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ON CONFORMALLY BIRECURRENT RICCI-RECURRENT MANIFOLDS

BY

STANISŁAW EWERT-KRZEMIENIEWSKI (SZCZECIN)

1. Introduction. Let M be a Riemannian manifold with a possibly indefinite metric g. A tensor field T of type (p,q) on M is called *recurrent* ([12]) if

$$T^{i_1\dots i_p}{}_{k_1\dots k_q,l}T^{h_1\dots h_p}{}_{j_1\dots j_q} - T^{i_1\dots i_p}{}_{k_1\dots k_q}T^{h_1\dots h_p}{}_{j_1\dots j_q,l} = 0 \tag{1}$$

where the comma denotes covariant differentiation with respect to g. If

$$T^{i_1\dots i_p}{}_{k_1\dots k_q,lm}T^{h_1\dots h_p}{}_{j_1\dots j_q} - T^{i_1\dots i_p}{}_{k_1\dots k_q}T^{h_1\dots h_p}{}_{j_1\dots j_q,lm} = 0\,, \tag{2}$$

then the tensor field T is called *birecurrent*. One can easily verify that (1) implies (2), but the converse is false in general. Moreover, (1) yields that at each $x \in M$ such that $T(x) \neq 0$ there exists a unique covariant vector b (called the *recurrence vector* of T) which satisfies

$$T^{i_1\dots i_p}{}_{j_1\dots j_q,l}(x) = b_l(x)T^{i_1\dots i_p}{}_{j_1\dots j_q}(x)\,. \tag{3}$$

Analogously, if $T(x) \neq 0$, then (2) yields that there exists a unique covariant tensor of type $(0,2)$ (called the *tensor of birecurrence*) which satisfies

$$T^{i_1\dots i_p}{}_{j_1\dots j_q,lm}(x) = a_{lm}(x)T^{i_1\dots i_p}{}_{j_1\dots j_q}(x)\,. \tag{4}$$

A Riemannian manifold of dimension $n > 2$ is called *Ricci-recurrent* ([11]) (*birecurrent* [8]) if its Ricci tensor is recurrent (if its curvature tensor is birecurrent). Following Adati and Miyazawa ([1]), an n-dimensional ($n \geq 4$) Riemannian manifold (M,g) will be called *conformally recurrent* if its Weyl conformal curvature tensor

$$\begin{aligned} C_{hijk} = R_{hijk} &- \frac{1}{n-2}[g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}] \\ &+ \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj}) \end{aligned} \tag{5}$$

is recurrent. In [12] the metric form of conformally recurrent Ricci-recurrent manifolds has been obtained.

In [2] and [9] the concept of conformally birecurrent manifold was introduced. Those are Riemannian manifolds of dimension $n \geq 4$ with birecurrent Weyl conformal curvature tensor. That class contains all birecurrent manifolds of dimension $n \geq 4$ as well as conformally recurrent ones. The existence of essentially conformally birecurrent manifolds, i.e., conformally birecurrent manifolds satisfying $C_{hijk,lm} \neq 0$ which are neither conformally recurrent nor birecurrent, was established in [3], [7], [5] for $n = 4$, $n = 2p$ and $n = 2p - 1$ respectively. In all known examples the Ricci tensor is recurrent.

In this paper we shall deal with conformally birecurrent and Ricci-recurrent manifolds M with both the Weyl conformal curvature tensor and the Ricci tensor nowhere vanishing. We shall prove that if $\dim M > 4$, then in some neighbourhood of a generic point there exists a non-trivial null parallel vector field. Moreover, an algebraic form of the curvature tensor will be given. These are generalizations of some results of [12]. In the next paper ([6]) we shall consider conformally birecurrent manifolds admitting some vector fields. Among other things we shall prove that for $n > 4$, if around a generic point there exists a non-trivial null parallel vector field, then in some neighbourhood the Ricci tensor is recurrent. Throughout this paper all manifolds are assumed to be connected and smooth and their metrics are not assumed to be definite.

2. Preliminaries. In the sequel we shall need the following lemmas.

LEMMA 1. *The Weyl conformal curvature tensor satisfies*

$$C_{hijk} = -C_{ihjk} = C_{jkhi}\,, \quad C^r{}_{rjk} = C^r{}_{irk} = C^r{}_{ijr} = 0\,,$$
$$C_{hijk} + C_{hjki} + C_{hkij} = 0\,,$$
$$C^r{}_{ijk,r} = \frac{n-3}{n-2}\left[R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)}(g_{ij}R_{,k} - g_{ik}R_{,j})\right]. \tag{6}$$

LEMMA 2 ([1], eq. 3.7 and [4], p. 91). *The Weyl conformal curvature tensor satisfies*

$$C_{hijk,l} + C_{hikl,j} + C_{hilj,k} = \frac{1}{n-3}[g_{hj}C^r{}_{ikl,r} + g_{hk}C^r{}_{ilj,r} + g_{hl}C^r{}_{ijk,r} - g_{ij}C^r{}_{hkl,r} - g_{ik}C^r{}_{hlj,r} - g_{il}C^r{}_{hjk,r}]\,. \tag{7}$$

LEMMA 3 ([10], Proposition 2). *Let M be a Riemannian manifold of dimension $n \geq 4$. Assume that $R_{ij,[lm]} = B_{lm}R_{ij}$ on a subset U with nowhere vanishing Ricci tensor, and $C_{hijk,[lm]} = A_{lm}C_{hijk}$ on a subset V with nowhere vanishing Weyl conformal curvature tensor. Then $B_{lm} = 0$ on U and $A_{lm} = 0$ on V.*

We shall often assume the following hypothesis:

(A) (M,g) is a conformally birecurrent Ricci-recurrent manifold of dimension $n \geq 4$ with Weyl conformal curvature tensor and Ricci tensor both nowhere vanishing.

