)$ $(m\ge 0,\ k\ge 1)$ can be uniquely represented in the form*

$$f=dv+i\lambda+\tilde f,\tag{1.5}$$

where $v \in H^{k+1}(S^{m-1}\tau'_M)$ *satisfies the conditions*

$$jv = 0, \quad v|_{\partial M} = 0, \tag{1.6}$$

$\lambda \in H^k(S^{m-2}\tau'_M)$, *and* $\tilde f \in H^k(S^m\tau'_M)$ *satisfies the conditions*

$$\delta \tilde f = 0, \quad j\tilde f = 0. \tag{1.7}$$

The summands in (1.5) *continuously depend on* f, *i.e., the stability estimates*

$$\|v\|_{H^{k+1}} \le C\|f\|_{H^k}, \quad \|\lambda\|_{H^k} \le C\|f\|_{H^k}, \quad \|\tilde f\|_{H^k} \le C\|f\|_{H^k} \tag{1.8}$$

hold with some constant C *independent of* f.

In General Relativity, conformal Killing tensor fields appear as polynomial first integrals of the equation for null geodesics [8]. Our interest in the conformal Killing tensor fields is motivated by the following question from Integral Geometry.

Given a Riemannian manifold (M, g), let

$$\Omega M = \{(x,\xi) \mid x \in M,\ \xi \in T_xM,\ |\xi|^2 = g_{ij}(x)\xi^i\xi^j = 1\}$$

denote the unit sphere bundle, and let $H : C^\infty(\Omega M) \to C^\infty(\Omega M)$ be the differentiation along the geodesic flow. In local coordinates,

$$H = \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x)\xi^j\xi^k \frac{\partial}{\partial \xi^i}, \tag{1.9}$$

where Γ^i_{jk} are the Christoffel symbols. As is seen from (1.9), if the function $U(x,\xi)$ polynomially depends on ξ then HU is also a polynomial in ξ. More precisely, for $u \in C^\infty(S^{m-1}\tau'_M)$,

$$H(u_{i_1\dots i_{m-1}}(x)\xi^{i_1}\dots\xi^{i_{m-1}}) = (du)_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}. \tag{1.10}$$

The question on validity of the converse statement is very important: *Is it true that every solution to the boundary value problem*

$$HU = v_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m} \quad \text{on} \quad \Omega M, \tag{1.11}$$

$$U|_{\partial(\Omega M)} = 0 \tag{1.12}$$

is a homogeneous polynomial of degree $m-1$ *in* ξ? The question is equivalent to the problem of inversion of the ray transform (see [15, Ch. 1] for a detailed discussion). The question is open in the general case and the positive answer is obtained only under certain curvature conditions.

Consider the following weaker version of the latter question. *Assume a solution* U *to the boundary value problem* (1.11)–(1.12) *to depend polynomially on* ξ. *Is* U *a restriction to* ΩM *of a homogeneous polynomial of degree* $m-1$? The question is not trivial since polynomials of different degrees can have the same restriction to ΩM in view of the identity $g_{ij}\xi^i\xi^j\big|_{\Omega M} = 1$. The positive answer to this question can be easily obtained from Theorem 1.3 even if (1.12) is replaced by the weaker condition

$$U(x,\xi)|_{x\in\Gamma} = 0, \tag{1.13}$$

where Γ is a relatively open subset of ∂M. Indeed, assume a solution $U(x,\xi)$ to the problem (1.11) and (1.13) to be a homogeneous polynomial in ξ of degree $m+2k-1$ (the case of a nonhomogeneous polynomial can be easily reduced to the considered one). This means the existence of $u \in C^\infty(S^{m+2k-1}\tau'_M)$ such that

$$U(x,\xi) = u_{i_1\dots i_{m+2k-1}}(x)\xi^{i_1}\dots\xi^{i_{m+2k-1}} \quad \text{on} \quad \Omega M, \quad u|_\Gamma = 0.$$

By (1.10), equation (1.11) takes the form

$$du = i^k v. \tag{1.14}$$

Applying Corollary 1.4, we find $w \in C^\infty(S^{m+2k-3}\tau'_M)$ such that $u = iw$ and $w\big|_\Gamma = 0$. Equation (1.14) can be written in terms of w as $i(dw) = i(i^{k-1}v)$. Since i is a monomorphism, this implies

$$dw = i^{k-1}v.$$

Repeating this argument by induction in k, we find the field $\widetilde{w} \in C^\infty(S^{m-1}\tau'_M)$ such that $u = i^k\widetilde{w}$. This means that $U(x,\xi)\big|_{\Omega M}$ coincides with the homogeneous polynomial $\widetilde{w}_{i_1\dots i_{m-1}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}}$ of degree $m-1$.

The questions under consideration are also important in the case of closed manifolds (i.e., compact manifolds with no boundary).

Theorem 1.6. *If (M,g) is a closed Riemannian manifold of dimension $n \ge 2$ of nonpositive sectional curvature then every trace-free conformal Killing symmetric tensor field u on M is absolutely parallel, i.e., $\nabla u = 0$, and every symmetric Killing tensor field is absolutely parallel.*

In addition, if M is connected and there is a point $x_0 \in M$ such that all sectional curvatures at x_0 are negative then there is no nonzero trace-free conformal Killing symmetric tensor field of any rank and every symmetric Killing tensor field is of the form cg^k for some constant c.

The classical theorem by Bochner-Yano states: there is no nontrivial conformal Killing vector field on a closed Riemannian manifold of negative Ricci curvature [22, Theorem 2.14]. Theorem 1.6 generalizes the last statement to arbitrary rank tensor fields, however, under the stronger hypothesis: The requirement of negative Ricci curvature is replaced by the requirement of negative sectional curvature.

Theorem 1.7. *Let (M,g) be a closed Riemannian manifold of dimension $n \ge 2$ without conjugate points. A vector field on M is conformal Killing if and only if it is a Killing vector field. A trace-free tensor field of rank 2 is conformal Killing if and only if it is the trace-free part of some Killing field. In addition, if the geodesic flow has a dense orbit in ΩM then there are neither nontrivial conformal Killing vector fields nor nontrivial trace-free conformal Killing fields of rank 2.*

The natural assumption is that, under hypotheses of Theorem 1.7, similar statements are valid for higher rank tensor fields. In the case of $\dim M = 2$, this easily follows from the uniformization theorem, invariance of the definition of conformal Killing tensor fields with respect to a conformal change of the metric, and from Theorem 1.6.

The following fact is well known: If the geodesic flow has a dense orbit in ΩM then (regardless of the dimension of M) every symmetric Killing tensor field is of the form cg^k with some constant c (see [3]).

The rest of the paper is organized as follows. Section 2 contains preliminaries from algebra of symmetric tensor fields. In particular, after deriving a commutation formula for the operators i and j, we show that the singular decomposition of the operator ji corresponds to the decomposition of polynomials in spherical harmonics.

In Section 3, we introduce the differential operators d and δ on symmetric tensor fields, prove some commutation formulas for these operators, and obtain some useful propositions. In this section, we derive Corollary 1.2 of Theorem 1.1.

Sections 4 and 5 contain the proofs of Theorems 1.3 and 1.5, respectively. These proofs are essentially based on Theorem 1.1.

In Section 6, we give the proofs of Theorems 1.6 and 1.7. It should be mentioned that this section differs from the others by the nature of the methods. Namely, here we use semi-basic tensor fields and the estimates for a solution to the kinetic equation which are based on Pestov's identity. We cannot present all necessary definitions here because of the volume limitation, so we refer the reader to [15, Chapter 3] for details. This does not concern other sections since they are independent of Section 6.

In Section 7, we discuss higher order differential operators on tensor fields paying a particular attention to the principle parts of operators.

In Section 8, we introduce the Laplace operator on symmetric tensor fields and prove some commutation formulas for powers of the Laplacian which are needed to prove Theorem 1.1.

The proof of Theorem 1.1 in the case of $\dim M \geq 3$ is presented in Section 9.

In Section 10, we derive the equations that relate the Fourier coefficients of a solution to the kinetic equation, to the Fourier coefficients of the right-hand side. We need these equations to prove Theorem 1.1 in the two-dimensional case. These equations are also of some independent interest since they constitute the basis of the so-called method of spherical harmonics for the numerical solution of the kinetic equation and the related linear transport equation. In the literature on the method of spherical harmonics, several versions of the equations are presented for different particular geometries (see [2]). However, the invariant form of the equations, as presented in Theorem 10.2, was probably unknown before.

The final Section 11 contains the proof of Theorem 1.1 in the two-dimensional case.

2. Algebra of symmetric tensors

We use the standard terminology of vector bundle theory. For a smooth manifold N, we denote the algebra of smooth real functions on N by $C^\infty(N)$. If $\xi = (E, \pi, N)$ is a smooth vector bundle and $U \subset N$ is an open set then $C^\infty(\xi; U)$ denotes the $C^\infty(U)$-module of smooth sections of ξ over U, and $C_0^\infty(\xi; U)$ denotes the submodule of compactly supported sections. We often reduce the notation $C^\infty(\xi; N)$ and $C_0^\infty(\xi; N)$ to $C^\infty(\xi)$ and $C_0^\infty(\xi)$, respectively. We deal here only with finite dimensional bundles with just one exception: Sometimes, we consider a graded vector bundle $\xi^* = \oplus_{m=0}^\infty \xi^m$, where each summand ξ^m has a finite dimension. Such an object can be thought as a sequence of finite-dimensional bundles. If $\eta = \oplus_{m=0}^\infty \eta^m$ is another graded bundle and $A \in \operatorname{Hom}(\xi, \eta)$ then A_m denotes the restriction of A to ξ^m. We say A has a degree k if $A(\xi^m) \subset \eta^{m+k}$.

Let (M, g) be a smooth Riemannian manifold of dimension $n \geq 2$. By $\tau_M = (TM, \pi, M)$ and $\tau'_M = (T'M, \pi, M)$, we denote the tangent bundle and the cotangent bundle, respectively. We often shorten these notation to $\tau = (T, \pi, M)$ and $\tau' = (T', \pi, M)$. Let $\otimes^m \tau' = (\otimes^m T', \pi, M)$ be the bundle of real covariant tensors of rank m and let $S^m \tau' = (S^m T', \pi, M)$ be its subbundle consisting of the symmetric tensors. There is the natural projection $\sigma \in \operatorname{Hom}(\otimes^m \tau', S^m \tau')$ acting as follows:

$$\sigma(v_1 \otimes \cdots \otimes v_m) = \frac{1}{m!} \sum_{\pi \in \Pi_m} v_{\pi(1)} \otimes \cdots \otimes v_{\pi(m)}, \tag{2.1}$$

where Π_m is the group of permutations of the set $\{1, \dots, m\}$. Note that $S^1\tau' = \tau'$ and $S^0 \tau' = M \times \mathbb{R}$. It is convenient to assume $\otimes^m \tau' = S^m \tau' = 0$ for $m < 0$. For a point $x \in M$, let T_x, T'_x, $\otimes^m T'_x$, and $S^m T'_x$ be the fibers over x of the corresponding bundles.

For $u \in S^m T'_x$ and $v \in S^l T'_x$, the *symmetric product* is defined by $uv = \sigma(u \otimes v)$. Thus, $S^* \tau' = \oplus_{m=0}^\infty S^m \tau'$ becomes a bundle of commutative graded algebras.

We will extensively use the coordinate representation of tensors. If $(x^1,\dots,x^n)$ is a local coordinate system in the domain $U\subset M$ then every tensor field $u\in C^\infty(\otimes^m\tau';U)$ is uniquely represented in the form

$$u=u_{i_1\dots i_m}dx^{i_1}\otimes\dots\otimes dx^{i_m}, \tag{2.2}$$

where the functions $u_{i_1\dots i_m}\in C^\infty(U)$ are the *covariant components* of the field u in this coordinate system. In (2.2) and below, we use the Einstein rule: The summation from 1 to n is assumed over an index repeated in the multivariate subscript and superscript of a monomial. Assuming the choice of a coordinate system to be clear from the context, we reduce formula (2.2) to

$$u=(u_{i_1\dots i_m}). \tag{2.3}$$

For $x\in U$ and $u\in\otimes^mT'_x$, formulas (2.2) and (2.3) also make sense but the components are real numbers in this case. *Contravariant components* are defined by

$$u^{i_1\dots i_m}=g^{i_1j_1}\dots g^{i_mj_m}u_{j_1\dots j_m},$$

where (g^{ij}) is the inverse matrix to (g_{ij}).

A tensor $u=(u_{i_1\dots i_m})\in\otimes^mT'_x$ belongs to $S^mT'_x$ if and only if its covariant and (or) contravariant components are symmetric with respect to all indices. We will also consider partially symmetric tensors. The partial symmetry of a tensor is denoted by

$$\mathrm{sym}\ u_{i_1\dots i_kj_1\dots j_l}:(i_1\dots i_{k-1})i_k(j_1\dots j_{l-1})j_l. \tag{2.4}$$

This means the tensor $(u_{i_1\dots i_kj_1\dots j_l})$ is symmetric in each group of indices in parentheses on the right-hand side of (2.4). Along with the full symmetrization σ, we will use partial symmetrization operators that are defined in coordinates by

$$\sigma(i_1\dots i_m)u_{i_1\dots i_mj_1\dots j_l}=\frac{1}{m!}\sum_{\pi\in\Pi_m}u_{i_{\pi(1)}\dots i_{\pi(m)}j_1\dots j_l}.$$

Lemma 2.1. *Let $m\ge1$, $p\ge1$, and $x\in M$. For every tensor $f\in\otimes^{2m+p}T'_x$ possessing the symmetry*

$$\mathrm{sym}\ f_{i_1\dots i_mj_1\dots j_pk_1\dots k_m}:(i_1\dots i_mj_1\dots j_p)(k_1\dots k_m), \tag{2.5}$$

there exists a unique solution to the equation

$$\sigma(i_1\dots i_mj_1\dots j_p)u_{i_1\dots i_mj_1\dots j_pk_1\dots k_m}=f_{i_1\dots i_mj_1\dots j_pk_1\dots k_m}, \tag{2.6}$$

possessing the symmetry

$$\mathrm{sym}\ u_{i_1\dots i_mj_1\dots j_pk_1\dots k_m}:(i_1\dots i_m)(j_1\dots j_pk_1\dots k_m). \tag{2.7}$$

The solution is expressed by the formula

$$\begin{aligned}u_{i_1\dots i_mj_1\dots j_pk_1\dots k_m}&=\sigma(i_1\dots i_m)\sigma(j_1\dots j_pk_1\dots k_m)\\&\quad\times\sum_{l=0}^m(-1)^l\binom{p+l-1}{l}\binom{m+p}{m-l}f_{i_1\dots i_{m-l}j_1\dots j_pk_1\dots k_mi_{m-l+1}\dots i_m},\end{aligned} \tag{2.8}$$

where $\binom{k}{l}=\frac{k!}{l!(k-l)!}$ are the binomial coefficients.

Since this is a purely algebraic statement, it suffices to prove it in the case of $M=\mathbb{R}^n$. For $p=1$, the statement is proved in [15, § 2.4]. For an arbitrary p, the proof is quite similar. The idea of the proof is as follows. Since the dimension of the space of tensors possessing symmetry (2.5) is equal to the dimension of the space of tensors possessing symmetry (2.7), it suffices to verify the equation obtained by substituting (2.8) into (2.6). This verification can be done by straightforward calculations that are omitted.

There is a natural inner product on $S^mT'_x$ defined in coordinates by

$$\langle u, v\rangle = u_{i_m\dots i_m}v^{i_m\dots i_m}. \tag{2.9}$$

We extend the product to $S^*T'_x = \oplus_{m=0}^{\infty} S^mT'_x$ assuming $S^mT'_x$ and $S^lT'_x$ to be orthogonal to each other for $m \neq l$. The product smoothly depends on x. Therefore, $S^*\tau'$ obtains the structure of a Riemannian vector bundle. So we can introduce the L^2-product on $C_0^\infty(S^*\tau')$ as follows:

$$(u, v)_{L^2} = \int\limits_M \langle u(x), v(x)\rangle dV(x), \tag{2.10}$$

where dV is the Riemannian volume form.

For $u \in S^*T'_x$, let $i_u : S^*T'_x \to S^*T'_x$ be the operator of symmetric multiplication by u, i.e., $i_uv = uv$, and let j_u be the adjoint operator of i_u. These operators are expressed by the formulas

$$(i_uv)_{i_1\dots i_{m+l}} = \sigma(i_1\dots i_{m+l})(u_{i_1\dots i_m}v_{i_{m+1}\dots i_{m+l}}),$$
$$(j_uv)_{i_1\dots i_{m-l}} = v_{i_1\dots i_l}u^{i_{l-m+1}\dots i_l}$$

for $u \in S^mT'_x$ and $v \in S^lT'_x$. The second formula makes sense only for $m \leq l$. If $m > l$ then $j_uv = 0$. For $u \in C^\infty(S^*\tau')$, the operators $i_u, j_u \in \mathrm{Hom}(S^*\tau', S^*\tau')$ are defined by $i_u(x) = i_{u(x)}$ and $j_u(x) = j_{u(x)}$. The operators i_g and j_g are of a particular importance in the present article, so we distinguish them by introducing the notation $i = i_g$ and $j = j_g$. These operators were already used in Introduction.

Lemma 2.2. *For $m \geq 0$ and $k \geq 1$, the following commutation formula holds on $S^m\tau'$:*

$$ji^k = \frac{2k(n+2m+2k-2)}{(m+2k-1)(m+2k)}i^{k-1} + \frac{m(m-1)}{(m+2k-1)(m+2k)}i^kj.$$

In the case of $k = 1$, the formula has the form

$$ji = \frac{2(n+2m)}{(m+1)(m+2)}E + \frac{m(m-1)}{(m+1)(m+2)}ij, \tag{2.11}$$

where E the identity operator. The latter formula is proved by a straightforward calculation in coordinates which is omitted. The general case easily follows from (2.11) with the help of induction in k.

Lemma 2.3. *For an arbitrary integer $m \geq 0$, the following decomposition formula holds*:

$$S^m\tau' = \bigoplus_{k=0}^{[m/2]} i^k(\mathrm{Ker}\, j_{m-2k}), \tag{2.12}$$

where $[m/2]$ is the integer part of $m/2$, and $\mathrm{Ker}\, j_{m-2k}$ is the kernel of the restriction j_{m-2k} of the operator j to $S^{m-2k}\tau'$. Each summand of the decomposition is a subbundle in $S^m\tau'$ and the summands are orthogonal to each other. The operator i is a monomorphism and its range is related to decomposition (2.12) *by the formula*

$$\mathrm{Ran}\, i_{m-2} = \bigoplus_{k=1}^{[m/2]} i^k(\mathrm{Ker}\, j_{m-2k}). \tag{2.13}$$

The product ji is a self-adjoint and positive definite operator. Each summand of (2.12) *is a proper subspace of the operator ji associated with the eigenvalue*

$$\lambda_k = \frac{2(k+1)(n+2m-2k)}{(m+1)(m+2)}.$$

The dimension of the summand equals $\alpha(m-2k)-\alpha(m-2k-2)$*, where* $\alpha(m)=\alpha(m,n)=\binom{n+m-1}{m}$ *is the dimension of* $S^m\tau'$*.*

Proof. The operator ij is self-adjoint and nonnegative since it is the product of two operators that are dual to each other. Therefore, (2.11) implies that ji is a positive self-adjoint operator. Hence, i is a monomorphism and the orthogonal decomposition

$$S^*\tau' = \operatorname{Ker} j \oplus \operatorname{Ran} i \tag{2.14}$$

holds with the summands on the right-hand side being sub-bundles of the left-hand side.

Let $u \in \operatorname{Ker} j_{m-2k}$. By Lemma 2.2, we get $(ji)(i^k u) = \lambda_k i^k u$. Therefore, each summand in (2.12) is the eigen-subspace of the operator ji associated with the eigenvalue λ_k. Since all λ_k are different, all summands are orthogonal to each other. The injectivity of i and equation (2.14) imply

$$\dim\,[i^k(\operatorname{Ker} j_{m-2k}] = \dim\,(\operatorname{Ker} j_{m-2k}) = \alpha(m-2k)-\alpha(m-2k-2).$$

Equality (2.12) is now proved by comparing dimensions of spaces on both sides of the equality. ☐

Let $p \in \operatorname{Hom}(S^*\tau', \operatorname{Ker} j)$ and $q \in \operatorname{Hom}(S^*\tau', \operatorname{Ran}\, i)$ be the orthogonal projections onto the summands of (2.14). One easily checks the equality

$$q = i(ji)^{-1}j. \tag{2.15}$$

Decomposition (2.12) is closely related to the expansion of functions on the sphere in Fourier series in spherical harmonics. In order to explain the relationship, we introduce some notation.

If $(x^1,\dots,x^n)$ is a local coordinate system with the domain U then, for $x\in U$, a vector $\xi \in T_x$ is uniquely represented as $\xi = \xi^i\frac{\partial}{\partial x^i}(x)$. The functions x^i,ξ^i $(i=1,\dots,n)$ form a local coordinate system on the manifold T with the domain $\pi^{-1}(U)$, where π is the projection of the tangent bundle. Strictly speaking, we should write $x^i\circ\pi$ instead of x^i. We use the shorter notation x^i and hope it will not cause any ambiguity. Thus, every function $\varphi \in C^\infty\big(\pi^{-1}(U)\big)$ can be written in coordinates as follows:

$$\varphi = \varphi(x^1,\dots,x^n,\xi^1,\dots,\xi^n) \quad (x\in U, \xi \in T_x).$$

Since T_x has the structure of an Euclidean space, the Laplace operator $\overset{v}{\Delta}_x : C^\infty(T_x)\to C^\infty(T_x)$ is well defined. It smoothly depends on x, and therefore, defines an operator $\overset{v}{\Delta} : C^\infty(T)\to C^\infty(T)$. $\big($Warning: Do not mix up $C^\infty(T)$ and $C^\infty(\tau)$!$\big)$ The operator is written in coordinates as

$$\overset{v}{\Delta}\varphi(x,\xi) = g^{ij}(x)\frac{\partial^2\varphi(x,\xi)}{\partial\xi^i\partial\xi^j}$$

and is called the *vertical* (or *fiberwise*) Laplacian.

The embedding $\varkappa_x : S^*T'_x \to C^\infty(T_x)$ is defined by

$$(\varkappa_x u)(\xi) = u_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m}$$

for $u\in S^mT'_x$. It smoothly depends on x, and hence, defines an embedding $\varkappa : C^\infty(S^*\tau')\to C^\infty(T)$ by the formula:

$$(\varkappa u)(x,\xi) = u_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}$$

for $u\in C^\infty(S^m\tau')$. Thus, $\varkappa$ identifies rank m symmetric tensor fields with homogeneous polynomials (with respect to ξ) of degree m on T.

Let $\Omega = \Omega M$ be the submanifold of T which consists of the unit vectors, and let $\Omega_x = \Omega \cap T_x$ be the unit sphere in T_x. Given $u \in S^*T'_x$, denote the restriction of $\varkappa_x u$ to Ω_x by $\lambda_x u$. The operator $\lambda_x : S^*T'_x \to C^\infty(\Omega_x)$ smoothly depends on x and defines an operator $\lambda : C^\infty(S^*\tau') \to C^\infty(\Omega)$.

We introduce an inner product $\langle\cdot\,,\cdot\rangle_\omega$ on the space $C^\infty(\Omega_x)$ by the formula

$$\langle\varphi,\psi\rangle_\omega = \int\limits_{\Omega_x} \varphi(\xi)\psi(\xi)d\omega(\xi),$$

where $d\omega$ is the volume form on Ω_x induced by the metric g. The index ω is used in the notation in order to distinguish this product from the product defined by (2.9).

Lemma 2.4. *The following equalities hold on* $S^mT'_x$:

$$\lambda_x i = \lambda_x, \tag{2.16}$$

$$m(m-1)\varkappa_x j = \overset{v}{\Delta}_x \varkappa_x. \tag{2.17}$$

For $u, v \in S^mT'_x$ *such that* $ju = jv = 0$, *the following equality holds*:

$$\langle\lambda_x u, \lambda_x v\rangle_\omega = \frac{m!\pi^{n/2}}{2^{m-1}\Gamma(n/2+m)}\langle u, v\rangle. \tag{2.18}$$

Proof. For $u \in S^mT'_x$ and $\xi \in \Omega_x$, we have

$$(\lambda_x iu)(\xi) = \Big(\sigma(i_1\dots i_{m+2})(u_{i_1\dots i_m} g_{i_{m+1}i_{m+2}})\Big)\xi^{i_1}\dots\xi^{i_{m+2}}.$$

Since the product $\xi^{i_1}\cdots\xi^{i_{m+2}}$ is symmetric with respect to $(i_1,\dots,i_{m+2})$, we can omit the symmetrization $\sigma(i_1\dots i_{m+2})$ here. So we get

$$(\lambda_x iu)(\xi) = (u_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m})(g_{i_{m+1}i_{m+2}}\xi^{i_{m+1}}\xi^{i_{m+2}}) = u_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m}$$

because $g_{i_{m+1}i_{m+2}}\xi^{i_{m+1}}\xi^{i_{m+2}} = 1$ on Ω_x. By definition, the right-hand side of the last formula equals $(\lambda_x u)(\xi)$. This proves (2.16). Formula (2.17) is also proved by direct calculations:

$$\begin{aligned}(\overset{v}{\Delta}_x\varkappa_x u)(\xi) &= g^{ij}\frac{\partial^2}{\partial\xi^i\partial\xi^j}\left(u_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m}\right)\\ &= m(m-1)g^{ij}u_{i_1\dots i_{m-2}ij}\xi^{i_1}\dots\xi^{i_{m-2}} = m(m-1)(\varkappa_x ju)(\xi).\end{aligned}$$

It suffices to prove (2.18) for $u = v$. The polynomial $\varkappa_x u$ can be written in the two ways:

$$(\varkappa_x u)(\xi) = u_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m} = \sum_{|\alpha|=m} u_\alpha\xi^\alpha, \tag{2.19}$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multiindex. The coefficients of (2.19) are related by the equality $u_\alpha = \frac{m!}{\alpha!}u_{i(\alpha)}$, with $i(\alpha) = (1\dots 1\, 2\dots 2\dots n\dots n)$ where 1 is repeated α_1 times, 2 is repeated α_2 times, etc.

We choose the coordinates in a neighborhood of x so that $g_{ij}(x) = \delta_{ij}$, where (δ_{ij}) is the Kronecker symbol. Then

$$\langle u, u\rangle = \frac{1}{m!}\sum_{|\alpha|=m}\alpha!|u_\alpha|^2. \tag{2.20}$$

Indeed,

$$\langle u, u\rangle = \sum_{i_1,\dots i_m=1}^{n}|u_{i_1\dots i_m}|^2 = \sum_{|\alpha|=m}\frac{m!}{\alpha!}|u_{i(\alpha)}|^2.$$

Taking the relations $u_{i(\alpha)} = \alpha! u_\alpha / m!$ into account, we obtain (2.20).

For $|\alpha| = m$, formula (2.19) implies

$$\partial_\xi^\alpha(\varkappa_x u) = \alpha! u_\alpha. \tag{2.21}$$

Therefore equality (2.20) can be written as

$$\langle u, u\rangle = \frac{1}{m!} \sum_{|\alpha|=m} \frac{1}{\alpha!} |\partial_\xi^\alpha(\varkappa_x u)|^2. \tag{2.22}$$

In virtue of (2.17), the condition $ju = 0$ means $\lambda_x u$ is a spherical harmonics of degree m. Ass well known, the spherical harmonics satisfy the equality

$$A_{m,m} = \frac{2^m \Gamma(n/2 + m)\Gamma(m+1)}{\Gamma(n/2)} A_{0,m}, \tag{2.23}$$

(for example, see [17, Lemma XI.1]), where

$$A_{0,m} = \int\limits_{\Omega_x} |\lambda_x u(\xi)|^2 d\omega(\xi) = \langle \lambda_x u, \lambda_x u\rangle_\omega, \tag{2.24}$$

$$A_{m,m} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int\limits_{\Omega_x} |\partial_\xi^\alpha(\varkappa_x u)|^2 d\omega(\xi).$$

By (2.21), the last integrand is constant and the last formula gives

$$A_{m,m} = m! \omega_n \sum_{|\alpha|=m} \alpha! |u_\alpha|^2,$$

where ω_n is the volume of the unit sphere in $\mathbb{R}^n$. Together with (2.20), the last formula implies

$$A_{m,m} = (m!)^2 \omega_n \langle u, u\rangle. \tag{2.25}$$

Finally, substituting (2.24), (2.25), and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ into (2.23), we obtain (2.18). $\square$

According to Lemma 2.4, the operator λ_x isomorphically maps the subspace $\operatorname{Ker} j_m \subset S^m T'_x$ onto the space of spherical harmonics of degree m on Ω_x. Moreover, this isomorphism is an isometry up to a constant factor if $\operatorname{Ker} j_m$ is equipped with the inner product $\langle\cdot\,,\cdot\rangle$ and the space of harmonics is equipped with the product $\langle\cdot\,,\cdot\rangle_\omega$.

As well known (for example, see [17]), spherical harmonics of different degrees are orthogonal to each other and every function $\varphi \in C^\infty(\Omega_x)$ can be expanded in the Fourier series in spherical harmonics of different degrees. The series converges absolutely and uniformly. The expansion smoothly depends on the point x, and we arrive to the following statement.

Lemma 2.5. *Every function $\varphi \in C^\infty(\Omega)$ can be uniquely represented by the series*

$$\varphi = \sum_{m=0}^{\infty} \lambda u_m, \tag{2.26}$$

where $u_m \in C^\infty(S^m\tau')$ satisfy the condition $ju_m = 0$. The series converges absolutely and uniformly on each compact subset of Ω.

By Lemma 2.3, a tensor field $u \in C^\infty(S^m\tau')$ is uniquely represented in the form

$$u = \sum_{k=0}^{[m/2]} i^k u_{m-2k}, \tag{2.27}$$

where every $u_{m-2k} \in C^\infty(S^{m-2k}\tau')$ satisfies $ju_{m-2k} = 0$. Expansion (2.27) coincides with the Fourier series (2.26) of the function $\lambda u \in C^\infty(\Omega)$. More precisely, the Fourier series of the function λu has a finite number of the summands, namely,

$$\lambda u = \sum_{k=0}^{[m/2]} \lambda u_{m-2k}$$

where the tensor fields u_{m-2k} are the same as in (2.27).

3. The operators d and δ

Given a Riemannian manifold (M, g), let $\nabla : C^\infty(\otimes^m\tau') \to C^\infty(\otimes^{m+1}\tau')$ be the covariant differentiation with respect to the Levi-Civita connection. For a tensor field $u = (u_{i_1\dots i_m})$, higher order covariant derivatives are denoted by $\nabla^k u = (\nabla_{j_1\dots j_k} u_{i_1\dots i_m})$. The *inner derivative* $d : C^\infty(S^*\tau') \to C^\infty(S^*\tau')$ is defined by $d = \sigma\nabla$. The *divergence* $\delta : C^\infty(S^*\tau') \to C^\infty(S^*\tau')$ is defined in local coordinates as $(\delta u)_{i_1\dots i_{m-1}} = g^{jk}\nabla_j u_{ki_1\dots i_{m-1}}$. These d and δ are the first order differential operators of degree 1 and -1, respectively. They were already mentioned in Introduction.

Theorem 3.1. *The operators d and $-\delta$ are dual to each other with respect to the L^2-product of symmetric tensor fields. Moreover, for a compact manifold M with boundary, Green's formula*

$$\int_M (\langle du, v\rangle + \langle u, \delta v\rangle)dV = \int_{\partial M} \langle i_\nu u, v\rangle dV'$$

holds for $u, v \in C^\infty(S^*\tau')$, where ν is the unit outward normal to the boundary, and dV and dV' are the Riemannian volume forms of M and of ∂M, respectively.

The proof is given in [15, § 3.3].

Lemma 3.2. *The following equalities hold on $C^\infty(S^*\tau')$:*

$$id = di, \qquad j\delta = \delta j, \tag{3.1}$$

$$pdp = pd, \qquad p\delta p = \delta p, \tag{3.2}$$

$$qdq = dq, \qquad q\delta q = q\delta, \tag{3.3}$$

$$pdq = 0, \qquad q\delta p = 0. \tag{3.4}$$

Proof. These formulas are written in pairs every of which is formed by two relatively dual relationships. It suffices to prove one formula in each pair. The second formula in (3.1) can be proved by direct calculation in coordinates and we omit it. The first formula in (3.3) is derived from (2.15) and (3.1) as follows:

$$qdq = i(ji)^{-1}jdi(ji)^{-1}j = i(ji)^{-1}jid(ji)^{-1}j = id(ji)^{-1}j = di(ji)^{-1}j = dq.$$

Formula (3.2) is obtained from (3.3) by the substitution $q = E - p$. The left multiplication of the first formula of (3.3) by p implies the first formula of (3.4). $\square$

Lemma 3.3. *The following equalities hold on $C^\infty(S^m\tau')$:*

$$\delta i = \frac{2}{m+2}d + \frac{m}{m+2}i\delta, \tag{3.5}$$

$$jd = \frac{2}{m+1}\delta + \frac{m-1}{m+1}dj. \tag{3.6}$$

Proof. It suffices to prove (3.5) since (3.6) is obtained by the duality. For $u \in C^\infty(S^m\tau')$, we have

$$(\delta iu)_{i_1\dots i_m j} = g^{kl}[\sigma(i_1\dots i_m jk)(g_{jk}\nabla_l u_{i_1\dots i_m})]$$

by the definition of i and δ, and by the equality $\nabla g = 0$. The expression in brackets represents the sum over Π_{m+2}. We divide all the terms of the sum into four groups as follows: The first group includes the products of the form $g_{rs}\nabla_l u_{t_1\dots t_m}$ for $\{r,s\} = \{j,k\}$; the second group (the third group) consists of the products such that $k \in \{r,s\}$ and $j \notin \{r,s\}$ ($k \notin \{r,s\}$ and $j \in \{r,s\}$), and the fourth group contains all the remaining terms. We thus obtain

$$\frac{(m+1)(m+2)}{2}(\delta iu)_{i_1\dots i_m j} = g^{kl}\Big[g_{jk}\nabla_l u_{i_1\dots i_m} + \sum_{a=1}^{m} g_{ki_a}\nabla_l u_{ji_1\dots\widehat{i_a}\dots i_m}$$
$$+ \sum_{a=1}^{m} g_{ji_a}\nabla_l u_{ki_1\dots\widehat{i_a}\dots i_m} + \sum_{1\le a<b\le m} g_{i_a i_b}\nabla_l u_{jki_1\dots\widehat{i_a}\dots\widehat{i_b}\dots i_m}\Big],$$

where $\wedge$ over an index means the index is omitted. Using the identity $g^{kl}g_{jk} = \delta^l_k$, where (δ^l_k) is the Kronecker tensor, we rewrite the last formula in the form

$$\frac{(m+1)(m+2)}{2}(\delta iu)_{i_1\dots i_m j} = \nabla_j u_{i_1\dots i_m} + \sum_{a=1}^{m}\nabla_{i_a} u_{ji_1\dots\widehat{i_a}\dots i_m}$$
$$+ \sum_{a=1}^{m} g_{ji_a}(\delta u)_{i_1\dots\widehat{i_a}\dots i_m} + \sum_{1\le a<b\le m} g_{i_a i_b}(\delta u)_{ji_1\dots\widehat{i_a}\dots\widehat{i_b}\dots i_m}.$$

The sum of the first two summands on the right-hand side of this formula equals $(m+1)(du)_{ji_1\dots i_m}$, and the sum of the last two summands equals $\frac{m(m+1)}{2}(i\delta u)_{ji_1\dots i_m}$. ☐

Lemma 3.4. *The following equalities hold on $C^\infty(S^m\tau')$:*

$$dq = qd - \frac{m}{n+2m-2}i\delta p, \quad q\delta = \delta q - \frac{m-1}{n+2m-4}pdj, \tag{3.7}$$

$$dp = pd + \frac{m}{n+2m-2}i\delta p, \quad p\delta = \delta p + \frac{m-1}{n+2m-4}pdj. \tag{3.8}$$

Proof. As above, the formulas are written in dual pairs and (3.8) is obtained from (3.7) by the substitution $q = E - p$. Hence it suffices to prove the first of formulas (3.7).