Under hypothesis (A), in view of (4) and (3) we have

$$C_{hijk,lm} = a_{lm} C_{hijk}\,, \tag{8}$$

$$R_{ij,l} = b_l R_{ij}\,, \qquad R_{ij,lm} = b_{lm} R_{ij}\,, \tag{9}$$

where $b_{lm} = b_{l,m} + b_l b_m$.

As a consequence of (8), (9), (5) and Lemma 3, we get

PROPOSITION. *Under hypothesis* (A) *we have*

$$C_{hijk,lm} - C_{hijk,ml} = 0\,, \tag{10}$$

$$R_{ij,lm} - R_{ij,ml} = 0\,, \tag{11}$$

$$R_{hijk,lm} - R_{hijk,ml} = 0\,. \tag{12}$$

Hence, the tensors a_{lm} and b_{lm} defined by (8) and (9) are symmetric.

LEMMA 4. *Under hypothesis* (A), *the manifold M is birecurrent iff $a_{lm} = b_{lm}$ everywhere on M.*

Proof. The "only if" part is obvious. On the other hand, by differentiating (5) twice and making use of (8) and (9) we get $R_{,l} = b_l R$, $R_{,lm} = b_{lm} R$ and

$$R_{hijk,lm} - a_{lm} R_{hijk} = \frac{1}{n-2}(b_{lm} - a_{lm})\Big[g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik} - \frac{R}{n-1}(g_{ij}g_{hk} - g_{ik}g_{hj})\Big]\,.$$

This completes the proof.

LEMMA 5 ([10], Proposition 1). *Let M be a Ricci-recurrent manifold such that $b_l(x) \neq 0$ for some $x \in M$. Then*

$$R_{ir}R^r{}_j = \tfrac{1}{2} R R_{ij} \tag{13}$$

on M.

LEMMA 6. *Under assumption* (A) *we have on M*

$$(a_{lm} - b_{lm}) R^{rs} C_{rijs} = 0\,, \tag{14}$$

$$(a_{lm} - b_{lm}) R = 0\,. \tag{15}$$

Proof. By a direct calculation, in view of (5), (11), (13) and the Ricci identity, we find

$$R_{rm}C^r{}_{ijk} + R_{ri}C^r{}_{mjk} = \frac{3-n}{2(n-1)(n-2)} R(g_{ij}R_{mk} - g_{ik}R_{mj} + g_{mj}R_{ik} - g_{mk}R_{ij})\,, \tag{16}$$

which, by contraction with g^{mk} and the use of Lemma 1, implies

$$R^{rs}C_{rijs} = \frac{3-n}{2(n-1)(n-2)} R(g_{ij}R - nR_{ij})\,. \tag{17}$$

Differentiating (17) covariantly and taking into account (9) and (17) we get

$$R^{rs}C_{rijs,l} = b_l R^{rs}C_{rijs}\,. \tag{18}$$

Differentiating (18), in virtue of (8), (9) and (18), we obtain (14). Moreover, substituting (17) into (14), we have $(a_{lm} - b_{lm})R(Rg_{ij} - nR_{ij}) = 0$. Transvecting with $R^i{}_k$ and applying (13) we easily obtain (15). This completes the proof.

Lemma 7. *Under assumption* (A) *we have on* M

$$(a_{lm} - b_{lm})R_{nr}C^r{}_{ijs}b^s{}_p = 0\,.$$

Proof. (15) and (16) yield

$$(a_{lm} - b_{lm})(R_{rn}C^r{}_{ijk} + R_{ri}C^r{}_{njk}) = 0\,. \tag{19}$$

Permuting cyclically the indices n, j, k in (19) and adding the resulting equations to (19) we get

$$(a_{lm} - b_{lm})(R_{rn}C^r{}_{ijk} + R_{rj}C^r{}_{ikn} + R_{rk}C^r{}_{inj}) = 0\,. \tag{20}$$

Since $R^r{}_{i,r} = b_r R^r{}_i = \frac{1}{2}b_i R$, which follows from (9), by transvecting (20) with $b^k{}_p = g^{kr}b_{rp}$, in virtue of (15), we obtain

$$(a_{lm} - b_{lm})(R_{rn}C^r{}_{ijs}b^s{}_p - R_{rj}C^r{}_{ins}b^s{}_p) = 0\,. \tag{21}$$

Symmetrizing in (j, i) and taking account of (19) we find

$$(a_{lm} - b_{lm})(R_{rn}C^r{}_{ijs}b^s{}_p + R_{rn}C^r{}_{jis}b^s{}_p) = 0\,. \tag{22}$$

Finally, adding (21) with i, n interchanged to (22) and using (19), we get our lemma.

Assume that there exists $x \in M$ such that

$$a_{lm}(x) - b_{lm}(x) \neq 0\,. \tag{B}$$

Lemma 8. *Under hypotheses* (A) *and* (B) *we have*

$$R = 0$$

and

$$R_{hr}C^r{}_{ijk} + R_{ir}C^r{}_{hjk} = 0$$

in some neighbourhood of x.

Proof. In view of hypothesis (B) this is a simple consequence of (15) and (16).

Lemma 9. *Under assumptions* (A) *and* (B) *we have*

$$a_{rp}R^r{}_x R_{ts}C^s{}_{ijl} = 0 \tag{23}$$

in some neighbourhood of x.