By (3.1) and (3.6),

$$jid = jdi = \left(\frac{2}{m+3}\delta + \frac{m+1}{m+3}dj\right)i.$$

The left and right multiplications of this formula by $(ji)^{-1}$ yield

$$d(ji)^{-1} = \frac{m+1}{m+3}(ji)^{-1}d + \frac{2}{m+3}(ji)^{-1}\delta i(ji)^{-1}.$$

In virtue of (2.15) and (3.6), this gives

$$\begin{aligned} dq = di(ji)^{-1}j = id(ji)^{-1}j &= i\Big(\frac{m-1}{m+1}(ji)^{-1}d + \frac{2}{m+1}(ji)^{-1}\delta i(ji)^{-1}\Big)j \\ &= \frac{m-1}{m+1}i(ji)^{-1}\Big(\frac{m+1}{m-1}jd - \frac{2}{m-1}\delta\Big) + \frac{2}{m+1}i(ji)^{-1}\delta q \\ &= qd - \frac{2}{m+1}i(ji)^{-1}\delta(E-q), \end{aligned}$$

i.e.,

$$dq = qd - \frac{2}{m+1}i(ji)^{-1}\delta p. \tag{3.9}$$

The equality $jp = 0$ follows from the definition of p. According to (2.11) and (3.1), this implies

$$(ji)\delta p = \frac{2(n+2m-2)}{m(m+1)}\delta p + \frac{(m-1)(m-2)}{m(m+1)}ij\delta p = \frac{2(n+2m-2)}{m(m+1)}\delta p.$$

Hence the equality

$$(ji)^{-1}\delta p = \frac{m(m+1)}{2(n+2m-2)}\delta p \tag{3.10}$$

holds on $C^\infty(S^m\tau')$. Substitution of (3.10) into (3.9) implies the first formula of (3.7). □

Let us now demonstrate how does Theorem 1.1 imply Corollary 1.2. Let u and v satisfy the hypotheses of Corollary 1.2, and let $\tilde u = pu$. Then

$$j\tilde u = 0. \tag{3.11}$$

Applying the operator p to (1.2), we obtain $pdu = 0$. Transformation of the left-hand side of this equality by (3.8) implies

$$\left(d - \frac{m}{n+2m-2}i\delta\right)\tilde u = 0.$$

We denote $\tilde v = \frac{m}{n+2m-2}\delta\tilde u$ and rewrite the last formula as

$$d\tilde u = i\tilde v. \tag{3.12}$$

As is seen from (2.15), the operator $p = E - q$ is represented in coordinates by a matrix whose elements are rational functions of the components g_{ij} of the metric tensor. Therefore conditions (1.4) imply the similar conditions for $\tilde u = pu$:

$$\tilde u(x_0) = 0, \quad \nabla\tilde u(x_0) = 0, \quad \dots, \quad \nabla^l\tilde u(x_0) = 0. \tag{3.13}$$

According to (3.11)–(3.13), $\tilde u$ satisfies the hypotheses of Theorem 1.1. Assuming the theorem to be valid, we obtain $\tilde u = pu = 0$. This means the existence of w such that $u = iw$. Theorem 1.3 implies Corollary 1.4 in a similar way.

4. Proof of Theorem 1.3

According to Theorem 1.1 whose proof will be given below, Theorem 1.3 follows from a weaker statement formulated in

Lemma 4.1. *Let Γ be a smooth hypersurface in a Riemannian manifold M. If tensor fields $u \in C^\infty(S^m\tau')$ and $v \in C^\infty(S^{m-1}\tau')$ satisfy the conditions*

$$du = iv, \quad ju = 0, \quad u\big|_\Gamma = 0$$

then u and v vanish on Γ together with all their derivatives.

Proof. The statement is trivial for $m = 0$. Let $m \geq 1$. We prove by induction in k the validity of the following statement:

$$
\begin{aligned}
&u|_\Gamma = 0,\ \nabla u|_\Gamma = 0,\ \dots, \nabla^k u|_\Gamma = 0,\\
&v|_\Gamma = 0,\ \nabla v|_\Gamma = 0,\ \dots, \nabla^{k-1} v|_\Gamma = 0.
\end{aligned}
$$

For $k = 0$, the statement coincides with the hypothesis $u\big|_\Gamma = 0$. Assume the required statement to be true for some $k \geq 0$.

We choose a coordinate system $(x^1, \dots, x^n) = (x'^1, \dots, x'^{n-1}, y)$ in some neighborhood of $x_0 \in \Gamma$ so that Γ is defined by the equation $y = 0$ and $g_{in} = \delta_{in}$. Here and below (δ_{ij}) is the Kronecker tensor. By the induction hypothesis,

$$
\begin{aligned}
&\frac{\partial^l}{\partial y^l}\partial^\beta_{x'} u_{i_1\dots i_m}\bigg|_{y=0} = 0 \quad \text{for} \quad l \leq k,\\
&\frac{\partial^l}{\partial y^l}\partial^\beta_{x'} v_{i_1\dots i_{m-1}}\bigg|_{y=0} = 0 \quad \text{for} \quad l \leq k-1
\end{aligned}
\tag{4.1}
$$

for an arbitrary $(n-1)$-variate index β.

The equality $du = iv$ has the following form in the chosen coordinates:

$$
\begin{aligned}
&\nabla_{i_1} u_{i_2\dots i_{m+1}} + \nabla_{i_2} u_{i_1 i_3\dots i_{m+1}} + \dots + \nabla_{i_{m+1}} u_{i_1\dots i_m}\\
&\qquad = (m+1)\sigma(i_1\dots i_{m+1})\big(g_{i_1 i_2} v_{i_3\dots i_{m+1}}\big).
\end{aligned}
$$

Applying the operator $\frac{\partial^k}{\partial y^k}\big|_{y=0}$ to this equality and taking (4.1) into account, we obtain

$$
\begin{aligned}
&\frac{\partial^k}{\partial y^k}\bigg(\frac{\partial u_{i_2\dots i_{m+1}}}{\partial x^{i_1}} + \dots + \frac{\partial u_{i_1\dots i_m}}{\partial x^{i_{m+1}}}\bigg)\bigg|_{y=0}\\
&\qquad = (m+1)\sigma(i_1\dots i_{m+1})\bigg(g_{i_1 i_2}\frac{\partial^k v_{i_3\dots i_{m+1}}}{\partial y^k}\bigg|_{y=0}\bigg).
\end{aligned}
\tag{4.2}
$$

Hereafter we use the following agreement: Greek indices vary from 1 to $n-1$, and the summation from 1 to $n-1$ is assumed over repeated Greek indices. Set $(i_1, \dots, i_{m+1}) = (\alpha_1, \dots, \alpha_{m+1})$ in (4.2). Then the left-hand side of (4.2) equals zero by (4.1) and we obtain

$$
\sigma(\alpha_1\dots\alpha_{m+1})\left(g_{\alpha_1\alpha_2}\frac{\partial^k v_{\alpha_3\dots\alpha_{m+1}}}{\partial y^k}\bigg|_{y=0}\right) = 0. \tag{4.3}
$$

We rewrite (4.2) in the form

$$
\begin{aligned}
&\frac{\partial^k}{\partial y^k}\bigg(\frac{\partial u_{i_2\dots i_{m+1}}}{\partial x^{i_1}} + \dots + \frac{\partial u_{i_1\dots i_m}}{\partial x^{i_{m+1}}}\bigg)\bigg|_{y=0}\\
&\qquad = \frac{1}{m!}\sum_{\pi\in\Pi_{m+1}} g_{i_{\pi(1)} i_{\pi(2)}}\frac{\partial^k v_{i_{\pi(3)}\dots i_{\pi(m+1)}}}{\partial y^k}\bigg|_{y=0}.
\end{aligned}
\tag{4.4}
$$

Let $0 \leq s \leq m$. We set $(i_1, \dots, i_{m-s}) = (\alpha_1, \dots, \alpha_{m-s})$ and $i_{m-s+1} = \dots = i_{m+1} = n$ in (4.4). By (4.1), the first $m-s$ summands on the left-hand side of (4.4) are equal to zero and the last $s+1$ summands coincide, i.e.,

$$
\frac{\partial^k}{\partial y^k}\left(\frac{\partial u_{i_2\dots i_{m+1}}}{\partial x^{i_1}} + \dots + \frac{\partial u_{i_1\dots i_m}}{\partial x^{i_{m+1}}}\right)\bigg|_{y=0} = (s+1)\,\frac{\partial^{k+1} u_{\alpha_1\dots\alpha_{m-s} n\dots n}}{\partial y^{k+1}}\bigg|_{y=0}. \tag{4.5}
$$

Let us analyze the right-hand side of (4.4) for the chosen indices. If $\pi(1) \le m-s$ and $\pi(2) > m-s$ then $g_{i_{\pi(1)}i_{\pi(2)}} = 0$. Similarly, $g_{i_{\pi(1)}i_{\pi(2)}} = 0$ if $\pi(1) > m-s$ and $\pi(2) \le m-s$. Therefore

$$\sum_{\pi\in\Pi_{m+1}} g_{i_{\pi(1)}i_{\pi(2)}} \frac{\partial^k v_{i_{\pi(3)}\dots i_{\pi(m+1)}}}{\partial y^k}\Big|_{y=0}$$
$$= \sum_{\pi\in\Pi_{m+1}(s)} \frac{\partial^k v_{i_{\pi(3)}\dots i_{\pi(m+1)}}}{\partial y^k}\Big|_{y=0} + \sum_{\pi\in\Pi'_{m+1}(s)} g_{i_{\pi(1)}i_{\pi(2)}} \frac{\partial^k v_{i_{\pi(3)}\dots i_{\pi(m+1)}}}{\partial y^k}\Big|_{y=0}, \tag{4.6}$$

where

$$\Pi_{m+1}(s) = \big\{\pi \in \Pi_{m+1} \mid \pi(1) > m-s,\ \pi(2) > m-s\big\},$$
$$\Pi'_{m+1}(s) = \big\{\pi \in \Pi_{m+1} \mid \pi(1) \le m-s,\ \pi(2) \le m-s\big\}.$$

All the summands of the first sum on the right-hand side of (4.6) coincide because v is symmetric. And the total amount of the summands is $(m-1)!s(s+1)$, i.e.,

$$\sum_{\pi\in\Pi_{m+1}(s)} \frac{\partial^k v_{i_{\pi(3)}\dots i_{\pi(m+1)}}}{\partial y^k}\Big|_{y=0} = (m-1)!s(s+1)\ \frac{\partial^k v_{\alpha_1\dots\alpha_{m-s}n\dots n}}{\partial y^k}\Big|_{y=0}. \tag{4.7}$$

For $s=0$, the right-hand side of (4.7) is equal to zero due to the factor s.

The second sum on the right-hand side of (4.6) is obviously equal to

$$c(m,s)\sigma(\alpha_1\dots\alpha_{m-s})\left(g_{\alpha_1\alpha_2}\ \frac{\partial^k v_{\alpha_3\dots\alpha_{m-s}n\dots n}}{\partial y^k}\Big|_{y=0}\right), \tag{4.8}$$

where $c(m,s) = (m-1)!(m-s)(m-s-1)$ is total amount of elements in $\Pi'_{m+1}(s)$. Substitute (4.7) and (4.8) into (4.6) to obtain

$$\frac{1}{(m-1)!}\sum_{\pi\in\Pi_{m+1}} g_{i_{\pi(1)}i_{\pi(2)}} \frac{\partial^k v_{i_{\pi(3)}\dots i_{\pi(m+1)}}}{\partial y^k}\Big|_{y=0} = s(s+1)\frac{\partial^k v_{\alpha_1\dots\alpha_{m-s}n\dots n}}{\partial y^k}\Big|_{y=0}$$
$$+ (m-s)(m-s-1)\sigma(\alpha_1\dots\alpha_{m-s})\left(g_{\alpha_1\alpha_2}\frac{\partial^k v_{\alpha_3\dots\alpha_{m-s}n\dots n}}{\partial y^k}\Big|_{y=0}\right). \tag{4.9}$$

Next, we substitute (4.5) and (4.9) into (4.4)

$$\frac{\partial^{k+1} u_{\alpha_1\dots\alpha_{m-s}n\dots n}}{\partial y^{k+1}}\Big|_{y=0} = \frac{s}{m}\frac{\partial^k v_{\alpha_1\dots\alpha_{m-s}n\dots n}}{\partial y^k}\Big|_{y=0}$$
$$+ \frac{(m-s)(m-s-1)}{m(s+1)}\sigma(\alpha_1\dots\alpha_{m-s})\left(g_{\alpha_1\alpha_2}\frac{\partial^k v_{\alpha_3\dots\alpha_{m-s}n\dots n}}{\partial y^k}\Big|_{y=0}\right). \tag{4.10}$$

We define the tensor fields

$$z^{(s)} \in C^\infty(S^{m-s}\tau'_\Gamma)\ \ (0 \le s \le m), \quad w^{(s)} \in C^\infty(S^{m-s}\tau'_\Gamma)\ \ (1 \le s \le m)$$

on Γ as follows:

$$z^{(s)}_{\alpha_1\dots\alpha_{m-s}} = \frac{\partial^{k+1} u_{\alpha_1\dots\alpha_{m-s}n\dots n}}{\partial y^{k+1}}\Big|_{y=0}, \quad w^{(s)}_{\alpha_1\dots\alpha_{m-s}} = \frac{\partial^k v_{\alpha_1\dots\alpha_{m-s}n\dots n}}{\partial y^k}\Big|_{y=0}. \tag{4.11}$$

For convenience, we also define $w^{(0)}=0$, $w^{(m+1)}=0$, and $w^{(m+2)}=0$. Then (4.10) can be written in the coordinate-free form

$$z^{(s)}=\frac{s}{m}w^{(s)}+\frac{(m-s)(m-s-1)}{m(s+1)}iw^{(s+2)}\quad(0\le s\le m),\tag{4.12}$$

and (4.3) can be written as $iw^{(1)}=0$. Since i is a monomorphism, this implies

$$w^{(1)}=0.\tag{4.13}$$

If $m=1$ then v is a scalar function and (4.13) gives $\frac{\partial^k v}{\partial y^k}\big|_{y=0}=0$. Equation (4.12) implies

$$z^{(0)}_\alpha=\frac{\partial^{k+1}u_\alpha}{\partial y^{k+1}}\bigg|_{y=0}=0,\quad \frac{\partial^{k+1}u_n}{\partial y^{k+1}}\bigg|_{y=0}=z^{(1)}=w^{(1)}=0.$$

This justifies the induction step in the case of $m=1$. Therefore, we assume $m\ge 2$ in the rest of the proof.

In the chosen coordinates, the equation $ju=0$ is written as follows:

$$u_{i_1\dots i_{m-2}nn}+g^{\beta\gamma}u_{\beta\gamma i_1\dots i_{m-2}}=0.$$

Differentiating this identity $k+1$ times with respect to y, we obtain

$$\frac{\partial^{k+1}u_{i_1\dots i_{m-2}nn}}{\partial y^{k+1}}\bigg|_{y=0}+g^{\beta\gamma}\,\frac{\partial^{k+1}u_{\beta\gamma i_1\dots i_{m-2}}}{\partial y^{k+1}}\bigg|_{y=0}=0.$$

We set $(i_1,\dots,i_{m-s})=(\alpha_1,\dots,\alpha_{m-s})$ and $i_{m-s+1}=\dots i_{m-2}=n$ in the last formula to obtain

$$z^{(s)}+jz^{(s-2)}=0\quad(2\le s\le m).$$

This implies

$$z^{(2s)}=(-j)^s z^{(0)}\quad\text{for}\quad 0\le 2s\le m,\qquad z^{(2s+1)}=(-j)^s z^{(1)}\quad\text{for}\quad 0\le 2s+1\le m.\tag{4.14}$$

Setting $s=m$ and then $s=m-1$ in (4.12), we get

$$w^{(m)}=z^{(m)},\quad w^{(m-1)}=\frac{m}{m-1}z^{(m-1)}.$$

Taking (4.14) into account, this implies

$$\begin{aligned} w^{(2m')}&=(-j)^{m'}z^{(0)} &&\text{for}\ \ m=2m',\\ w^{(2m')}&=\frac{2m'+1}{2m'}(-j)^{m'}z^{(0)} &&\text{for}\ \ m=2m'+1,\end{aligned}\tag{4.15}$$

$$\begin{aligned} w^{(2m'-1)}&=\frac{2m'}{2m'-1}(-j)^{m'-1}z^{(1)} &&\text{for}\ \ m=2m',\\ w^{(2m'+1)}&=(-j)^{m'}z^{(1)} &&\text{for}\ \ m=2m'+1.\end{aligned}\tag{4.16}$$

Now, we are going to prove the representations

$$w^{(2s)}=(-1)^s\sum_{l=0}^{[m/2]-s}a(m,s,l)i^l j^{s+l}z^{(0)}\qquad\text{for}\ \ 0<2s\le m,\tag{4.17}$$

$$w^{(2s+1)}=(-1)^s\sum_{l=0}^{[m-1/2]-s}b(m,s,l)i^l j^{s+l}z^{(1)}\ \ \text{for}\ \ 0<2s+1\le m,\tag{4.18}$$

with some *positive* coefficients $a(m,s,l)$ and $b(m,s,l)$, where $[\cdot]$ denotes, as usual, the integer part of a number. For $0\le m-2s\le 1$, formula (4.17) coincides with (4.15). We shall prove (4.17) by induction in $m-2s$. Let $s>0$ and $m-2s\ge 2$. If we take $s:=2s$ in (4.12) then we get

$$z^{(2s)}=\frac{2s}{m}w^{(2s)}+\frac{(m-2s)(m-2s-1)}{m(2s+1)}iw^{(2s+2)}.$$

We express $w^{(2s)}$ from this equation

$$w^{(2s)}=\frac{m}{2s}z^{(2s)}-\frac{(m-2s)(m-2s-1)}{2s(2s+1)}iw^{(2s+2)}.$$

We replace the first term on the right-hand side by its value (4.14) and replace the second term by its expression from the inductive hypothesis (4.17)

$$w^{(2s)}=(-1)^s\Big(\frac{m}{2s}j^sz^{(0)}+\frac{(m-2s)(m-2s-1)}{2s(2s+1)}\sum_{l=0}^{[\frac{m}{2}]-s-1}a(m,s+1,l)i^{l+1}j^{s+l+1}z^{(0)}\Big).$$

Changing the summation index, we transform this expression to the form

$$w^{(2s)}=(-1)^s\Big(\frac{m}{2s}j^sz^{(0)}+\frac{(m-2s)(m-2s-1)}{2s(2s+1)}\sum_{l=1}^{[\frac{m}{2}]-s}a(m,s+1,l-1)i^{l}j^{s+l}z^{(0)}\Big).$$

This is equivalent to (4.17) with

$$a(m,s,l)=\begin{cases} m/2s \quad \text{for} \quad l=0,\\ \frac{(m-2s)(m-2s-1)}{2s(2s+1)}a(m,s+1,l-1) \quad \text{for} \quad l\ge 1.\end{cases}$$

Thus, representation (4.17) is proved. The proof of (4.18) is quite similar.