Proof. Substituting (6) into (7) and applying Lemma 8 we have

$$\begin{aligned} &C_{hijk,l} + C_{hikl,j} + C_{hilj,k} \\ &\quad = \frac{1}{n-2}[g_{hj}(R_{ik,l} - R_{il,k}) + g_{hk}(R_{il,j} - R_{ij,l}) \\ &\qquad + g_{hl}(R_{ij,k} - R_{ik,j}) - g_{ij}(R_{hk,l} - R_{hl,k}) - g_{ik}(R_{hl,j} - R_{hj,l}) \\ &\qquad - g_{il}(R_{hj,k} - R_{hk,j})]\,. \end{aligned}$$

Differentiating and making use of (8) and (9) we get

$$\begin{aligned} &a_{lm}C_{hijk} + a_{jm}C_{hikl} + a_{km}C_{hilj} \\ &\quad = \frac{1}{n-2}[g_{hj}(R_{ik}b_{lm} - R_{il}b_{km}) + g_{hk}(R_{il}b_{jm} - R_{ij}b_{lm}) \\ &\qquad + g_{hl}(R_{ij}b_{km} - R_{ik}b_{jm}) - g_{ij}(R_{hk}b_{lm} - R_{hl}b_{km}) \\ &\qquad - g_{ik}(R_{hl}b_{jm} - R_{hj}b_{lm}) - g_{il}(R_{hj}b_{km} - R_{hk}b_{jm})]\,, \end{aligned} \tag{24}$$

which, by transvection with ${C^k}_{pqt}$, yields

$$\begin{aligned} &a_{lm}C_{hijr}{C^r}_{pqt} - a_{jm}C_{hilr}{C^r}_{pqt} + a_{rm}{C^r}_{pqt}C_{hilj} \\ &\quad = \frac{1}{n-2}[(g_{hj}b_{lm} - g_{hl}b_{jm})R_{ir}{C^r}_{pqt} - (g_{ij}b_{lm} - g_{il}b_{jm})R_{hr}{C^r}_{pqt} \\ &\qquad + C_{hpqt}(R_{il}b_{jm} - R_{ij}b_{lm}) - C_{ipqt}(R_{hl}b_{jm} - R_{hj}b_{lm}) \\ &\qquad + (-g_{hj}R_{il} + g_{hl}R_{ij} + g_{ij}R_{hl} - g_{il}R_{hj})b_{rm}{C^r}_{pqt}]\,. \end{aligned} \tag{25}$$

Changing in (25) the indices (h, i, j) to (t, q, p) respectively we get

$$\begin{aligned} &a_{lm}C_{tqpr}{C^r}_{jih} - a_{pm}C_{tqlr}{C^r}_{jih} + a_{rm}{C^r}_{jih}C_{tqlp} \\ &\quad = \frac{1}{n-2}[(g_{tp}b_{lm} - g_{tl}b_{pm})R_{qr}{C^r}_{jih} - (g_{qp}b_{lm} - g_{ql}b_{pm})R_{tr}{C^r}_{jih} \\ &\qquad + C_{tjih}(R_{ql}b_{pm} - R_{qp}b_{lm}) - C_{qjih}(R_{tl}b_{pm} - R_{tp}b_{lm}) \\ &\qquad + (-g_{tp}R_{ql} + g_{tl}R_{qp} + g_{qp}R_{tl} - g_{ql}R_{tp})b_{rm}{C^r}_{jih}]\,. \end{aligned} \tag{26}$$

Interchanging j and l gives

$$\begin{aligned} &a_{jm}C_{tqpr}{C^r}_{lih} - a_{pm}C_{tqjr}{C^r}_{lih} + a_{rm}{C^r}_{lih}C_{tqjp} \\ &\quad = \frac{1}{n-2}[(g_{tp}b_{jm} - g_{tj}b_{pm})R_{qr}{C^r}_{lih} - (g_{qp}b_{jm} - g_{qj}b_{pm})R_{tr}{C^r}_{lih} \\ &\qquad + C_{tlih}(R_{qj}b_{pm} - R_{qp}b_{jm}) - C_{qlih}(R_{tj}b_{pm} - R_{tp}b_{jm}) \\ &\qquad + (-g_{tp}R_{qj} + g_{tj}R_{qp} + g_{qp}R_{tj} - g_{qj}R_{tp})b_{rm}{C^r}_{lih}]\,. \end{aligned} \tag{27}$$

Adding (25) to (27) and subtracting (26) we get

$$\begin{aligned} &a_{pm}(C_{tqlr}{C^r}_{jih} - C_{tqjr}{C^r}_{lih}) \\ &\quad = a_{rm}{C^r}_{jih}C_{tqlp} - a_{rm}{C^r}_{pqt}C_{hilj} - a_{rm}{C^r}_{lih}C_{tqjp} \end{aligned} \tag{28}$$

$$
\begin{aligned}
&+\frac{1}{n-2}[(g_{hj}b_{lm}-g_{hl}b_{jm})R_{ir}C^{r}{}_{pqt}-(g_{ij}b_{lm}-g_{il}b_{jm})R_{hr}C^{r}{}_{pqt}\\
&+C_{hpqt}(R_{il}b_{jm}-R_{ij}b_{lm})-C_{ipqt}(R_{hl}b_{jm}-R_{hj}b_{lm})\\
&+(-g_{hj}R_{il}+g_{hl}R_{ij}+g_{ij}R_{hl}-g_{il}R_{hj})b_{rm}C^{r}{}_{pqt}\\
&+(g_{tp}b_{jm}-g_{tj}b_{pm})R_{qr}C^{r}{}_{lih}-(g_{qp}b_{jm}-g_{qj}b_{pm})R_{tr}C^{r}{}_{lih}\\
&+C_{tlih}(R_{qj}b_{pm}-R_{qp}b_{jm})-C_{qlih}(R_{tj}b_{pm}-R_{tp}b_{jm})\\
&+(-g_{tp}R_{qj}+g_{tj}R_{qp}+g_{qp}R_{tj}-g_{qj}R_{tp})b_{rm}C^{r}{}_{lih}\\
&-(g_{tp}b_{lm}-g_{tl}b_{pm})R_{qr}C^{r}{}_{jih}+(g_{qp}b_{lm}-g_{ql}b_{pm})R_{tr}C^{r}{}_{jih}\\
&-C_{tjih}(R_{ql}b_{pm}-R_{qp}b_{lm})+C_{qjih}(R_{tl}b_{pm}-R_{tp}b_{lm})\\
&-(-g_{tp}R_{ql}+g_{tl}R_{qp}+g_{qp}R_{tl}-g_{ql}R_{tp})b_{rm}C^{r}{}_{jih}]\,,
\end{aligned}
$$

since $C_{tqlr}C^{r}{}_{jih}=C_{hijr}C^{r}{}_{lqt}$.

On the other hand, applying the Ricci identity, (10), (5) and Lemma 8, we have

$$
\begin{aligned}
(29)\qquad &C_{rijl}C^{r}{}_{hqt}+C_{hrjl}C^{r}{}_{iqt}+C_{hirl}C^{r}{}_{jqt}+C_{hijr}C^{r}{}_{lqt}\\
&=\frac{-1}{n-2}[g_{hq}R_{t}{}^{r}C_{rijl}-g_{ht}R_{q}{}^{r}C_{rijl}+R_{hq}C_{tijl}-R_{ht}C_{qijl}\\
&\quad+g_{iq}R_{t}{}^{r}C_{hrjl}-g_{it}R_{q}{}^{r}C_{hrjl}+R_{iq}C_{htjl}-R_{it}C_{hqjl}\\
&\quad+g_{jq}R_{t}{}^{r}C_{hirl}-g_{jt}R_{q}{}^{r}C_{hirl}+R_{jq}C_{hitl}-R_{jt}C_{hiql}\\
&\quad+g_{lq}R_{t}{}^{r}C_{hijr}-g_{lt}R_{q}{}^{r}C_{hijr}+R_{lq}C_{hijt}-R_{lt}C_{hijq}]\,.
\end{aligned}
$$

Symmetrizing (28) in (h,i) and (l,j), substituting (29), then contracting the resulting equation with g^{hq} (cf. [11], Lemma 9) and applying Lemmas 5–8, we get

$$
\begin{aligned}
(30)\qquad &-a_{pm}R_{tr}C^{r}{}_{ijl}\\
&\quad=\frac{n-3}{n-2}[b_{lm}R_{jr}C^{r}{}_{tip}+b_{pm}R_{ir}C^{r}{}_{tjl}\\
&\qquad+b_{jm}R_{lr}C^{r}{}_{tpi}+b_{im}R_{pr}C^{r}{}_{tlj}]\\
&\qquad-\frac{n-3}{n-2}R_{pt}b_{rm}C^{r}{}_{ijl}+2\frac{n-3}{n-2}b_{tm}R_{pr}C^{r}{}_{ijl}\\
&\qquad+\frac{1}{n-2}[R_{ij}((n-2)b_{rm}C^{r}{}_{tpl}+2b_{rm}C^{r}{}_{plt})\\
&\qquad-R_{il}((n-2)b_{rm}C^{r}{}_{tpj}+2b_{rm}C^{r}{}_{pjt})\\
&\qquad+R_{tl}b_{rm}C^{r}{}_{jpi}+R_{tj}b_{rm}C^{r}{}_{lip}+R_{ti}b_{rm}C^{r}{}_{pjl}]\,,
\end{aligned}
$$

which, by further transvection with $R^{m}{}_{x}$, implies (23).

LEMMA 10. *Let a_{jm}, T_{pjih}, b_{jm}, W_{pjih} be numbers satisfying*

$$
(31)\qquad T_{pjih}=-T_{jpih}\,,\qquad W_{pjih}=-W_{jpih}\,,
$$

(32) $$a_{jm}T_{pkih} - a_{km}T_{pjih} = b_{jm}W_{pkih} - b_{km}W_{pjih}\,.$$

Then $a_{jm}T_{pkih} = b_{jm}W_{pkih}$.

Proof. Symmetrizing (32) in (p,j) and using (31) we get

(33) $$a_{jm}T_{pkih} + a_{pm}T_{jkih} = b_{jm}W_{pkih} + b_{pm}W_{jkih}\,,$$

whence, by interchanging j and k,

(34) $$a_{km}T_{pjih} + a_{pm}T_{kjih} = b_{km}W_{pjih} + b_{pm}W_{kjih}\,.$$

Adding (32), (33), (34) and applying (31) we get the assertion.