Set $s=0$ in (4.12)

$$z^{(0)}=(m-1)iw^{(2)}. \tag{4.19}$$

By (4.17),

$$w^{(2)}=-\sum_{l=0}^{[\frac{m}{2}]-1}a(m,1,l)i^lj^{l+1}z^{(0)}.$$

Substitution of the last expression into (4.19) gives

$$\Big[E+(m-1)\sum_{l=1}^{[m/2]}a(m,1,l-1)i^lj^l\Big]z^{(0)}=0, \tag{4.20}$$

where E is the identity operator. The operator in the brackets is nondegenerate since the coefficients of the sum are positive and the operator i^lj^l is nonnegative. Hence, (4.20) implies $z^{(0)}=0$. So, according to (4.14) and (4.17), $z^{(2s)}=0$ and $w^{(2s)}=0$ for all s.

By (4.13), $w^{(1)}=0$. On the other hand, setting $s=0$ in (4.18), we see

$$w^{(1)}=\Big[\sum_{l=0}^{[\frac{m-1}{2}]}b(m,0,l)i^lj^l\Big]z^{(1)}=0.$$

Since the operator in the brackets is nondegenerate, $z^{(1)}=0$. Together with (4.14) and (4.18), this gives $z^{(2s+1)}=0$ and $w^{(2s+1)}=0$ for all s.

We have proved $z^{(s)}=0$ and $w^{(s)}=0$ for all s. Recalling definition (4.11), we see

$$\frac{\partial^{k+1}u_{i_1\dots i_m}}{\partial y^{k+1}}\Big|_{y=0}=0,\qquad \frac{\partial^{k}v_{i_1\dots i_m}}{\partial y^{k}}\Big|_{y=0}=0,$$

and this is the finish of the inductive step. $\square$

5. Proof of Theorem 1.5

We start with the following observation: the tensor fields λ and $\tilde f$ can be eliminated from the system (1.5)–(1.7). Indeed, let $f, \tilde f \in H^k(S^m\tau')$, $v \in H^{k+1}(S^{m-1}\tau')$, and $\lambda \in H^k(S^{m-2}\tau')$ satisfy (1.5)–(1.7). Applying the operator j to (1.5), we get

$$jf = jdv + ji\lambda.$$

Express λ from this

$$\lambda = (ji)^{-1}j(f - dv), \tag{5.1}$$

and substitute the result into (1.5)

$$f = dv + i(ji)^{-1}j(f - dv) + \tilde f.$$

Due to (2.15), this equality can be written in the form

$$f = dv + q(f - dv) + \tilde f$$

or

$$(E - q)f = (E - q)dv + \tilde f,$$

where E is the identity operator. Since $E - q = p$,

$$pf = pdv + \tilde f. \tag{5.2}$$

To eliminate $\tilde f$, we apply the operator δ to equation (5.2). Taking $\delta\tilde f = 0$ into account, we obtain

$$\delta pf = \delta pdv.$$

Hence, v is a solution to the boundary value problem

$$(\delta pd)v = h, \quad v|_{\partial M} = 0 \tag{5.3}$$

with

$$h = \delta pf \in H^{k-1}(S^{m-1}\tau'). \tag{5.4}$$

Recall that the subbundle $\operatorname{Ker} j$ of the vector bundle $S^*\tau'$ was defined in Section 2. The right-hand side h of equation (5.3) belongs to $H^{k-1}(\operatorname{Ker} j)$ by (5.4) and (3.1). The desired solution v to problem (5.3) must be a section of $\operatorname{Ker} j$ since the requirement $jv = 0$ is contained in (1.6). Finally, δpd can be considered as a differential operator on the vector bundle $\operatorname{Ker} j$, i.e.,

$$\delta pd : C^\infty(\operatorname{Ker} j) \to C^\infty(\operatorname{Ker} j),$$

since $q(\delta pd) = (q\delta p)d = 0$ in view of (3.4). So, (5.3) can be considered as a boundary value problem on the bundle $\operatorname{Ker} j$. We shall prove this is an elliptic problem with zero kernel and co-kernel. Then, applying the theorem on regular solvability of elliptic problems, we shall deduce that, for every $h \in H^k(\operatorname{Ker} j)$ $(k \geq 0)$, problem (5.3) has a unique solution $v \in H^{k+2}(\operatorname{Ker} j)$ satisfying the stability estimate

$$\|v\|_{H^{k+2}} \leq C\|h\|_{H^k}.$$

Setting $h = \delta pf$ and defining λ and $\tilde f$ by formulas (5.1) and (5.2), we get (1.5)–(1.8). Theorem 1.5 is thus reduced to the following proposition:

Theorem 5.1. *Let M be a compact connected Riemannian manifold with nonempty boundary. Being considered on the vector bundle* $\operatorname{Ker} j$*, the boundary value problem* (5.3) *is elliptic and has zero kernel and co-kernel.*

Proof. We start with checking ellipticity of the operator δpd on $\operatorname{Ker} j$. The principal symbols $\sigma_1(d)$ and $\sigma_1(\delta)$ of the operators d and δ at a point $(x,\xi)\in T'$ are

$$\sigma_1(d)=\sqrt{-1}i_\xi \quad \sigma_1(\delta)=\sqrt{-1}j_\xi;$$

here $\sqrt{-1}$ is the imaginary unit. Hence,

$$\sigma_2(\delta pd)=-j_\xi p i_\xi.$$

For $x\in M$, let $\operatorname{Ker}^m_x j=\{f\in S^mT'_x \mid jf=0\}$. We have to prove the operator

$$j_\xi p i_\xi : \operatorname{Ker}^m_x j\rightarrow \operatorname{Ker}^m_x j \tag{5.5}$$

is an isomorphism for every $m\ge 0$ and every $0\ne\xi\in T'_x$.

The operator $j_\xi p i_\xi$ is easily seen to be nonnegative. Indeed,

$$\langle j_\xi p i_\xi f,f\rangle=\langle p i_\xi f,i_\xi f\rangle=\langle p i_\xi f,p i_\xi f\rangle=|p i_\xi f|^2.$$

Therefore verification of the ellipticity of δpd reduces to the following proposition.

Lemma 5.2. *If a tensor $f\in S^mT'_x$ satisfies the conditions*

$$jf=0,\quad p i_\xi f=0$$

for some $0\ne\xi\in T'_x$ then $f=0$.

To prove Lemma 5.2, we need the following:

Lemma 5.3. *The commutation formula*

$$p i_\xi=i_\xi p-\frac{2}{m+1}i(ji)^{-1}j_\xi p$$

holds on $S^m\tau'$.

Proof. The commutation formula

$$ji_\xi=\frac{2}{m+1}j_\xi+\frac{m-1}{m+1}i_\xi j\quad\text{on}\quad S^m\tau' \tag{5.6}$$

is checked by direct calculations in coordinates, and we omit them. Using (2.15) and (5.6), we obtain

$$qi_\xi=i(ji)^{-1}ji_\xi=i(ji)^{-1}(ji_\xi)=i(ji)^{-1}\left(\frac{2}{m+1}j_\xi+\frac{m-1}{m+1}i_\xi j\right).$$

Hence,

$$qi_\xi=i(ji)^{-1}\left(\frac{2}{m+1}j_\xi+\frac{m-1}{m+1}i_\xi j\right)\quad\text{on}\quad S^m\tau'. \tag{5.7}$$

Using (5.6) again, we get

$$jii_\xi=(ji_\xi)i=\left(\frac{2}{m+3}j_\xi+\frac{m+1}{m+3}i_\xi j\right)i=\frac{2}{m+3}j_\xi i+\frac{m+1}{m+3}i_\xi(ji).$$

Multiplying the extreme parts of this formula by $(ji)^{-1}$ from the left and from the right, we obtain

$$i_\xi(ji)^{-1}=\frac{2}{m+3}(ji)^{-1}j_\xi i(ji)^{-1}+\frac{m+1}{m+3}(ji)^{-1}i_\xi.$$

Hence,

$$(ji)^{-1}i_\xi=\frac{m+3}{m+1}i_\xi(ji)^{-1}-\frac{2}{m+1}(ji)^{-1}j_\xi i(ji)^{-1}\quad\text{on}\quad S^m\tau'. \tag{5.8}$$

We transform the second summand on the right-hand side of (5.7) with the htlp of (5.8). The summand equals zero in the case of $m=0$ and of $m=1$ due to the factor j on its right. Therefore we assume $m \ge 2$. Since the operator j acts before $(ji)^{-1}i_\xi$, the value m in (5.8) should be changed to $m-2$. Thus, the result of the transformation is as follows:

$$qi_\xi = \frac{2}{m+1}i(ji)^{-1}j_\xi + \frac{m-1}{m+1}i\left(\frac{m+1}{m-1}i_\xi(ji)^{-1} - \frac{2}{m-1}(ji)^{-1}j_\xi i(ji)^{-1}\right)j.$$

We arrange this formula as

$$qi_\xi = i_\xi i(ji)^{-1}j + \frac{2}{m+1}i(ji)^{-1}j_\xi(E - i(ji)^{-1}j)$$

and use (2.15) again to obtain

$$qi_\xi = i_\xi q + \frac{2}{m+1}i(ji)^{-1}j_\xi p \quad \text{on} \quad S^m\tau'. \tag{5.9}$$

We have thus proved (5.9) in the case of $m \ge 2$. Actually, (5.9) is valid for any $m \ge 0$. Indeed, both sides of this formula are equal to zero in the case of $m = 0$. In the case of $m = 1$, (5.9) has the form

$$qi_\xi = \frac{1}{n}ij_\xi \quad \text{on} \quad \tau'$$

and can be easily checked.

Substituting $q = E - p$ into (5.9), we complete the proof of Lemma 5.3. $\square$

Proof of Lemma 5.2. The statement of the lemma is trivial for $m = 0$ since $pi_\xi f = i_\xi f$ in the latter case and i_ξ is a monomorphism for $\xi \ne 0$. So we assume $m \ge 1$.

Let $f \in S^mT'_x$ satisfy the equalities $jf = 0$ and $pi_\xi f = 0$. By Lemma 5.3, the second equality implies

$$0 = pi_\xi f = i_\xi pf - \frac{2}{m+1}i(ji)^{-1}j_\xi pf.$$

Since $pf = f$, this equality is simplified to the following one:

$$i_\xi f - \frac{2}{m+1}i(ji)^{-1}j_\xi f = 0.$$

Taking the scalar product of this with $i_\xi f$, we get

$$\langle i_\xi f, i_\xi f\rangle - \frac{2}{m+1}\langle i(ji)^{-1}j_\xi f, i_\xi f\rangle = 0$$

or

$$\langle j_\xi i_\xi f, f\rangle - \frac{2}{m+1}\langle (ji)^{-1}j_\xi f, ji_\xi f\rangle = 0. \tag{5.10}$$

The operators i_ξ and j_ξ satisfy the commutation formula

$$j_\xi i_\xi f = \frac{|\xi|^2}{m+1}f + \frac{m}{m+1}i_\xi j_\xi f \quad \text{for} \quad f \in S^m\tau' \tag{5.11}$$

(see [15, Lemma 3.3.3]).

Since $jf = 0$, formula (5.6) implies

$$ji_\xi f = \frac{2}{m+1}j_\xi f. \tag{5.12}$$

Using (2.11) and taking $jf = 0$ into account, we deduce

$$(ji)(j_\xi f) = \frac{2(n+2m-2)}{m(m+1)}j_\xi f.$$

Applying the operator $(ji)^{-1}$ to this equation, we infer

$$(ji)^{-1}(j_\xi f) = \frac{m(m+1)}{2(n+2m-2)} j_\xi f. \tag{5.13}$$

Substitute (5.11)–(5.13) into (5.10) to obtain

$$|\xi|^2|f|^2 + m\left(1 - \frac{2}{n+2m-2)}\right)|j_\xi f|^2 = 0.$$

Both coefficients of the equality are positive in the case of $n \geq 2$, $m \geq 1$, and $\xi \neq 0$. Hence, $f = 0$. □

We have thus proved the ellipticity of the principal symbol $j_\xi p i_\xi$ of the operator $-\delta pd$ on the bundle $\operatorname{Ker} j$. Actually, we have shown the principal symbol is positive. This implies the ellipticity of the boundary value problem (5.3). Indeed, as is known [19, Chapter 5, Proposition 11.10], the positivity of the principal symbol implies the Lopatinskiĭ condition for the Dirichlet problem.

Next, we are going to prove the triviality of the kernel of the boundary value problem (5.3). Let $v \in H^k(\operatorname{Ker} j)$ $(k \geq 2)$ be a solution to the homogeneous problem

$$(\delta pd)v = 0, \quad v|_{\partial M} = 0. \tag{5.14}$$

Due to the ellipticity, v is smooth: $v \in C^\infty(\operatorname{Ker} j)$. Applying Green's formula from Theorem 3.1, we have

$$(pdv, pdv)_{L^2} = (pdv, dv)_{L^2} = -(\delta pdv, v)_{L^2} = 0,$$

i.e., $pdv = 0$. Hence, v is a trace-free conformal Killing field. According to Theorem 1.3, if such a field satisfies the homogeneous boundary condition $v\big|_{\partial M} = 0$ then it is identically zero.

Let $S^m\tau'\big|_{\partial M}$ denote the restriction of the bundle $S^m\tau'$ to the boundary. To prove the triviality of the co-kernel of the boundary value problem (5.3), we need the following proposition.

Lemma 5.4. *If a tensor field $u \in C^\infty\big(S^m\tau'\big|_{\partial M}\big)$ satisfies the condition $ju = 0$ then there exists $v \in C^\infty(S^m\tau')$ satisfying the conditions $v\big|_{\partial M} = 0$, $jv = 0$ and such that*

$$j_\nu pdv|_{\partial M} = u, \tag{5.15}$$

where ν is the outward normal vector to the boundary.

The proof of Lemma 5.4 will be given below. We now finish the proof of Theorem 5.1 with the help of the lemma.

Assume a field $w \in C^\infty(\operatorname{Ker} j)$ to be orthogonal to the range of the operator of the boundary value problem (5.3), i.e.,

$$(w, \delta pdv)_{L^2} = 0 \tag{5.16}$$

for every $v \in C^\infty(\operatorname{Ker} j)$ satisfying the boundary condition $v\big|_{\partial M} = 0$. We have to show $w \equiv 0$. We first choose v such that $\operatorname{supp} v \subset M \setminus \partial M$. Green's formula and (5.16) imply

$$(\delta pdw, v)_{L^2} = (w, \delta pdv)_{L^2} = 0.$$

Since $v \in C_0^\infty(\operatorname{Ker} j)$ is arbitrary, this means

$$\delta pdw = 0. \tag{5.17}$$

For an arbitrary $u \in C^\infty\big(\operatorname{Ker} j\big|_{\partial M}\big)$, Lemma 5.4 guaranties the existence of some $v \in C^\infty(\operatorname{Ker} j)$ which satisfies (5.15) and vanishes on the boundary. With the help of Green's formula, (5.15)–(5.17) yield

$$0 = (w, \delta p d v)_{L^2} = (\delta p d w, v)_{L^2} + \int\limits_{\partial M} \langle w, j_\nu p d v\rangle dV' = \int\limits_{\partial M} \langle w, u\rangle dV'.$$

This means $w\big|_{\partial M} = 0$ since u is arbitrary. So, w belongs to the kernel of the boundary problem operator. As was already proved, such w must be identically equal to zero. Theorem 5.1 is proved. □

Proof of Lemma 5.4. In order to simplify the notation, we give here the proof only in the case of an odd m. The case of an even m is considered in a similar way. Both the cases can be considered simultaneously but with much more complicated notation.

In virtue of the condition $v\big|_{\partial M} = 0$, (5.15) can be considered as an algebraic equation in the unknown $\partial v/\partial\nu\big|_{\partial M}$. We are going to prove the existence and uniqueness of a solution to the equation under the condition $ju = 0$. Moreover, the solution satisfies $j\partial v/\partial\nu\big|_{\partial M} = 0$ as will be shown. Then the proof of the existence is realized by choosing a section v of the vector bundle $\operatorname{Ker} j$ with prescribed boundary values $v\big|_{\partial M} = 0$ and $\partial v/\partial\nu\big|_{\partial M}$.

We choose normal boundary coordinates $(x^1, \dots, x^n) = (x^1, \dots, x^{n-1}, y)$ in a neighborhood of a boundary point so that $g_{in} = \delta_{in}$ and the boundary is defined by the equation $y = 0$. Below the Greek indices change from 1 to $n-1$. We define the tensor fields $u^{(s)}, v^{(s)} \in C^\infty(S^s\tau'_{\partial M})$ for $0 \le s \le 2m+1$ by the formulas

$$v^{(s)}_{\alpha_1\dots\alpha_s} = \frac{\partial v_{\alpha_1\dots\alpha_s n\dots n}}{\partial y}\bigg|_{y=0}, \quad u^{(s)}_{\alpha_1\dots\alpha_s} = 2(m+1)(n+4m)u_{\alpha_1\dots\alpha_s n\dots n}. \tag{5.18}$$

The condition $ju = 0$ is expressed in terms of $u^{(s)}$ as follows:

$$u^{(s)} + ju^{(s+2)} = 0.$$

From this,

$$u^{(2s)} = (-j)^{m-s}u^{(2m)}, \quad u^{(2s+1)} = (-j)^{m-s}u^{(2m+1)}. \tag{5.19}$$

Using (3.8), we transform (5.15) to the form

$$j_\nu\left(dpv - \frac{2m+1}{n+4m} i\delta pv\right)\bigg|_{\partial M} = u.$$

If $jv = 0$ then $pv = v$ and the equation simplifies to the following one:

$$j_\nu\left(dv - \frac{2m+1}{n+4m} i\delta v\right)\bigg|_{\partial M} = u. \tag{5.20}$$

We are going to derive some recurrent formulas from (5.20) which uniquely determine the tensors $v^{(s)}$.