LEMMA 11. *Under assumptions* (A) *and* (B) *the relations*

(35) $$a_{rm}C^r{}_{ijk} = \frac{n-3}{n-2}(R_{ij}b_{km} - R_{ik}b_{jm})\,,$$

(36) $$\begin{aligned}(n-3)(b_{qm}R_{tr}C^r{}_{jih} - b_{tm}R_{qr}C^r{}_{jih} + b_{hm}R_{ir}C^r{}_{jqt} - b_{im}R_{hr}C^r{}_{jqt})\\ = 2b_{jm}(R_{hr}C^r{}_{iqt} + R_{qr}C^r{}_{tih}) + R_{ij}b_{rm}C^r{}_{hqt}\\ - R_{hj}b_{rm}C^r{}_{iqt} + R_{tj}b_{rm}C^r{}_{qih} - R_{qj}b_{rm}C^r{}_{tih}\,,\end{aligned}$$

and

(37) $$\begin{aligned}&-a_{pm}R_{tr}C^r{}_{ijl} + a_{tm}R_{pr}C^r{}_{ijl}\\ &= \frac{n-3}{n-2}(b_{lm}R_{jr}C^r{}_{itp} - b_{jm}R_{lr}C^r{}_{itp} + b_{tm}R_{pr}C^r{}_{ijl} - b_{pm}R_{tr}C^r{}_{ijl})\\ &\quad + 2\frac{n-3}{n-2}(b_{im}R_{tr}C^r{}_{pjl} + b_{tm}R_{pr}C^r{}_{ijl} + b_{pm}R_{ir}C^r{}_{tjl})\\ &\quad + \frac{n}{n-2}(R_{ij}b_{rm}C^r{}_{lpt} - R_{il}b_{rm}C^r{}_{jpt})\\ &\quad + \frac{1}{n-2}(R_{tl}b_{rm}C^r{}_{jpi} - R_{pl}b_{rm}C^r{}_{jti} + R_{tj}b_{rm}C^r{}_{lip}\\ &\qquad - R_{pj}b_{rm}C^r{}_{lit} + R_{ti}b_{rm}C^r{}_{pjl} - R_{pi}b_{rm}C^r{}_{tjl})\end{aligned}$$

are satisfied in some neighbourhood of x.

Proof. Differentiating (6), then using (8), (9) and Lemma 8 we get (35). Contracting (28) with g^{lp} and making use of (35), Lemmas 7 and 8 we obtain (36). Finally, alternating (30) in (t,p), we have (37), which completes the proof.

LEMMA 12. *Under conditions* (A) *and* (B), *if* $a_{rm}R^r{}_p = 0$, *then* $a_{tm}R_{qr}C^r{}_{ijk} = 0$ *on* M.

Proof. Assume $a_{pm}(x) \neq 0$. Transvecting (24) with $R^l{}_p$, in virtue of Lemmas 5 and 8, we get

$$a_{jm}R_{pr}C^r{}_{kih} - a_{km}R_{pr}C^r{}_{jih}$$

$$= \frac{-1}{n-2}[b_{jm}(R_{ph}R_{ki} - R_{pi}R_{kh}) - b_{km}(R_{ph}R_{ji} - R_{pi}R_{jh})]\,.$$

It is easy to see that if we put

$$R_{pr}C^r{}_{kih} = T_{pkih}\,, \qquad \frac{-1}{n-2}(R_{ph}R_{ki} - R_{pi}R_{kh}) = W_{pkih}\,,$$

then, in view of Lemma 8, the assumptions of Lemma 10 are satisfied. Thus we have

$$a_{jm}R_{pr}C^r{}_{kih} = \frac{-1}{n-2}b_{jm}(R_{ph}R_{ki} - R_{pi}R_{kh})\,, \tag{38}$$

whence, alternating in (p, k) and (h, i), we get

$$a_{jm}(R_{pr}C^r{}_{kih} - R_{hr}C^r{}_{ikp}) = 0\,. \tag{39}$$

Applying this in (36), permuting cyclically the indices h, i, j and adding the three resulting equations we obtain

$$\begin{aligned} 2(n-3)(b_{hm}R_{ir}C^r{}_{jqt} + b_{im}R_{jr}C^r{}_{hqt} + b_{jm}R_{hr}C^r{}_{iqt}) &\\ = R_{tj}b_{rm}C^r{}_{qih} - R_{qj}b_{rm}C^r{}_{tih} + R_{th}b_{rm}C^r{}_{qji} &\\ - R_{qh}b_{rm}C^r{}_{tji} + R_{ti}b_{rm}C^r{}_{qhj} - R_{qi}b_{rm}C^r{}_{thj}\,. & \end{aligned} \tag{40}$$

Now, changing in (36) and (40) the indices (q, t, j, i, h) to (l, j, i, t, p) respectively and substituting the obtained expressions into the first and second rows of the right-hand side of (37) we get

$$a_{tm}R_{pr}C^r{}_{ijl} - a_{pm}R_{tr}C^r{}_{ijl} = R_{ij}b_{rm}C^r{}_{lpt} - R_{il}b_{rm}C^r{}_{jpt}\,. \tag{41}$$

On the other hand, applying the Ricci identity to (10) and transvecting with $a^h{}_t$, in virtue of (35) and (11), we find

$$a_{rt}R^r{}_{slm}C^s{}_{ijk} = \frac{n-3}{n-2}(R_{ij}b_{rt}R^r{}_{klm} - R_{ik}b_{rt}R^r{}_{jlm})\,.$$

Hence, by the use of (5), (35) and Lemma 8, we have

$$\begin{aligned} (a_{tm}+(n-3)b_{tm})R_{lr}C^r{}_{ijk} - (a_{tl} + (n-3)b_{tl})R_{mr}C^r{}_{ijk} &\\ = (n-3)(R_{ij}b_{rt}C^r{}_{klm} - R_{ik}b_{rt}C^r{}_{jlm}) &\\ + \frac{n-3}{n-2}[b_{tm}(R_{ij}R_{kl} - R_{ik}R_{jl}) - b_{tl}(R_{ij}R_{km} - R_{ik}R_{jm})]\,. & \end{aligned}$$