In the normal boundary coordinates, the vector ν has the coordinates $(0, \dots, 0, 1)$ and equation (5.20) takes the form

$$(dv)_{ni_1\dots i_{2m+1}}|_{y=0} - \frac{2m+1}{n+4m}(i\delta v)_{ni_1\dots i_{2m+1}}|_{y=0} = u_{i_1\dots i_{2m+1}}. \tag{5.21}$$

Set $(i_1,\dots,i_s)=(\alpha_1,\dots,\alpha_s)$ and $i_{s+1}=\dots=i_{2m+2}=n$ in the equality

$$(dv)_{i_1\dots i_{2m+2}}=\frac{1}{2m+2}\left(\frac{\partial v_{i_2\dots i_{2m+2}}}{\partial x^{i_1}}+\frac{\partial v_{i_1 i_3\dots i_{2m+2}}}{\partial x^{i_2}}+\dots+\frac{\partial v_{i_1\dots i_{2m+2}}}{\partial x^{i_{2m+1}}}\right).$$

The first s summands on the right-hand side vanish on ∂M since $v\big|_{\partial M}=0$. The last $2m-s+2$ summands are pairwise equal. Hence,

$$(dv)_{\alpha_1\dots\alpha_s n\dots n}\big|_{y=0}=\frac{2m-s+2}{2(m+1)}v^{(s)}_{\alpha_1\dots\alpha_s}. \tag{5.22}$$

Similarly, we deduce

$$(\delta v)_{\alpha_1\dots\alpha_s n\dots n}\big|_{y=0}=v^{(s)}_{\alpha_1\dots\alpha_s}. \tag{5.23}$$

Setting $(i_1,\dots,i_s)=(\alpha_1,\dots,\alpha_s)$ and $i_{s+1}=\dots=i_{2m+2}=n$ in the equality

$$(i\delta v)_{i_1\dots i_{2m+2}}=\frac{1}{(2m+2)!}\sum_{\pi\in\Pi_{2m+2}}g_{i_{\pi(1)}i_{\pi(2)}}(\delta v)_{i_{\pi(3)}\dots i_{\pi(2m+2)}}$$

and analyzing the right-hand side in the same way as has been used for deriving (4.9), we obtain

$$\begin{aligned}(i\delta v)_{\alpha_1\dots\alpha_s n\dots n}\Big|_{y=0}=&\frac{(2m-s+1)(2m-s+2)}{(2m+1)(2m+2)}(\delta v)_{\alpha_1\dots\alpha_s n\dots n}\Big|_{y=0}\\&+\frac{s(s-1)}{(2m+1)(2m+2)}\sigma(\alpha_1\dots\alpha_s)\big(g_{\alpha_1\alpha_2}(\delta v)_{\alpha_3\dots\alpha_s n\dots n}\big|_{y=0}\big).\end{aligned}$$

With the help of (5.23), this gives

$$\begin{aligned}(i\delta v)_{\alpha_1\dots\alpha_s n\dots n}\Big|_{y=0}=&\frac{(2m-s+1)(2m-s+2)}{(2m+1)(2m+2)}v^{(s)}_{\alpha_1\dots\alpha_s}\\&+\frac{s(s-1)}{(2m+1)(2m+2)}\big(iv^{(s-2)}\big)_{\alpha_1\dots\alpha_s}.\end{aligned} \tag{5.24}$$

We set $(i_1,\dots,i_s)=(\alpha_1,\dots,\alpha_s)$ and $i_{s+1}=\dots=i_{2m+2}=n$ in (5.21). Then we substitute values (5.22) and (5.24) for the summands on the left-hand side of (5.21) and value (5.18) for the right-hand side of (5.21). In such the way we obtain the recurrent formula

$$(2m-s+2)(n+2m+s-1)v^{(s)}-s(s-1)iv^{(s-2)}=u^{(s)}.$$

In view of (5.19), this formula can be rewritten as

$$2(m-s+1)(n+2m+2s-1)v^{(2s)}-2s(2s-1)iv^{(2s-2)}=(-j)^{m-s}u^{(2m)}, \tag{5.25}$$

$$(2m-2s+1)(n+2m+2s)v^{(2s+1)}-2s(2s+1)iv^{(2s-1)}=(-j)^{m-s}u^{(2m+1)}. \tag{5.26}$$

Formulas (5.25) and (5.26) imply the following representations:

$$v^{(2s)}=\sum_{k=0}^{s}a(s,k)i^k j^{m-s+k}u^{(2m)}, \tag{5.27}$$

$$v^{(2s+1)}=\sum_{k=0}^{s}b(s,k)i^k j^{m-s+k}u^{(2m+1)}, \tag{5.28}$$

with some coefficients $a(s,k)$ and $b(s,k)$ which depend on n,m,s, and k only. The dependence on n and m is not indicated explicitly since the values of these two parameters are fixed in the proof.

Substitution of (5.27) and (5.28) into (5.25) and (5.26), respectively, imply the following recurrent relations:

$$\begin{aligned} a(s,0) &= \frac{(-1)^{m-s}}{2(m-s+1)(n+2m+2s-1)}, \\ a(s,k) &= \frac{(2s-1)}{(m-s+1)(n+2m+2s-1)}a(s-1,k-1) \quad \text{for} \quad 1 \le k \le s, \end{aligned} \tag{5.29}$$

$$\begin{aligned} b(s,0) &= \frac{(-1)^{m-s}}{(2m-2s+1)(n+2m+2s)}, \\ b(s,k) &= \frac{2s(2s+1)}{(2m-2s+1)(n+2m+2s)}b(s-1,k-1) \quad \text{for} \quad 1 \le k \le s. \end{aligned} \tag{5.30}$$

The coefficients $a(s,k)$ and $b(s,k)$ are uniquely determined by equations (5.29) and (5.30), and formulas (5.27) and (5.28) show that the tensors $v^{(2s)}$ and $v^{(2s+1)}$ are uniquely determined by $u^{(2m)}$ and $u^{(2m+1)}$.

Finally, we have to prove the tensors $v^{(2s)}$ and $v^{(2s+1)}$ satisfy the equations

$$v^{(2s)} + jv^{(2s+2)} = 0 \quad \text{for} \quad 0 \le s \le m-1, \tag{5.31}$$

$$v^{(2s+1)} + jv^{(2s+3)} = 0 \quad \text{for} \quad 0 \le s \le m-1 \tag{5.32}$$

that are equivalent to the relation $j\frac{\partial v}{\partial \nu}\big|_{\partial M} = 0$ in view of the first formula in (5.18).

Applying the operator j to equation (5.27), we obtain

$$jv^{(2s+2)} = \sum_{k=0}^{s+1} a(s+1,k) j i^k j^{m-s+k-1} u^{(2m)}. \tag{5.33}$$

We transpose the factors j and i^k on the right-hand side of (5.33) with the help of Lemma 2.2. We have to set $n := n-1$ and $m := 2s-2k+2$ in the statement of the ltmma since $j^{m-s+k-1}u^{(2m)}$ is the tensor of rank $2s-2k+2$ on the $(n-1)$-dimensional manifold ∂M. So we have

$$\begin{aligned} jv^{(2s+2)} = \sum_{k=0}^{s+1} a(s+1,k)\bigg(&\frac{k(n+4s-2k+1)}{(s+1)(2s+1)} i^{k-1} j^{m-s+k-1} \\ &+ \frac{(s-k+1)(2s-2k+1)}{(s+1)(2s+1)} i^k j^{m-s+k}\bigg) u^{(2m)}. \end{aligned}$$

This equality can be transformed as follows:

$$\begin{aligned} jv^{(2s+2)} = \sum_{k=0}^{s} \bigg(&\frac{(s-k+1)(2s-2k+1)}{(s+1)(2s+1)} a(s+1,k) \\ &+ \frac{(k+1)(n+4s-2k-1)}{(s+1)(2s+1)} a(s+1,k+1)\bigg) i^k j^{m-s+k} u^{(2m)}. \end{aligned} \tag{5.34}$$

Substitution of (5.27) and (5.34) into (5.31) gives

$$\begin{aligned} \sum_{k=0}^{s} \bigg[a(s,k) &+ \frac{(s-k+1)(2s-2k+1)}{(s+1)(2s+1)} a(s+1,k) \\ &+ \frac{(k+1)(n+4s-2k-1)}{(s+1)(2s+1)} a(s+1,k+1)\bigg] i^k j^{m-s+k} u^{(2m)} = 0. \end{aligned}$$

Since $u^{(2m)}$ is an arbitrary tensor, the expression in the brackets must be equal to zero for all s and k, i.e.,

$$\begin{aligned} a(s,k) &+ \frac{(s-k+1)(2s-2k+1)}{(s+1)(2s+1)}a(s+1,k) \\ &+ \frac{(k+1)(n+4s-2k-1)}{(s+1)(2s+1)}a(s+1,k+1) = 0 \quad \text{for} \quad 0 \le k \le s \le m-1. \end{aligned} \tag{5.35}$$

Similarly, (5.32) is equivalent to the equation

$$\begin{aligned} b(s,k) &+ \frac{(s-k+1)(2s-2k+3)}{(s+1)(2s+3)}b(s+1,k) \\ &+ \frac{(k+1)(n+4s-2k+1)}{(s+1)(2s+1)}b(s+1,k+1) = 0 \quad \text{for} \quad 0 \le k \le s \le m-1. \end{aligned} \tag{5.36}$$

We have to prove the following statement: Being defined by recurrent formulas (5.29) and (5.30), the coefficients $a(s,k)$ and $b(s,k)$ satisfy equations (5.35) and (5.36), respectively. This can be proved with the help of the following explicit formulas for the coefficients:

$$\begin{aligned} a(s,k) = \frac{(-1)^{m-s-k}}{2} & \frac{s!(2s-1)!!(m-s)!}{(n+2m+2s-1)!!} \\ & \times \frac{(n+2m+2s-2k-3)!!}{(s-k)!(m-s+k+1)!(2s-2k-1)!!}, \end{aligned} \tag{5.37}$$

$$\begin{aligned} b(s,k) = (-1)^{m-s-k} & \frac{s!(2s+1)!!(2m-2s-1)!!}{(n+2m+2s)!!} \\ & \times \frac{2^k(n+2m+2s-2k-2)!!}{(s-k)!(2m-2s+2k+1)!!(2s-2k+1)!!}. \end{aligned} \tag{5.38}$$

Here we use the standard notation:

$$(2k)!! = 2^k k!, \quad (2k+1)!! = (2k+1)(2k-1)\dots 1, \quad (-1)!! = 1.$$

Formulas (5.37) and (5.38) are proved by substituting them into recurrent formulas (5.29) and (5.30) and checking the validity of the resulting equations. Then the validity of (5.35) and (5.36) is proved by substitution of values (5.37) and (5.38) followed by a direct calculation. □

6. Proof of Theorems 1.6 and 1.7

In this section, for a Riemannian manifold, we use the notions of a semibasic tensor field and the vertical and horizontal derivatives of such a field. The corresponding definitions are presented in [15, Ch. 3] (see also [4, § 4] where the case of a Finsler manifold is considered as well). We denote the space of smooth semibasic (r,s)-tensor fields on TM by $C^\infty(\beta^r_s M)$, and $\overset{v}{\nabla}$, $\overset{h}{\nabla} : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M)$ denote the vertical and horizontal derivatives, respectively.

Proof of Theorem 1.6. Let $u \in C^\infty(S^m\tau'_M)$ be a trace-free conformal Killing tensor field, i.e., $ju = 0$ and $du = iv$ for some $v \in C^\infty(S^{m-1}\tau'_M)$. We assume here $m \ge 1$ since the statement of the theorem is trivial in the case of $m = 0$. Define the function $U \in C^\infty(TM) = C^\infty(\beta^0_0 M)$ as follows:

$$U(x,\xi) = u_{i_1\dots i_m}(x)\xi^{i_1}\cdots\xi^{i_m}. \tag{6.1}$$

The function is homogeneous with respect to ξ,

$$U(x,t\xi)=t^mU(x,\xi), \tag{6.2}$$

and satisfies the kinetic equation

$$HU(x,\xi)=|\xi|^2v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}}, \tag{6.3}$$

where H denotes differentiation along the geodesic flow, and $HU(x,\xi)=\xi^i\overset{h}{\nabla}_iU$. Since $ju=0$, the function U satisfies the equation

$$\overset{v}{\Delta}U=0, \tag{6.4}$$

where $\overset{v}{\Delta}=\overset{v}{\nabla}{}^i\overset{v}{\nabla}_i$ is the vertical Laplacian (see Lemma 2.4).

Let us derive the commutation formula for $\overset{v}{\Delta}$ and H. Since the vertical and horizontal derivatives commute,

$$\overset{v}{\Delta}HU=\overset{v}{\nabla}{}^i\overset{v}{\nabla}_i\big(\xi^j\overset{h}{\nabla}_jU\big)=\overset{v}{\nabla}{}^i\big(\overset{h}{\nabla}_iU+\xi^j\overset{h}{\nabla}_j\overset{v}{\nabla}_iU\big)=2\overset{v}{\nabla}{}^i\overset{h}{\nabla}_iU+H\overset{v}{\Delta}U.$$

By (6.4), this gives

$$\overset{v}{\Delta}HU=2\overset{v}{\nabla}{}^i\overset{h}{\nabla}_iU. \tag{6.5}$$

We write the Pestov identity for the function U (see [15])

$$2\big\langle\overset{h}{\nabla}U,\overset{v}{\nabla}HU\big\rangle-\overset{v}{\nabla}_i\big(\overset{h}{\nabla}{}^iU\cdot HU\big)=\big|\overset{h}{\nabla}U\big|^2+\overset{h}{\nabla}_iw^i-R_\xi\big(\overset{v}{\nabla}U\big), \tag{6.6}$$

where

$$R_\xi\big(\overset{v}{\nabla}U\big)=R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^jU\cdot\overset{v}{\nabla}{}^lU \tag{6.7}$$

and w is some semibasic vector field on TM. It depends on U quadratically but its value is not relevant now. Since the sectional curvature is nonpositive, we have

$$R_\xi\big(\overset{v}{\nabla}U\big)\le 0. \tag{6.8}$$

We transform the first summand on the left-hand side of equation (6.6) with the help of (6.5) as follows:

$$\begin{aligned}
2\big\langle\overset{h}{\nabla}U,\overset{v}{\nabla}HU\big\rangle&=2\overset{h}{\nabla}{}^iU\cdot\overset{v}{\nabla}_i(HU)\\
&=\overset{v}{\nabla}_i\big(2\overset{h}{\nabla}{}^iU\cdot HU\big)-2\overset{v}{\nabla}{}^i\overset{h}{\nabla}_iU\cdot HU\\
&=-\overset{v}{\Delta}HU\cdot HU+\overset{v}{\nabla}_i\big(2\overset{h}{\nabla}{}^iU\cdot HU\big)\\
&=-\overset{v}{\nabla}_i\overset{v}{\nabla}{}^iHU\cdot HU+\overset{v}{\nabla}_i\big(2\overset{h}{\nabla}{}^iU\cdot HU\big)\\
&=-\overset{v}{\nabla}_i\big(\overset{v}{\nabla}{}^iHU\cdot HU\big)+\overset{v}{\nabla}{}^iHU\cdot\overset{v}{\nabla}_iHU+\overset{v}{\nabla}_i\big(2\overset{h}{\nabla}{}^iU\cdot HU\big)\\
&=\big|\overset{v}{\nabla}HU\big|^2+\overset{v}{\nabla}_i\big(2\overset{h}{\nabla}{}^iU\cdot HU-\overset{v}{\nabla}{}^iHU\cdot HU\big).
\end{aligned}$$

Substitute this value into (6.6)

$$\big|\overset{v}{\nabla}HU\big|^2+\overset{v}{\nabla}_i\big(\overset{h}{\nabla}{}^iU\cdot HU-\overset{v}{\nabla}{}^iHU\cdot HU\big)=\big|\overset{h}{\nabla}U\big|^2+\overset{h}{\nabla}_iw^i-R_\xi\big(\overset{v}{\nabla}U\big).$$

We integrate this equality over ΩM versus to the Liouville volume form $d\Sigma$ and transform the integrals of divergent terms by the Gauss-Ostrogradsky formulas (see [15, Theorem 3.6.3])

$$\int_{\Omega M}\Big(\big|\overset{v}{\nabla}HU\big|^2+(n+2m)\langle\xi,\overset{h}{\nabla}U-\overset{v}{\nabla}HU\rangle HU\Big)\,d\Sigma=\int_{\Omega M}\Big(\big|\overset{h}{\nabla}U\big|^2-R_\xi\big(\overset{v}{\nabla}U\big)\Big)\,d\Sigma. \tag{6.9}$$

The coefficient $(n+2m)$ appears here because the semibasic vector field $\big(\overset{h}{\nabla}U-\overset{v}{\nabla}HU\big)HU$ is homogeneous of degree $2m+1$ in ξ. With the help of the Euler formula for homogeneous functions

$$\big\langle\xi,\overset{v}{\nabla}HU\big\rangle=(m+1)HU \tag{6.10}$$

and of $\big\langle\xi,\overset{h}{\nabla}U\big\rangle=HU$, formula (6.9) takes the form

$$\int_{\Omega M}\Big(\big|\overset{v}{\nabla}HU\big|^2-m(n+2m)|HU|^2\Big)\,d\Sigma=\int_{\Omega M}\Big(\big|\overset{h}{\nabla}U\big|^2-R_\xi\big(\overset{v}{\nabla}U\big)\Big)\,d\Sigma. \tag{6.11}$$

Now, we estimate the left-hand side of (6.11) as follows. At an arbitrary point $(x,\xi)\in\Omega M$, we represent the vector $\overset{v}{\nabla}HU$ in the form

$$\overset{v}{\nabla}HU=\lambda\xi+\overset{v}{\nabla}{}^{\perp}HU,\quad \big\langle\xi,\overset{v}{\nabla}{}^{\perp}HU\big\rangle=0. \tag{6.12}$$

Here $\lambda=\lambda(x,\xi)$ is some scalar function. The second summand of the representation has a clear geometrical sense: If $\psi_x=HU\big|_{\Omega_xM}$ is a restriction of HU to the unit sphere Ω_xM then $\overset{v}{\nabla}{}^{\perp}HU(x,\xi)=\nabla\psi_x(\xi)$ is the gradient of the function ψ_x at the point $\xi\in\Omega_xM$. Formula (6.3) implies that $\psi_x(\xi)=v_{i_1\dots v_{i_{m-1}}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}}$. Applying the Euler formula (6.10), we see that $\lambda=(m+1)HU$. Thus, (6.12) implies

$$\big|\overset{v}{\nabla}HU\big|^2=(m+1)^2|HU|^2+|\nabla\psi_x|^2. \tag{6.13}$$

By Green's formula,

$$\int_{\Omega_xM}|\nabla\psi_x|^2\,d\omega(\xi)=-\int_{\Omega_xM}\psi_x\Delta_\omega\psi_x\,d\omega(\xi),$$

where Δ_ω is the spherical Laplacian on Ω_xM. The eigenvalues of $-\Delta_\omega$ are $\lambda_k=k(n+k-2)$, $k=0,1,\dots$, and the spherical harmonics of degree k are the eigenfunctions corresponding to λ_k. Since ψ_x is a polynomial of degree $m-1$, the last integral can be estimated as follows:

$$\begin{aligned}\int_{\Omega_xM}|\nabla\psi_x|^2\,d\omega(\xi)&=-\int_{\Omega_xM}\psi_x\Delta_\omega\psi_x\,d\omega(\xi)\\&\le\sup_{k\le m-1}\lambda_k\int_{\Omega_xM}|\psi_x|^2\,d\omega(\xi)\\&=(m-1)(n+m-3)\int_{\Omega_xM}|HU|^2\,d\omega(\xi).\end{aligned}$$

Together with (6.13), this imply

$$\int_{\Omega M}\Big(\big|\overset{v}{\nabla}HU\big|^2-m(n+2m)|HU|^2\Big)\,d\Sigma\le-(2m+n-4)\int_{\Omega M}|HU|^2\,d\Sigma.$$

Taking this inequality into account, we derive from (6.11)

$$-(2m+n-4)\int_{\Omega M}|HU|^2\,d\Sigma \ge \int_{\Omega M}\Big(\big|\overset{h}{\nabla}U\big|^2 - R_\xi\big(\overset{v}{\nabla}U\big)\Big)\,d\Sigma. \tag{6.14}$$

The coefficient $(2m+n-4)$ is nonnegative since $m \ge 1$ and $n \ge 2$. Hence the left-hand side of (6.14) is nonpositive. At the same time, the right-hand side of (6.14) is nonnegative in virtue of (6.8). Thus, both sides of (6.14) are equal to zero. In particular, $\big|\overset{h}{\nabla}U\big|^2 - R_\xi\big(\overset{v}{\nabla}U\big) = 0$ on ΩM. Applying (6.8) once more, we obtain

$$R_\xi(\overset{v}{\nabla}U) = 0 \tag{6.15}$$

and $\overset{h}{\nabla}U = 0$ on ΩM. Hence, $\overset{h}{\nabla}U$ is identically zero on TM. Now, (6.1) implies $0 = \overset{h}{\nabla}_i U = \nabla_i u_{i_1\dots i_m}\xi^{i_1\dots i_m}$. Thus, ∇u is identically zero on M, i.e., u is absolutely parallel.

Now, we prove the statement: $u(x_0) = 0$ if all sectional curvatures at the point x_0 are negative. This implies $u \equiv 0$ since u is absolutely parallel.

Given $\xi \in \Omega_{x_0}$, like in (6.12), we represent the vector $\overset{v}{\nabla}U(x_0,\xi)$ in the form

$$\overset{v}{\nabla}U(x_0,\xi) = \mu(\xi)\xi + \overset{v}{\nabla}{}^{\perp}U(x_0,\xi), \quad \big\langle \xi, \overset{v}{\nabla}{}^{\perp}U(x_0,\xi)\big\rangle = 0. \tag{6.16}$$

Here μ is some scalar function. We claim $\overset{v}{\nabla}{}^{\perp}U(x_0,\xi) = 0$ for all $\xi \in \Omega_{x_0}$. Indeed, assume $\overset{v}{\nabla}{}^{\perp}U(x_0,\xi) \ne 0$ for some ξ. Then, substituting (6.16) into (6.7) and using symmetries of the curvature tensor, we infer

$$R_\xi\big(\overset{v}{\nabla}{}^{\perp}U(x_0,\xi)\big) = K\big(x_0, \xi\wedge\overset{v}{\nabla}U(x_0,\xi)\big)\big|\overset{v}{\nabla}{}^{\perp}U\big|^2 < 0.$$

Here $K\big(x_0,\xi\wedge\overset{v}{\nabla}U(x_0,\xi)\big)$ is the value of the sectional curvature at the point x_0 in the two-dimensional direction $\xi\wedge\overset{v}{\nabla}U(x_0,\xi)$. The last inequality contradicts (6.15).

Hence, $\overset{v}{\nabla}{}^{\perp}U(x_0,\xi) = 0$ for all $\xi \in \Omega_{x_0}$. This means that

$$U\big|_{\Omega_{x_0}M} = c = \text{const.} \tag{6.17}$$

In the case of an odd m, the constant c must be equal to zero since the function $U(x,\xi)$ is odd in ξ. In the case of $m = 2l > 0$, (6.1) and (6.17) imply $u(x_0) = cg^l$. The condition $ju = 0$ implies $c = 0$. Thus, $u(x_0) = 0$ for all m.

We have proved the statements of Theorem 1.6 concerning trace-free conformal Killing tensor fields. We now prove the statements of the theorem concerning a Killing field by induction in the rank m of the field.

The statements are valid in the cases of $m = 0$ and of $m = 1$ since a Killing vector field is a trace-free conformal Killing field as well. Assume $m \ge 2$ and let $u \in C^\infty(S^m\tau_M)$ be a Killing tensor field of rank m. Represent u in the form

$$u = \tilde u + iv, \tag{6.18}$$

where $\tilde u$ satisfies the condition $j\tilde u = 0$. So, $\tilde u$ is a trace-free conformal Killing field, and thus, $\nabla\tilde u = 0$. Applying the operator d to (6.18), we obtain $idv = 0$. Hence, $dv = 0$, i.e., v is a Killing field. We obtain $\nabla v = 0$ by the inductive assumption. So, both summands on the right-hand side of (6.18) are absolutely parallel, and u is also an absolutely parallel field.

The remaining statement on Killing fields is proved in a similar way. □

Proof of Theorem 1.7. We now need the following corollary of Theorem A from [5] (see also the remark after formulation of the theorem in [5]).

Proposition 6.1. *Let (M,g) be a closed Riemannian manifold without conjugate points. Let $h\in C^\infty(M)$ and $\theta\in C^\infty(\tau'_M)$. If the equation*

$$HU(x,\xi)=h(x)+\theta_i(x)\xi^i,\quad (x,\xi)\in\Omega M,$$

has a solution $U\in C^\infty(\Omega M)$ then $h=0$ and θ is an exact 1-form.

Assume now u to be a conformal Killing covector field, i.e.,

$$du=iv \tag{6.19}$$

for some function v on M. Define the function $U\in C^\infty(\Omega M)$ by

$$U(x,\xi)=u_i(x)\xi^i,\quad (x,\xi)\in\Omega M.$$

As follows from (6.19), U satisfies the kinetic equation

$$HU(x,\xi)=v(x)\ \text{ on }\ \Omega M.$$

Applying Proposition 6.1, we obtain $v\equiv 0$. Since

$$\big(du(x)\big)_{ij}\xi^i\xi^j=HU(x,\xi)=v(x)=0,$$

we see that $du=0$, i.e., u is a Killing covector field. If the geodesic flow of (M,g) has a dense orbit in ΩM then $HU=0$ implies $U\equiv\text{const}$. This means that $u\equiv 0$.

Assume now u to be a trace-free conformal Killing symmetric field of rank 2, i.e.,

$$du=iv,\quad ju=0$$

for some covector field v. Define the function

$$U(x,\xi)=u_{ij}(x)\xi^i\xi^j$$

on ΩM. It satisfies the equation

$$HU(x,\xi)=v_i(x)\xi^i,\quad (x,\xi)\in\Omega M. \tag{6.20}$$

By Proposition 6.1, v is an exact 1-form, i.e.,

$$v=d\varphi \tag{6.21}$$

for some function φ on M. Formulas (6.20) and (6.21) imply

$$d(u-\varphi g)=0. \tag{6.22}$$

Since $ju=0$, this means u is the trace-free part of the Killing field $u-\varphi g$. On the other hand, the trace-free part of a Killing tensor field is obviousely a trace-free conformal Killing field.

If the geodesic flow (M,g) has a dense in ΩM orbit then $u-\varphi g=cg$ for some constant c, as follows from (6.22). Together with the condition $ju=0$, this means $u=0$. Theorem 1.6 is proved. ∎

7. Comparison of differential operators modulo low order terms

In our studying differential operators on tensor fields, we will usually ignore low order terms. In order to simplify the exposition, we introduce the following notation: If A and B are two differential expressions, we write

$$Au = Bu \quad (\mathrm{mod}\ \nabla^k u) \ \text{ on } \ \mathcal{A}$$

if there exists a differential operator $L^{(k)}$ of order k such that

$$Au = Bu + L^{(k)}u$$

for all tensor fields u belonging to the subspace $\mathcal{A}$ of the space $C^\infty(\otimes^*\tau')$. The choice of the subspace will be mostly clear from the context. Similar notation is used for differential operators depending on several variables.

Lemma 7.1. *For $k \ge 1$ and $u \in C^\infty(\otimes^m\tau')$,*

$$\nabla_{j_1\dots j_k} u_{i_1\dots i_m} = \sigma(j_1\dots j_k)\big(\nabla_{j_1\dots j_k} u_{i_1\dots i_m}\big) \quad (\mathrm{mod}\ \nabla^{k-2}u).$$

We omit the proof that can be easily carried out by induction in k starting from the following formula for the second order derivatives:

$$\big(\nabla_{jk} - \nabla_{kj}\big) u_{i_1\dots i_m} = \sum_{a=1}^{m} R^p_{i_a kj} u_{i_1\dots i_{a-1} p i_{a+1}\dots i_m}, \tag{7.1}$$

where $\big(R^p_{ijk}\big)$ is the curvature tensor.

Lemma 7.2. *For $v \in C^\infty(S^m\tau')$ and $p \ge 0$,*

$$\begin{aligned}\nabla_{j_1\dots j_{m+p}} v_{i_1\dots i_m} = \sigma(i_1\dots i_m)\sigma(j_1\dots j_{m+p}) &\sum_{l=0}^{m} (-1)^l \binom{p+l-1}{l}\binom{m+p}{m-l} \\ &\times \nabla_{i_{m-l+1}\dots i_m j_{l+p+1}\dots j_{m+p}} (d^p v)_{i_1\dots i_{m-l} j_1\dots j_{l+p}} \quad (\mathrm{mod}\ \nabla^{m+p-2}v).\end{aligned}$$

Proof. Define the tensors u and f as follows:

$$\begin{aligned} u_{i_1\dots i_m j_1\dots j_p k_1\dots k_m} &= \sigma(j_1\dots j_p k_1\dots k_m)\nabla_{j_1\dots j_p k_1\dots k_m} v_{i_1\dots i_m}, \\ f_{i_1\dots i_m j_1\dots j_p k_1\dots k_m} &= \sigma(i_1\dots i_m j_1\dots j_p)\sigma(k_1\dots k_m)\nabla_{j_1\dots j_p k_1\dots k_m} v_{i_1\dots i_m}. \end{aligned} \tag{7.2}$$

By Lemma 7.1, we have

$$\nabla_{j_1\dots j_p k_1\dots k_m} v_{i_1\dots i_m} = u_{i_1\dots i_m j_1\dots j_p k_1\dots k_m} \quad (\mathrm{mod}\ \nabla^{m+p-2}v). \tag{7.3}$$

Applying the operator $\sigma(i_1\dots i_m j_1\dots j_p)\sigma(k_1\dots k_m)$ to this equation and using (7.2), we obtain

$$\sigma(i_1\dots i_m j_1\dots j_p) u_{i_1\dots i_m j_1\dots j_p k_1\dots k_m} = f_{i_1\dots i_m j_1\dots j_p k_1\dots k_m} \quad (\mathrm{mod}\ \nabla^{m+p-2}v).$$

Hence, u and f satisfy the hypotheses of Lemma 2.1. Application of the lemma gives

$$\begin{aligned} u_{i_1\dots i_m j_1\dots j_{m+p}} = \sigma(i_1\dots i_m)\sigma(j_1\dots j_{m+p}) &\sum_{l=0}^{m} (-1)^l \binom{p+l-1}{m-l}\binom{m+p}{m-l} \\ &\times f_{i_1\dots i_{m-l} j_1\dots j_{m+p} i_{m-l+1}\dots i_m} \quad (\mathrm{mod}\ \nabla^{m+p-2}v). \end{aligned} \tag{7.4}$$

From (7.2) and Lemma 7.1, we deduce

$$f_{i_1\dots i_m j_1\dots j_p k_1\dots k_m} = \sigma(k_1\dots k_m)\big(\nabla_{k_1\dots k_m}(d^p v)_{i_1\dots i_m j_1\dots j_p}\big) \quad (\mathrm{mod}\ \nabla^{m+p-2}v).$$

Substituting the last expression into (7.4) and using equality (7.3), we obtain the statement of the lemma. $\square$

Lemma 7.3. *Let $u_i \in C^\infty(\otimes^{m_i}\tau')$ for $1 \le i \le k$. If, for every i, there exists $p_i \ge 1$ such that*

$$\nabla^{p_i} u_i = 0 \quad (\mathrm{mod}\ \nabla^{p_1-1}u_1, \dots, \nabla^{p_k-1}u_k)$$

and $u_i(x_0) = 0$, $\nabla u_i(x_0) = 0$, $\dots$, $\nabla^{p_i-1}u_i(x_0) = 0$ at some point x_0 of a connected manifold M then all the fields u_i are identically equal to zero.

Proof. We present only the scheme of the proof without details. In order to show that u_i vanish at some point x_1, we connect the points x_0 and x_1 by a smooth curve $x(t)$. The components of tensors $u_i(t) = u_i\big(x(t)\big)$ satisfy a linear homogeneous system of ordinary differential equations with homogeneous initial conditions. The order of the system with respect to $u_i(t)$ equals p_i and the system is solved with respect to the highest order derivatives. This implies the statement of the lemma. □

The last two lemmas imply the following proposition.

Lemma 7.4. *Assume M to be connected and $u_i \in C^\infty(S^{m_i}\tau')$, $1 \le i \le k$. If, for every i, there exists p_i such that*

$$d^{p_i} u_i = 0 \quad (\mathrm{mod}\ \nabla^{p_1-1}u_1, \dots, \nabla^{p_k-1}u_k)$$

and

$$u_i(x_0) = 0, \quad \nabla u_i(x_0) = 0, \quad \dots, \quad \nabla^{m_i+p_i-1}u_i(x_0) = 0$$

at some point x_0 then all u_i are identically equal to zero.

8. Commutation formula for d and δ. The operator Δ

Let the operator $\Delta : C^\infty(S^m\tau') \to C^\infty(S^m\tau')$ be defined as follows:

$$(\Delta u)_{i_1\dots i_m} = g^{jk}\nabla_{jk}u_{i_1\dots i_m}. \tag{8.1}$$

This differential operator has the order 2 and the degree 0, and acts on sections of the fiber bundle $S^*\tau'$. Probably, (7.2) is not the best definition of the Laplacian, and some zero order terms should be added to the right-hand side like for the Laplacian on differential forms. However, the most of our statements concerning Δ are formulated modulo low order terms, and such statements are independent of low order terms on the right-hand side of (8.1).

Lemma 8.1. *The operator Δ is formally self-adjoint and satisfies the relations*

$$\Delta i = i\Delta, \tag{8.2}$$

$$\Delta j = j\Delta, \tag{8.3}$$

$$\Delta^l d^k u = d^k \Delta^l u \quad (\mathrm{mod}\ \nabla^{k+2l-2}u), \tag{8.4}$$

$$\Delta^l \delta^k u = \delta^k \Delta^l u \quad (\mathrm{mod}\ \nabla^{k+2l-2}u). \tag{8.5}$$

Proof. As follows from Green's formula, for $u, v \in C_0^\infty(S^*\tau')$,

$$(\Delta u, v)_{L^2} = -(\nabla u, \nabla v)_{L^2}.$$

This implies the first statement of the lemma since the right-hand side of the last formula is symmetric in u and v. Equality (8.2) is proved by a direct calculation in coordinates which is omitted, and (8.3) follows from (8.2) since these relations are dual to each other.

We now prove (8.4) in the case of $k = l = 1$. For $u \in C^\infty(S^m\tau')$, we have

$$(\Delta du)_{i_1\dots i_{m+1}} = \sigma(i_1 \dots i_{m+1})(g^{jk}\nabla_{jki_{m+1}}u_{i_1\dots i_m}),$$

$$(d\Delta u)_{i_1\dots i_{m+1}} = \sigma(i_1 \dots i_{m+1})(g^{jk}\nabla_{i_{m+1}jk}u_{i_1\dots i_m}).$$

By Lemma 7.1, the right-hand sides of these equalities coincide modulo ∇u. This proves (8.4) for $k=l=1$. In the general case, (8.4) is proved by induction in k and l. Formula (8.5) follows from (8.4) by conjugation. $\square$

We define the operator $R\in\mathrm{Hom}(S^*\tau',S^*\tau')$ by setting

$$(Ru)_{i_1\dots i_m}=\sum_{a=1}^{m}g^{ij}R_{ii_a}u_{ji_1\dots\widehat{i_a}\dots i_m}+2\sum_{1\le a<b\le m}g^{ip}g^{jq}R_{ii_aji_b}u_{pqi_1\dots\widehat{i_a}\dots\widehat{i_b}\dots i_m}$$

for $u\in S^m\tau'$. Here (R_{ijkl}) is the curvature tensor, and $\big(R_{ij}=g^{kl}R_{kijl}\big)$ is the Ricci tensor. The second sum on the right-hand side is absent in the case of $m=1$, .