Since (41) and (38) hold, the right-hand side of the above equation vanishes. Symmetrizing the resulting equation in (m, i), in virtue of Lemma 8, we obtain $(a_{tm} + (n-3)b_{tm})R_{lr}C^r{}_{ijk} = 0$. Assume that at some $x \in M$ we have $a_{tm} + (n-3)b_{tm} = 0$. Then (B) and (38) lead to

$$(n-3)R_{pr}C^r{}_{kih} = \frac{1}{n-2}(R_{ki}R_{ph} - R_{kh}R_{pi})\,,$$

whence, by covariant differentiation and the use of (8) and (9), we have

$$(n-3)a_{lm}R_{pr}C^{r}{}_{kih} = \frac{1}{n-2}b_{lm}(R_{ki}R_{ph} - R_{kh}R_{pi})\,.$$

Comparing the last result with (38) we get $R_{pr}C^{r}{}_{kih} = 0$ at x. This completes the proof.

LEMMA 13. *Under hypotheses* (A) *and* (B) *suppose that* $R_{ri}C^{r}{}_{jkl} = 0$. *Then*

$$R_{ij}b_{rm}C^{r}{}_{lpt} - R_{il}b_{rm}C^{r}{}_{jpt} = 0\,. \tag{42}$$

If, *moreover*, $a_{hp}(x) \neq 0$, *then*

$$a_{rp}R^{r}{}_{q}C_{tijl} = \frac{n-3}{n-2}b_{qp}(R_{ij}R_{tl} - R_{il}R_{tj}) \tag{43}$$

on some open U.

Proof. We set $M_{mijk} = b_{rm}C^{r}{}_{ijk}$. Then $M_{mijk} = -M_{mikj}$ and $M_{mijk} + M_{mjki} + M_{mkij} = 0$. In view of the assumptions, (36) and (37) can be rewritten as

$$R_{ij}M_{mhqt} - R_{hj}M_{miqt} = R_{tj}M_{mqhi} - R_{qj}M_{mthi}\,, \tag{44}$$

$$\begin{aligned} n(R_{ij}M_{mlpt} - R_{il}M_{mjpt}) + R_{tl}M_{mjpi} - R_{pl}M_{mjti} \\ + R_{tj}M_{mlip} - R_{pj}M_{mlit} + R_{ti}M_{mpjl} - R_{pi}M_{mtjl} = 0\,. \end{aligned} \tag{45}$$

Changing in (44) the indices (i,j,h,q,t) to (t,i,p,j,l) respectively and applying the obtained expression to the last two terms in (45) we get

$$\begin{aligned} (n-1)(R_{ij}M_{mlpt} - R_{il}M_{mjpt}) \\ + R_{tl}M_{mjpi} - R_{pl}M_{mjti} + R_{tj}M_{mlip} - R_{pj}M_{mlit} = 0\,. \end{aligned} \tag{46}$$

Alternating (46) in (t,p) and (j,l) we have

$$\begin{aligned} (n-1)(R_{ij}M_{mlpt} - R_{il}M_{mjpt} + R_{ip}M_{mtlj} - R_{it}M_{mplj}) \\ - R_{tl}M_{mijp} + R_{pj}M_{mitl} - R_{lp}M_{mitj} + R_{jt}M_{milp} = 0\,. \end{aligned}$$

Applying (44) to the first pair of terms in the (second) brackets we find that the bracketed expression vanishes and, consequently,

$$R_{tl}M_{mijp} - R_{pl}M_{mijt} = R_{tj}M_{milp} - R_{pj}M_{milt}\,. \tag{47}$$

Moreover, commuting in (47) i into j and l,j,i into j,i,l respectively, we obtain

$$\begin{aligned} R_{tl}M_{mjpi} - R_{pl}M_{mjti} = R_{ti}M_{mjpl} - R_{pi}M_{mjtl}\,, \\ R_{tj}M_{mlip} - R_{pj}M_{mlit} = R_{ti}M_{mljp} - R_{pi}M_{mljt}\,. \end{aligned} \tag{48}$$

Finally, commuting in (44) the indices (j,q,h,i) into (i,p,j,l), we get

$$R_{li}M_{mjpt} - R_{ji}M_{mlpt} = R_{ti}M_{mpjl} - R_{pj}M_{mtjl}\,. \tag{49}$$

Substituting (48) into (46) and taking into account equation (49) we obtain

$$(n-2)(R_{ij}M_{mlpt} - R_{il}M_{mjpt}) = 0\,, \tag{50}$$

whence (42) follows.

On the other hand, transvecting (29) with $a^h{}_p$ and making use of (35) and (42), we find

$$\begin{aligned} &a_{rp}R^r{}_q C_{tijl} - a_{rp}R^r{}_t C_{qijl} \\ &\quad = \frac{n-3}{n-2}[b_{qp}(R_{ij}R_{tl} - R_{il}R_{tj}) - b_{tp}(R_{ij}R_{ql} - R_{il}R_{qj})]\,, \end{aligned}$$

which, in virtue of Lemma 10, implies (43).