Lemma 8.2. *The following commutation formula holds on $C^\infty(S^m\tau')$:*

$$\delta d=\frac{1}{m+1}(md\delta+\Delta-R).$$

Proof. For $u\in C^\infty(S^m\tau')$, we have

$$\begin{aligned}(m+1)(\delta du)_{i_1\dots i_m}&=(m+1)g^{i_{m+1}i_{m+2}}\nabla_{i_{m+2}}(du)_{i_1\dots i_{m+1}}\\&=g^{i_{m+1}i_{m+2}}\left(\nabla_{i_{m+2}i_{m+1}}u_{i_1\dots i_m}+\sum_{a=1}^{m}\nabla_{i_{m+2}i_a}u_{i_1\dots\widehat{i_a}\dots i_{m+1}}\right).\end{aligned}$$

This can be written in the form

$$\begin{aligned}(m+1)(\delta du)_{i_1\dots i_m}=(\Delta u)_{i_1\dots i_m}&+\sum_{a=1}^{m}\nabla_{i_a}\big(g^{i_{m+1}i_{m+2}}\nabla_{i_{m+2}}u_{i_1\dots\widehat{i_a}\dots i_{m+1}}\big)\\&-\sum_{a=1}^{m}g^{i_{m+1}i_{m+2}}\big(\nabla_{i_ai_{m+2}}-\nabla_{i_{m+2}i_a}\big)u_{i_1\dots\widehat{i_a}\dots i_{m+1}}.\end{aligned}$$

Denote the last sum on the right-hand side of this equality by $A_{i_1\dots i_m}$ and rewrite the formula as

$$(m+1)(\delta du)_{i_1\dots i_m}=(\Delta u)_{i_1\dots i_m}+m(d\delta u)_{i_1\dots i_m}-A_{i_1\dots i_m}. \tag{8.6}$$

According to (7.1), we have

$$A_{i_1\dots i_m}=g^{i_{m+1}i_{m+2}}\sum_{a=1}^{m}\sum_{\substack{b=1\\b\ne a}}^{m+1}R^p_{i_bi_{m+2}i_a}u_{i_1\dots\widehat{i_a}\dots i_{b-1}pi_{b+1}\dots i_{m+1}}.$$

We distinguish the summands corresponding to $b=m+1$. Then

$$\begin{aligned}A_{i_1\dots i_m}=&\sum_{a=1}^{m}g^{i_{m+1}i_{m+2}}R^p_{i_{m+1}i_{m+2}i_a}u_{i_1\dots\widehat{i_a}\dots i_mp}\\&+2\sum_{1\le a<b\le m}g^{i_{m+1}i_{m+2}}R^p_{i_bi_{m+2}i_a}u_{i_1\dots\widehat{i_a}\dots\widehat{i_b}\dots i_{m+1}p}=(Ru)_{i_1\dots i_m}.\end{aligned}$$

The statement of the lemma immediately follows by substituting the last expression into (8.6). $\square$

We are going to use only the following corollary of Lemma 8.2:

$$\delta du=\frac{m}{m+1}d\delta u+\frac{1}{m+1}\Delta u\pmod u\quad\text{for}\quad u\in C^\infty(S^m\tau'). \tag{8.7}$$

Lemma 8.3. *For arbitrary nonnegative integers m, k, and l, the equality*

$$\delta^l d^k u = \frac{l!(m+k-l)!}{(m+k)!}\sum_p \binom{k}{p}\binom{m}{l-p} d^{k-p}\Delta^p\delta^{l-p}u \pmod{\nabla^{k+l-2}u} \tag{8.8}$$

holds for all $u \in C^\infty(S^m\tau')$. The summation in (8.8) *is taken over all integers p under the agreement:*

$$\binom{i}{j} = 0 \ \ \textit{for } j<0 \ \textit{ or } \ i<j, \ \ \textit{and } i! = 0 \ \ \textit{for } i<0. \tag{8.9}$$

Proof. Equality (8.8) is trivial for $k=0$ or $l=0$. For $k=l=1$, it coincides with (8.7). In the case of $l=1$ and and of an arbitrary k, (8.8) looks as follows:

$$\delta d^k u = \frac{m}{m+k} d^k\delta u + \frac{k}{m+k} d^{k-1}\Delta u \pmod{\nabla^{k-1}u}. \tag{8.10}$$

The trivial case $m=k=0$ is not considered here. We prove (8.10) by induction in k. Assume (8.10) to be valid for some $k \ge 1$. Then

$$\begin{aligned}\delta d^{k+1}u = \delta d^k(du) &= \left(\frac{m+1}{m+k+1}d^k\delta + \frac{k}{m+k+1}d^{k-1}\Delta \pmod{\nabla^{k-1}}\right) du\\ &= \frac{m+1}{m+k+1}d^k\delta\, du + \frac{k}{m+k+1}d^{k-1}\Delta\, du \pmod{\nabla^k u}.\end{aligned}$$

Taking (8.4) and (8.7) into account, this gives

$$\begin{aligned}\delta d^{k+1}u &= \frac{m+1}{m+k+1}d^k\left(\frac{m}{m+1}\,d\delta u + \frac{1}{m+1}\,\Delta\, u \pmod{u}\right)\\ &\quad + \frac{k}{m+k+1}d^k\Delta\, u \pmod{\nabla^k u}\\ &= \frac{m}{m+k+1}d^{k+1}\,\delta u + \frac{k+1}{m+k+1}d^k\,\Delta\, u \pmod{\nabla^k u}.\end{aligned}$$

The last relation coincides with (8.10) for $k := k+1$. Hence, (8.10) is proved.

Equality (8.8) is trivial for $l > k+m$ since, in this case, both its sides are equal to zero. Hence it suffices to prove (8.8) for $1 \le l \le k+m$. We use induction in l. For $l=1$, equality (8.8) is already established. Assume now that (8.8) is satisfied for some $1 \le l < k+m$. Then

$$\delta^{l+1}d^k u = \delta(\delta^l d^k u) = \frac{l!(m+k-l)!}{(m+k)!}\sum_p \binom{k}{p}\binom{m}{l-p}(\delta d^{k-p})\Delta^p\delta^{l-p}u \pmod{\nabla^{k+l-1}u}.$$

Using (8.10) and (8.5), we transform the last formula to the following one:

$$\begin{aligned}\delta^{l+1}d^k u = &\frac{l!(m+k-l)!}{(m+k)!}\sum_p \binom{k}{p}\binom{m}{l-p}\\ &\times\left[\frac{m-l+p}{m-l+k}d^{k-p}\Delta^p\delta^{l-p+1}u + \frac{k-p}{m-l+k}d^{k-p-1}\Delta^{p+1}\delta^{l-p}u\right] \pmod{\nabla^{k+l-1}u}.\end{aligned}$$

Combining the first summand in the brackets of the pth term and the second summand of the $(p-1)$th term, we arrive to (8.8) for $l := l+1$. $\square$

Lemma 8.4. *For arbitrary nonnegative integers m and k, the equality*

$$\delta^k iu = \frac{1}{(m+1)(m+2)}\Big(2k(m-k+2)d\delta^{k-1}u + k(k-1)\Delta\delta^{k-2}u + (m-k+1)(m-k+2)i\delta^k u\Big) \pmod{\nabla^{k-2}u} \tag{8.11}$$

is valid for all $u\in C^\infty(S^m\tau')$. If $m+k\ge 2$ then

$$jd^k u = \frac{1}{(m+k-1)(m+k)}\Big(2kmd^{k-1}\delta u + k(k-1)d^{k-2}\Delta u + m(m-1)d^k ju\Big) \pmod{\nabla^{k-2}u}. \tag{8.12}$$

Proof. These equalities are dual to each other. Hence it suffices to prove the second one.

We prove (8.12) by induction in k. This equality is trivial for $k=0$ and coincides with (3.6) for $k=1$. Assume (8.12) to be valid for some $k\ge 1$. Then

$$jd^{k+1}u = (jd^k)du = \frac{1}{(m+k)(m+k+1)}\Big(2k(m+1)d^{k-1}\delta du + k(k-1)d^{k-2}\Delta du + m(m+1)d^k jdu\Big) \pmod{\nabla^{k-1}u}.$$

Transforming each summand on the right-hand side according to Lemmas 8.3, 8.1, and 3.3, respectively, we arrive to (8.12) for $k:=k+1$. □

9. Proof of theorem 1.1 in the case of $n=\dim M\ge 3$

In the case of $m=0$, equation (1.3) reduces to $du=0$ for a scalar function u, and Theorem 1.1 is obvious in this case. Hence we assume $m\ge 1$ in this section.

Roughly speaking, the next lemma allows us to eliminate u from equations (1.2)–(1.3).

Lemma 9.1. *Let $u\in C^\infty(S^m\tau')$ and $v\in C^\infty(S^{m-1}\tau')$ satisfy (1.2)–(1.3). Then*

$$(n+2m-4)d^{m+1}v = i\big((m-1)d^m\delta v - d^{m-1}\Delta v\big) \pmod{\nabla^m u}, \tag{9.1}$$

$$jv = 0, \tag{9.2}$$

$$v = 0 \pmod{\nabla u}. \tag{9.3}$$

Proof. Applying the operator j to equation (1.3) and using Lemma 3.3, we obtain

$$(m+1)jiv = (m+1)jdu = 2\delta u + (m-1)dju.$$

Since $ju=0$, this gives

$$v = \frac{2}{m+1}(ji)^{-1}\delta u. \tag{9.4}$$

Observe that $j(\delta u)=\delta ju=0$. Applying Lemma 2.2 to δu, we obtain

$$(ji)^{-1}\delta u = \frac{m(m+1)}{2(n+2m-2)}\delta u.$$

Substitute this expression into (9.4) to obtain

$$v = \frac{m}{n+2m-2}\delta u. \tag{9.5}$$

In particular, this implies (9.2) and (9.3).

In order to prove (9.1), we introduce the temporary notation $f=iv$. Equation (1.3) can be written as $du=f$. The latter equation can be solved in $\nabla^{m+1}u$. Indeed, applying Lemma 7.2 with $p=1$, we obtain

$$\nabla_{j_1\dots j_{m+1}}u_{i_1\dots i_m}=\sigma(i_1\dots i_m)\sigma(j_1\dots j_{m+1})\sum_{l=0}^{m}(-1)^l\binom{m+1}{l+1}$$
$$\times\nabla_{i_{m-l+1}\dots i_m j_{l+2}\dots j_{m+1}}f_{i_1\dots i_{m-l}j_1\dots j_{l+1}}\pmod{\nabla^{m-1}u}.\qquad(9.6)$$

Our further arguments are different in the cases of $m=1$ and of $m>1$. We first assume $m>1$. We contract equation (9.6) with $g^{i_{m-1}i_m}$ (i.e., multiply this equation by $g^{i_{m-1}i_m}$ and take the sum over i_{m-1} and i_m). In virtue of the equality

$$g^{i_{m-1}i_m}\nabla_{j_1\dots j_{m+1}}u_{i_1\dots i_m}=\nabla_{j_1\dots j_{m+1}}(ju)_{i_1\dots i_{m-2}}=0,$$

we obtain

$$\sigma(j_1\dots j_{m+1})\sum_{l=1}^{m+1}(-1)^l\binom{m+1}{l}g^{i_{m-1}i_m}$$
$$\times\sigma(i_1\dots i_m)\nabla_{i_{m-l+2}\dots i_m j_{l+1}\dots j_{m+1}}f_{i_1\dots i_{m-l+1}j_1\dots j_l}=0\pmod{\nabla^{m-1}u}.\qquad(9.7)$$

Using Lemma 7.1 and the equality $f=iv=0\pmod{\nabla u}$ which follows from (9.3), we can permute the indices in each factor of the product

$$\nabla_{i_{m-l+2}\dots i_m j_{l+1}\dots j_{m+1}}f_{i_1\dots i_{m-l+1}j_1\dots j_l}$$

without violating equation (9.7). Using this observation, we divide all summands of the sum in (9.7) to three groups so that the indices i_{m-1} and i_m belong to the first (second) factor in all summands of the first (third) group, and belong to different factors in the summands of the second group. Rename these indices as $i_{m-1}=p_1$ and $i_m=p_2$ for clarity. In such the way we obtain

$$\sigma(i_1\dots i_{m-2})\sigma(j_1\dots j_{m+1})\sum_{l=1}^{m+1}(-1)^l\binom{m+1}{l}g^{p_1p_2}$$
$$\times\Big((l-1)(l-2)\nabla_{i_{m-l+2}\dots i_{m-2}j_{l+1}\dots j_{m+1}p_1p_2}f_{i_1\dots i_{m-l+1}j_1\dots j_l}$$
$$+2(l-1)(m-l+1)\nabla_{i_{m-l+1}\dots i_{m-2}j_{l+1}\dots j_{m+1}p_2}f_{i_1\dots i_{m-l}j_1\dots j_lp_1}$$
$$+(m-l)(m-l+1)\nabla_{i_{m-l}\dots i_{m-2}j_{l+1}\dots j_{m+1}}f_{i_1\dots i_{m-l-1}j_1\dots j_lp_1p_2}\Big)=0\pmod{\nabla^{m-1}u}.$$

Now, we symmetrize this equation in all free indices (i.e., apply the operator $\sigma(i_1\dots i_{m-2}j_1\dots j_{m+1})$). Changing simultaneously notations as $j_1=i_{m-1},\dots,j_{m+1}=i_{2m-1}$, we get

$$\sigma(i_1\dots i_{2m-1})\sum_{l=1}^{m+1}(-1)^l\binom{m+1}{l}g^{p_1p_2}$$
$$\times\Big[(l-1)(l-2)\nabla_{i_{m+2}\dots i_{2m-1}p_1p_2}f_{i_1\dots i_{m+1}}$$
$$+2(l-1)(m-l+1)\nabla_{i_{m+1}\dots i_{2m-1}p_2}f_{i_1\dots i_mp_1}$$
$$+(m-l)(m-l+1)\nabla_{i_m\dots i_{2m-1}}f_{i_1\dots i_{m-1}p_1p_2}\Big]=0\pmod{\nabla^{m-1}u}.$$

Observe that, for different values of l, the values of the first (second, third) summand in the brackets differ only by some factors. So the equation is transformed to the following form:

$$\sigma(i_1\dots i_{2m-1})\Big(a\nabla_{i_{m+2}\dots i_{2m-1}p_1p_2}f_{i_1\dots i_{m+1}}+b\nabla_{i_{m+1}\dots i_{2m-1}p_2}f_{i_1\dots i_mp_1}$$
$$+c\nabla_{i_m\dots i_{2m-1}}f_{i_1\dots i_{m-1}p_1p_2}\Big)g^{p_1p_2}=0\pmod{\nabla^{m-1}u},\qquad(9.8)$$

where
$$a=\sum_{l=1}^{m+1}(-1)^l\binom{m+1}{l}(l-1)(l-2)=-2,$$
$$b=2\sum_{l=1}^{m+1}(-1)^l\binom{m+1}{l}(l-1)(m-l+1)=2(m+1),$$
$$c=\sum_{l=1}^{m+1}(-1)^l\binom{m+1}{l}(m-l)(m-l+1)=-m(m+1).$$
Substituting these values for the coefficients, we write (9.8) in the coordinate-free form
$$2d^{m-2}\Delta f-2(m+1)d^{m-1}\delta f+m(m+1)d^m jf=0 \pmod{\nabla^{m-1}u}. \tag{9.9}$$
We recall that $f=iv$ and express all summands of (9.9) in terms of v. Lemma 2.2 and (9.2) imply
$$jf=jiv=\frac{2(n+2m-2)}{m(m+1)}v. \tag{9.10}$$
With the help of Lemma 3.3, we deduce
$$\delta f=\delta iv=\frac{2}{m+1}\,dv+\frac{m-1}{m+1}i\delta v. \tag{9.11}$$
Since the operators i and Δ commute, we have
$$\Delta f=\Delta iv=i\Delta v. \tag{9.12}$$
Substituting (9.10)–(9.12) into (9.9) and using the commutation formula $di=id$, we arrive to the relation
$$(n+2m-4)d^m v=i\big((m-1)d^{m-1}\delta v-d^{m-2}\Delta v\big) \pmod{\nabla^{m-1}u}. \tag{9.13}$$
Applying the operator d to this equation, we obtain (9.1).

Let us now consider the case of $m=1$. Equation (9.6) takes the form
$$\nabla_{j_2j_3}u_i=\nabla_{j_2}f_{ij_3}+\nabla_{j_3}f_{ij_2}-\nabla_i f_{j_2j_3} \pmod{u}.$$
After differentiation we obtain
$$\nabla_{j_1j_2j_3}u_i=\nabla_{j_1j_2}f_{ij_3}+\nabla_{j_1j_3}f_{ij_2}-\nabla_{j_1i}f_{j_2j_3} \pmod{\nabla u}. \tag{9.14}$$
According to Lemma 9.1, the third order derivatives satisfy the relation
$$\nabla_{j_1j_2j_3}u_i-\nabla_{j_2j_1j_3}u_i=0 \pmod{\nabla u}.$$
Inserting (9.14) into the last equation, we obtain
$$\nabla_{j_1j_3}f_{ij_2}+\nabla_{ij_2}f_{j_1j_3}-\nabla_{j_2j_3}f_{ij_1}-\nabla_{ij_2}f_{j_2j_3}=0 \pmod{\nabla u}.$$
Now, substituting $f=iv$, we deduce
$$g_{ij_2}\nabla_{j_1j_3}v+g_{j_1j_3}\nabla_{ij_2}v-g_{ij_1}\nabla_{j_2j_3}v-g_{j_2j_3}\nabla_{ij_1}v=0 \pmod{\nabla u}.$$
Contracting this equation with $g^{j_1j_3}$, we arrive to the formula
$$(n-2)\nabla_{ij}v=-(\Delta v)g_{ij} \pmod{\nabla u}$$
that coincides with (9.1) for $m=1$. □

Observe that, in the case of $m>1$, we have established relation (9.13) which is stronger than that of Lemma (9.1).

The next statement plays the main role in the proof of Theorem 1.1.

Lemma 9.2. *Assume $u \in C^\infty(S^m\tau')$ and $v \in C^\infty(S^{m-1}\tau')$ to satisfy (1.2) and (1.3) for $m \ge 1$. Then, for every integer l such that $0 \le l \le 2m$, the equation*

$$\sum_{p=p_1}^{p_2} a_p d^{m-l+p+1}\Delta^{l-p}\delta^p v = i\sum_{p=p_3}^{p_4} b_p d^{m-l+p-1}\Delta^{l-p+1}\delta^p v \pmod{\nabla^{m+l}u} \tag{9.15}$$

is valid with some rational coefficients $a_p = a_p(n,m,l)$ and $b_p = b_p(m,l)$. Here the summation limits are defined as follows:

$$\begin{aligned} p_1 &= p_1(m,l) = \max(0, l-m-1),\\ p_2 &= p_2(m,l) = \min(m-1, l),\\ p_3 &= p_3(m,l) = \max(0, l-m+1),\\ p_4 &= p_4(m,l) = \min(m-1, l+1). \end{aligned} \tag{9.16}$$

The coefficients a_{p_1} and a_{p_2} are not equal to zero.