Now we assume the following hypothesis:

(C) (M,g) is a conformally birecurrent and Ricci recurrent manifold of dimension $n > 4$ with Weyl conformal curvature tensor and Ricci tensor both nowhere vanishing. Moreover, there is $x \in M$ such that

$$a_{lm}(x) - b_{lm}(x) \neq 0\,.$$

LEMMA 14. *Under hypothesis* (C) *let* $R_{ir}C^r{}_{jkl} = 0$. *Then*

$$a_{rm}R^r{}_p = 0 \tag{51}$$

on some open $V \ni x$.

P r o o f. We can assume $a_{lm}(x) \neq 0$. Then, by Lemma 13, (43) is satisfied on some $U \ni x$. For the set of points at which b_{qp} vanishes (51) is obvious. Let $y \in U$ and $b_{qp}(y) \neq 0$. Transvecting (43) with $C^t{}_{abc}$ we have $a_{rp}R^r{}_q C_{sijl}C^s{}_{abc} = 0$. Suppose that at y

$$C_{sijl}C^s{}_{abc} = 0\,. \tag{52}$$

Differentiating (7) covariantly, making use of (8), then transvecting the obtained equation with $C^l{}_{abc}$, in virtue of (52), we get

$$a_{rm}C^r{}_{abc}C_{hijk} = \frac{1}{n-3}(C_{habc}a_{rm}C^r{}_{ijk} - C_{iabc}a_{rm}C^r{}_{hjk})\,.$$

Hence, by further transvection with $a^h{}_p$ and symmetrization in (m,p), we have

$$\frac{n-4}{n-3}(a_{rm}C^r{}_{abc}a_{sp}C^s{}_{ijk} + a_{rp}C^r{}_{abc}a_{sm}C^s{}_{ijk}) = 0\,.$$

This yields $a_{rm}C^r{}_{ijk} = 0$ at y, which, in virtue of (35), is equivalent to

$$R_{ij}b_{km} - R_{ik}b_{jm} = 0\,. \tag{53}$$

Since $b_{km}(y) \neq 0$, one can choose at y a vector t^k such that $b_{kp}t^kt^p = e$, $|e| = 1$. Transvecting (53) with t^kt^m we get

$$R_{ij} = ek_ik_j\,. \tag{54}$$

Applying this to (43) gives (51) at y. This completes the proof.

LEMMA 15. *Suppose that* (C) *and* $a_{lm}(x) \neq 0$ *hold. Then*

$$a_{rm}R^r{}_p = 0 \tag{55}$$

and

$$R_{ir}C^r{}_{jkl} = 0 \tag{56}$$

in some neighbourhood of x.

Proof. This follows immediately from Lemmas 9, 12 and 14.

3. Main results. We are now in a position to prove

THEOREM 1. *Suppose that under hypothesis* (C) *the inequalities* $R_{ij,l}(x) \neq 0$ *and* $R_{ij,lm}(x) \neq 0$ *hold. Then* $\operatorname{rank} R_{ij} = 1$ *and in some neighbourhood of* x *there exists a non-trivial null parallel vector field.*

Proof. Suppose $a_{lm}(x) = 0$. Then, by (35), we have (53) at x, which implies $\operatorname{rank} R_{ij}(x) = 1$. Thus assume that $a_{lm}(x) \neq 0$. Then, by Lemma 15, we have (55) in some neighbourhood of x. Substituting (55) into (43) we easily obtain $\operatorname{rank} R_{ij}(x) = 1$. Because of the recurrence of the Ricci tensor its rank must be constant on M. But it was proved by Roter (cf. [12], Proposition 1) that if a manifold admits a $(0,2)$ symmetric recurrent tensor of rank 1 and the recurrence vector is locally a gradient, then M admits locally a parallel vector field. Together with (11), (54), (9) and (15), this completes the proof.

Remark. The null parallel vector field we look for is of the form

$$v_i = \exp\left(-\tfrac{1}{2}b\right) k_i\,,$$

where k_i is defined by (54), $b_{,j} = b_j$, b_j is the recurrence vector of R_{ij}.

COROLLARY. *Under the assumptions of Theorem* 1 *the scalar curvature of* M *vanishes.*

LEMMA 16. *Suppose that under hypothesis* (C) *the inequalities* $a_{lm}(x) \neq 0$, $R_{ij,l}(x) \neq 0$, $R_{ij,lm}(x) \neq 0$ *hold. Then*

$$\begin{aligned} Q_{hpjiqt} = {} & R_{hp}C_{jiqt} - R_{hj}C_{piqt} + R_{ip}C_{hjqt} - R_{ij}C_{hpqt} \\ & + R_{qp}C_{hijt} - R_{qj}C_{hipt} + R_{tp}C_{hiqj} - R_{tj}C_{hiqp} = 0 \end{aligned} \tag{57}$$

in some neighbourhood of x.

Proof. Applying (56) and (42) to (28) we obtain

$$\begin{aligned} & a_{pm}(C_{tqlr}C^r{}_{jih} - C_{tqjr}C^r{}_{lih}) \\ & = a_{rm}C^r{}_{jih}C_{tqlp} - a_{rm}C^r{}_{pqt}C_{hilj} - a_{rm}C^r{}_{lih}C_{tqjp} \\ & \quad + \frac{1}{n-2}[C_{hpqt}(R_{il}b_{jm} - R_{ij}b_{lm}) - C_{ipqt}(R_{hl}b_{jm} - R_{hj}b_{lm}) \\ & \quad + (-g_{hj}R_{il} + g_{hl}R_{ij} + g_{ij}R_{hl} - g_{il}R_{hj})b_{rm}C^r{}_{pqt} \end{aligned} \tag{58}$$

$$
\begin{aligned}
&+ C_{tlih}(R_{qj}b_{pm} - R_{qp}b_{jm}) - C_{qlih}(R_{tj}b_{pm} - R_{tp}b_{jm}) \\
&+ (g_{tj}R_{qp} - g_{qj}R_{tp})b_{rm}C^{r}{}_{lih} \\
&- C_{tjih}(R_{ql}b_{pm} - R_{qp}b_{lm}) + C_{qjih}(R_{tl}b_{pm} - R_{tp}b_{lm}) \\
&- (g_{tl}R_{qp} - g_{ql}R_{tp})b_{rm}C^{r}{}_{jih}]\,.
\end{aligned}
$$