Proof. Apply the operator δ^l to equation (9.1)

$$(n+2m-4)\delta^l d^{m+1}v = \delta^l i\big((m-1)d^m\delta v - d^{m-1}\Delta v\big) \pmod{\nabla^{m+l}u}.$$

We transform the right-hand side of this equality with the help of Lemma 8.4 and obtain

$$\begin{aligned} &2m(2m-1)(n+2m-4)\delta^l d^{m+1}v - 2l(2m-l)(m-1)d\delta^{l-1}d^m\delta v\\ &\quad - l(l-1)(m-1)\Delta\delta^{l-2}d^m\delta v + 2l(2m-l)d\delta^{l-1}d^{m-1}\Delta v + l(l-1)\Delta\delta^{l-2}d^{m-1}\delta v\\ &\quad = (2m-l)(2m-l-1)i\big[(m-1)\delta^l d^m\delta v - \delta^l d^{m-1}\Delta v\big] \pmod{\nabla^{m+l}u}. \end{aligned}$$

Taking (9.3) into account, we transform each summand on the left-hand side and the summands in the brackets to the form $d^r\Delta^s\delta^t v$ by using the commutation formulas for powers of d, δ, and Δ (see Lemmas 8.1 and 8.3). Elementary but cumbersome calculations lead us to equation (9.15) where the summation is taken over all integers p under the agreement $d^k = \delta^k = \Delta^k = 0$ for $k < 0$, and the coefficients are as follows:

$$\begin{aligned} a_p = &\left[(n+2m-4)\binom{m+1}{l-p} + 2\binom{m-1}{l-p-1} + \binom{m-1}{l-p-2}\right]\binom{m-1}{p}\\ &- (m-1)\left[2\binom{m}{l-p} + \binom{m}{l-p-1}\right]\binom{m-2}{p-1}, \end{aligned} \tag{9.17}$$

$$b_p = (m-1)\binom{m}{l-p+1}\binom{m-2}{p-1} - \binom{m-1}{l-p}\binom{m-1}{p}. \tag{9.18}$$

Agreement (8.9) is used in (9.17) and (9.18).

Elementary arithmetical analysis of formula (9.17) shows that the coefficients a_p can be nonzero for $p_1 \le p \le p_2$ only, where p_1 and p_2 are defined in (9.16), and a_{p_1} and a_{p_2} are definitely nonzero. Similarly, (9.18) implies that b_p can be nonzero for $p_3 \le p \le p_4$ only. □

Proof of Theorem 1.1. Recall that we assume $m \ge 1$. First we prove by induction in k the equality

$$\Delta^{m+k}\delta^{m-k}v = 0 \pmod{\nabla^{3m+k-1}u} \quad \text{for} \quad 0 \le k \le m. \tag{9.19}$$

The equality is trivial for $k = 0$ since $\delta^m v = 0$. Assume (9.19) to be valid for $k = 0, \dots, s-1 < m$. We write (9.15) for $l = 2m - s + 1$ as follows:

$$\sum_{p=m-s}^{m-1} a_p d^{s+p-m} \Delta^{2m-s+p+1} \delta^p v = i \sum_{p=m-s+2}^{m-1} b_p d^{s+p-m-2} \Delta^{2m-s-p+2} \delta^p v \pmod{\nabla^{3m-s+1} u}.$$

We apply the operator Δ^{s-1} to this equality and transform all terms of the resulting formula to the form $d^r \Delta^s \delta^t v$ with the help of (8.4). By Lemma 9.2, $a_{m-s} \neq 0$. We distinguish the first summand on the left-hand side and write the result as follows:

$$\begin{aligned} \Delta^{m+s}\delta^{m-s} v &+ \sum_{p=m-s+1}^{m-1} a'_p d^{s+p-m} \Delta^{2m-p} \delta^p v \\ &= i \sum_{p=m-s+2}^{m-1} b'_p d^{s+p-m-2} \Delta^{2m-p+1} \delta^p v \pmod{\nabla^{3m+s-1} u}. \end{aligned} \tag{9.20}$$

By the inductive hypothesis,

$$\Delta^{m+k}\delta^{m-k} v = 0 \pmod{\nabla^{3m+k-1} u} \text{ for } 0 \le k \le s-1.$$

Setting $k = m - p$ here, we have

$$\Delta^{2m-p}\delta^p v = 0 \pmod{\nabla^{4m-p-1} u} \text{ for } m-s+1 \le p \le m-1.$$

Applying the operators d^{s+p-m} and $d^{s+p-m-2}\Delta$ to this equation, we obtain

$$d^{s+p-m}\Delta^{2m-p}\delta^p v = 0 \pmod{\nabla^{3m+s-1} u} \text{ for } m-s+1 \le p \le m-1,$$
$$d^{s+p-m-2}\Delta^{2m-p+1}\delta^p v = 0 \pmod{\nabla^{3m+s-1} u} \text{ for } m-s+2 \le p \le m-1.$$

Both sums on (9.20) are equal to zero $\pmod{\nabla^{3m+s-1} u}$, as follows from the last two equations. Hence,

$$\Delta^{m+s}\delta^{m-s} v = 0 \pmod{\nabla^{3m+s-1} u}.$$

This coincides with (9.19) in the case of $k = s$. This completes the inductive step. Thus, (9.19) is proved.

Now we prove the equality

$$d^{m+2r+k-1}\Delta^{2m-r-k}\delta^k v = 0 \pmod{\nabla^{5m-2} u} \text{ for } 0 \le r \le 2m,\ 0 \le k \le 2m-r \tag{9.21}$$

by double induction in r and k.

Setting $k := m - k$ in (9.19), we have

$$\Delta^{2m-k}\delta^k v = 0 \pmod{\nabla^{4m-k-1} u} \text{ for } 0 \le k \le m.$$

Applying the operator d^{m+k-1} to this equality, we obtain (9.21) for $r = 0$.

Assume now (9.21) to be valid valid for $0 \le r \le s-1 < 2m$, i.e.,

$$d^{m+2r+k-1}\Delta^{2m-r-k}\delta^k v = 0 \pmod{\nabla^{5m-2} u} \text{ for } 0 \le r \le s-1,\ 0 \le k \le 2m-r. \tag{9.22}$$

We are going to prove (9.21) for $r = s$. To this end we write down (9.15) for $l = 0$ $\big($this is exactly (9.1)$\big)$ as follows:

$$(n+2m-4) d^{m+1} v = i\big((m-1) d^m \delta v - d^{m-1}\Delta v\big) \pmod{\nabla^m u}.$$

Applying the operator $d^{2s-2}\Delta^{2m-s}$ to this equality, we obtain

$$\begin{aligned} &(n+2m-4) d^{m+2s-1}\Delta^{2m-s} v \\ &\quad = i\big((m-1) d^{m+2s-2}\Delta^{2m-s}\delta v - d^{m-2s-3}\Delta^{2m-s+1} v\big) \pmod{\nabla^{5m-2} u}. \end{aligned}$$

Setting $k = 0$ and $r = s - 1$ in (9.22), and then setting $k = 1$ and $r = s - 1$ in (9.22), we see that the right-hand side of the last formula equals zero $\pmod{\nabla^{5m-2}u}$. We have thus proved (9.21) for $r = s$ and $k = 0$.

Assume now (9.21) to be valid for $r = s$ and $0 \le k \le t - 1 < 2m - s$, i.e.,

$$d^{m+2s+k-1}\Delta^{2m-s-k}\delta^k v = 0 \pmod{\nabla^{5m-2}u} \ \text{ for } \ 0 \le k \le t-1. \tag{9.23}$$

We are going to prove (9.21) for $k = t$.

If $t \ge m$, then (9.21) is obviously true for $k = t$, since $\delta^t v = 0$ in this case. Therefore we assume

$$t \le \min(2m - s, m - 1) \tag{9.24}$$

Recall also that

$$1 \le s \le 2m. \tag{9.25}$$

Let us write down (9.15) for $l = t$. By (9.24) and (9.25), this equation takes the form

$$\sum_{p=0}^{t} a_p d^{m-t+p+1}\Delta^{t-p}\delta^p v = i\sum_{p=0}^{p_4} b_p d^{m-t+p-1}\Delta^{t-p+1}\delta^p v \pmod{\nabla^{m+t}u}, \tag{9.26}$$

where

$$p_4 = \min(m - 1, t + 1). \tag{9.27}$$

According to Lemma 9.2, the coefficient a_t in (9.26) is not zero. We distinguish the last summand on the left-hand side of (9.26) and apply the operator $d^{2s+t-2}\Delta^{2m-s-t}$ to this equation

$$\begin{aligned} d^{m+2s+t-1}\Delta^{2m-s-t}\delta^t v + \sum_{p=0}^{t-1} a'_p d^{m+2s+p-1}\Delta^{2m-s-p}\delta^p v \\ = i\sum_{p=0}^{p_4} b'_p d^{m+2s+p-3}\Delta^{2m-s-p+1}\delta^p v \pmod{\nabla^{5m-2}u}. \end{aligned} \tag{9.28}$$

The sum on the left-hand side of (9.28) equals zero $\pmod{\nabla^{5m-2}u}$ by the inductive hypothesis (9.23). We shall prove the same for the right-hand side.

Setting $r = s - 1$ and $k = p$ in (9.22), we obtain

$$d^{m+2s+p-3}\Delta^{2m-s-p+1}\delta^p v = 0 \pmod{\nabla^{5m-2}u} \ \text{ for } \ 0 \le p \le 2m - s + 1. \tag{9.29}$$

Inequalities (9.24) and (9.27) imply $p_4 \le 2m - s + 1$. Therefore all the summands on the right-hand side of (9.28) are equal to zero $\pmod{\nabla^{5m-2}u}$ according to (9.29). Hence, (9.28) implies

$$d^{m+2s+t-1}\Delta^{2m-s-t}\delta^t v = 0 \pmod{\nabla^{5m-2}u}.$$

We have thus proved (9.23) for $k = t$. This completes the inductive step in k and r. Thus, (9.21) is proved.

Setting $r = 2m$ and $k = 0$ in (9.21), we obtain

$$d^{5m-1}v = 0 \pmod{\nabla^{5m-2}u}. \tag{9.30}$$

According to (1.3), we have $du = 0 \pmod{v}$. Applying the operator d^{5m-2} to this equality, we deduce

$$d^{5m-1}u = 0 \pmod{\nabla^{5m-2}v}. \tag{9.31}$$

We write the initial conditions (1.4) in the form

$$u(x_0) = 0, \quad \nabla u(x_0) = 0, \quad \dots, \quad \nabla^{6m-2}u(x_0) = 0. \tag{9.32}$$

From (9.3) and (9.32),

$$v(x_0) = 0, \quad \nabla v(x_0) = 0, \quad \dots, \quad \nabla^{6m-3} v(x_0) = 0. \tag{9.33}$$

We have thus proved that u and v satisfy equations (9.30) and (9.31) and the initial conditions (9.32) and (9.33). Applying Lemma 7.4, we obtain $u \equiv v \equiv 0$ This finishes the proof of Theorem 1.1 in the case of $n \geq 3$. □

Recall that the highest order of derivatives in the initial conditions (1.4) is denoted by $l(m)$. We have shown in the proof that

$$l(m) \leq 6m - 2 \ \text{ if } \ m > 0.$$

As was mentioned after the statement of Theorem 1.1, this estimate is not sharp. The exact value $l(m) = 2m$ was found in [21] for $m = 2$, and in [1, 6] for an arbitrary m. In the same papers, the upper bound

$$\frac{(n+m-3)!(n+m-2)!(n+2m-2)(n+2m-1)(n+2m)}{m!(m+1)!(n-2)!n!}$$

was established for the dimension of the space of trace-free conformal Killing symmetric tensor fields of the rank m on a manifold of dimension $n \geq 3$. Both the estimates are sharp and become equalities in the case of a conformal flat manifold.

10. Spherical harmonics Fourier series expansion of solution to the kinetic equation

Let M be a Riemannian manifold. Recall the operator

$$\lambda : C^\infty(S^*\tau') \to C^\infty(\Omega)$$

was defined in Section 2. Let H be the vector field on $T = TM$ which generates the geodesic flow. The field is expressed by (1.9) in local coordinates. The field is tangent to the submanifold $\Omega \subset T$ at points of the latter submanifold. Hence the field can be considered as a differential operator $H : C^\infty(\Omega) \to C^\infty(\Omega)$ on the submanifold.

Lemma 10.1. *The following equality holds on $C^\infty(S^*\tau')$:*

$$\lambda d = H\lambda. \tag{10.1}$$

Since λ and H are restrictions to Ω of some operators defined on T, it suffices to prove the equality

$$\varkappa d = H\varkappa, \tag{10.2}$$

where H is considered as an operator on T, and the operator $\varkappa : C^\infty(S^*\tau') \to C^\infty(T)$ has been defined in § 2. The last equality can be easily checked by calculations in coordinates, and we omit the calculations.

Now, assume functions $U, F \in C^\infty(\Omega)$ to be linked by the *kinetic equation*

$$HU = F. \tag{10.3}$$

From (10.3), we will deduce some equations that relate the Fourier series of the functions U and F.

By Lemma 2.5, the functions U and F can be uniquely represented by the series

$$U = \sum_{m=0}^{\infty} \lambda u_m, \quad u_m \in C^\infty(S^m\tau'), \quad j u_m = 0, \tag{10.4}$$

$$F = \sum_{m=0}^{\infty} \lambda f_m, \quad f_m \in C^\infty(S^m\tau'), \quad j f_m = 0. \tag{10.5}$$

As well known [17], the Fourier series of a sufficiently smooth function on a sphere can be termwise differentiated with respect to the coordinates of a point of the sphere. The same is true for the differentiation with respect to the coordinates of a point $x \in M$ which play the role of parameters in the series. Hence, (10.4) implies

$$HU = \sum_{m=0}^{\infty} H\lambda u_m \tag{10.6}$$

and the series converges absolutely and uniformly on any compact subset of Ω.

According to Lemma 10.1, we have

$$H\lambda u_m = \lambda d u_m. \tag{10.7}$$

The condition $ju_m = 0$ is equivalent to $pu_m = u_m$. The last equality and Lemma 3.4 imply

$$du_m = dpu_m = pdu_m + \frac{m}{n+2m-2} i\delta pu_m = pdu_m + \frac{m}{n+2m-2} i\delta u_m.$$

Apply the operator λ to this equality and use Lemma 2.4 to obtain

$$\lambda du_m = \lambda pdu_m + \frac{m}{n+2m-2}\lambda\delta u_m.$$

Comparing this formula with (10.7), we see

$$H\lambda u_m = \lambda pdu_m + \frac{m}{n+2m-2}\lambda\delta u_m.$$

Substitute this expression into (10.6) to obtain

$$HU = \sum_{m=0}^{\infty} \lambda\Big(pdu_{m-1} + \frac{m+1}{n+2m}\delta u_{m+1}\Big). \tag{10.8}$$

For convenience, we assume here $u_{-1} = 0$. The expression in parentheses in (10.8) belongs to the kernel of j since δ and j commute. Hence, (10.8) is the Fourier series of the function $HU = F$ with respect to the spherical harmonics, i.e., (10.8) must coincide with (10.5). We have thus proved the following

Theorem 10.2. *Let $U \in C^\infty(\Omega)$ be a solution to the kinetic equation $HU = F$, and let* (10.4) *and* (10.5) *be the spherical Fourier series of U and F, respectively. Then*

$$\delta u_1 = nf_0,$$

$$pdu_m + \frac{m+2}{n+2m+2}\delta u_{m+2} = f_{m+1} \quad \text{for} \quad m = 0, 1, 2, \dots,$$

where $n = \dim M$.

11. Proof of THeorem 1.1 in the two-dimensional case

We assume here $n = \dim\, M = 2$. As well known, an isothermic coordinate system (x, y) exists in some neighborhood of every point of a two-dimensional Riemannian manifold. In such a coordinate system, the Riemannian metric has the form

$$ds^2 = e^{2\mu(x,y)}(dx^2 + dy^2). \tag{11.1}$$

It suffices to prove Theorem 1.1 under the assumption that such a coordinate system is defined on the whole of M.

Define the coordinate system (x, y, θ) on the three-dimensional manifold Ω such that θ is the angle between the unit vector $\xi \in \Omega$ and the coordinate line $y = \text{const}$. The spherical harmonics series expansion of a function $U(x, y, \theta) \in C^\infty(\Omega)$ coincides with the Fourier series with respect to θ. Hence the next statement is a particular case of Lemma 2.5 (cf. the remark after statement of the lemma).

Lemma 11.1. *Let $u \in C^\infty(S^m\tau')$ and let*

$$(\lambda u)(x,y,\theta) = \frac{1}{2}a_0(x,y) + \sum_{k=1}^{m}\big(a_k(x,y)\cos k\theta + b_k(x,y)\sin k\theta\big)$$

be the Fourier series of the function $\lambda u \in C^\infty(\Omega)$. Then

$$(\lambda pu)(x,y,\theta) = a_m(x,y)\cos m\theta + b_m(x,y)\sin m\theta.$$

Assume u and v to satisfy the hypotheses of Theorem 1.1. The assumption $ju = 0$ is equivalent to $pu = u$. Then, according to Lemma 11.1, there exist functions $a, b \in C^\infty(M)$ such that

$$(\lambda u)(x,y,\theta) = a(x,y)\cos m\theta + b(x,y)\sin m\theta. \tag{11.2}$$

The condition $du = iv$ is equivalent to $pdu = 0$. According to Lemma 11.1, this means that the Fourier series of the function λdu does not contain the harmonics of order $m+1$. Due to Lemma 10.1, we have $\lambda du = H\lambda u$. Hence the coefficients at $\cos(m+1)\theta$ and $\sin(m+1)\theta$ in the Fourier series of the function $H\lambda u$ are equal to zero identically in (x, y).

The operator H has the following form in the coordinates (x, y, θ):

$$H = e^{-\mu}\bigg(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} + (-\mu_x\sin\theta + \mu_y\cos\theta)\frac{\partial}{\partial\theta}\bigg). \tag{11.3}$$

This can be derived from (1.9) and (11.1) by a direct calculation that is omitted.

Now, we express $H\lambda u$ in terms of a and b using (11.2) and (11.3), and then expand $H\lambda u$ in the Fourier series in θ. Equating to zero the coefficients of the series at $\cos(m+1)\theta$ and $\sin(m+1)\theta$, we arrive to the following equations:

$$\begin{aligned} &a_x - b_y - m(\mu_x a - \mu_y b) = 0,\\ &a_y + b_x - m(\mu_y a + \mu_x b) = 0. \end{aligned} \tag{11.4}$$

Introducing the notation

$$z = x + iy, \quad w = a + ib, \quad \frac{\partial}{\partial\bar z} = \frac{1}{2}\bigg(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\bigg),$$

we write (11.4) in the complex form

$$\frac{\partial}{\partial\bar z}(e^{-m\mu}w) = 0.$$

So, $e^{-m\mu}w$ is a holomorphic function. According to the hypotheses of Theorem 1.1, this function vanishes together with all its derivatives at some point. Hence it is identically zero. Now, (11.2) shows that $\lambda u \equiv 0$. Hence, $u \equiv 0$ and Theorem 1.1. is proved.

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