Transvecting (58) with $a^{l}{}_{v}$ and using (56) and (35) we get

$$
\begin{aligned}
&a_{rm}C^{r}{}_{jih}a_{sv}C^{s}{}_{ptq} - a_{rm}C^{r}{}_{pqt}a_{sv}C^{s}{}_{jhi} \\
&+ \frac{1}{n-2}a_{sv}b^{s}{}_{m}(-R_{ij}C_{hpqt} + R_{hj}C_{ipqt} + R_{qp}C_{tjih} - R_{tp}C_{qjih}) \\
&+ \frac{1}{n-2}[(a_{hv}R_{ij} - a_{iv}R_{hj})b_{rm}C^{r}{}_{pqt} - a_{sv}C^{s}{}_{tih}(R_{qj}b_{pm} - R_{qp}b_{jm}) \\
&+ a_{sv}C^{s}{}_{qih}(R_{tj}b_{pm} - R_{tp}b_{jm}) - (a_{tv}R_{qp} - a_{qv}R_{tp})b_{rm}C^{r}{}_{jih}] = 0\,.
\end{aligned}
$$

Alternating in (p, j) and using (42) we find

$$
\begin{aligned}
&- a_{rm}C^{r}{}_{jih}a_{sv}C^{s}{}_{pqt} + a_{rm}C^{r}{}_{pih}a_{sv}C^{s}{}_{jqt} \\
&+ a_{rm}C^{r}{}_{pqt}a_{sv}C^{s}{}_{jih} - a_{rm}C^{r}{}_{jqt}a_{sv}C^{s}{}_{pih} \\
&+ \frac{1}{n-2}a_{sv}b^{s}{}_{m}(R_{hp}C_{jiqt} - R_{hj}C_{piqt} + R_{ip}C_{hjqt} - R_{ij}C_{hpqt} \\
&+ R_{qp}C_{hijt} - R_{qj}C_{hipt} + R_{tp}C_{hiqj} - R_{tj}C_{hiqp}) \\
&+ \frac{2}{n-2}[-a_{sv}C^{s}{}_{tih}(R_{qj}b_{pm} - R_{qp}b_{jm}) + a_{sv}C^{s}{}_{qih}(R_{tj}b_{pm} - R_{tp}b_{jm})] = 0\,,
\end{aligned}
$$

which, by (35) and Theorem 1, yields

$$
\begin{aligned}
a_{sv}b^{s}{}_{m}(&R_{hp}C_{jiqt} - R_{hj}C_{piqt} + R_{ip}C_{hjqt} - R_{ij}C_{hpqt} \\
&+ R_{qp}C_{hijt} - R_{qj}C_{hipt} + R_{tp}C_{hiqj} - R_{tj}C_{hiqp}) = 0\,.
\end{aligned}
$$

Assume that at some $x \in M$

(59) $$a_{sv}b^{s}{}_{m} = 0\,.$$

We shall prove that at x

(60) $$b_{rv}b^{r}{}_{m} = 0\,.$$

Transvecting (58) with $a^{t}{}_{v}$, by (35), (42), (55) and (56), we get

(61)
$$
\begin{aligned}
&\frac{(n-3)^2}{n-2}[b_{hm}(R_{il}b_{jv} - R_{ij}b_{lv}) - b_{im}(R_{hl}b_{jv} - R_{hj}b_{lv})] \\
&\quad + a_{jv}b_{rm}C^{r}{}_{lih} - a_{lv}b_{rm}C^{r}{}_{jih} \\
&\quad + \frac{n-3}{n-2}b_{sv}b^{s}{}_{m}(-g_{hj}R_{il} + g_{hl}R_{ij} + g_{ij}R_{hl} - g_{il}R_{hj}) = 0\,,
\end{aligned}
$$

since $\operatorname{rank} R_{ij} = 1$.

On the other hand, transvecting (42) with ${b^l}_v$, we have $R_{ij}b_{rm}b_{sv}{C^{rs}}_{pt} = 0$. Therefore, transvecting (61) with ${b^j}_p$, we find

$$b_{rv}{b^r}_m(R_{hl}b_{ip} - R_{il}b_{hp}) = (n-3)b_{rv}{b^r}_p(R_{hl}b_{im} - R_{il}b_{hm})\,,$$

whence we easily obtain

$$(n-4)b_{rv}{b^r}_m(R_{hl}b_{ip} - R_{il}b_{hp}) = 0\,, \tag{62}$$

since $b_{rv}{b^r}_m = b_{rm}{b^r}_v$. Finally, transvecting (62) with ${b^i}_q$, we get (60).

Now, in view of (60), relation (61) can be rewritten as

$$\frac{(n-3)^2}{n-2}[b_{hm}(R_{il}b_{jv} - R_{ij}b_{lv}) - b_{im}(R_{hl}b_{jv} - R_{hj}b_{lv})] + a_{jv}b_{rm}{C^r}_{lih} - a_{lv}b_{rm}{C^r}_{jih} = 0\,.$$

On the other hand, transvecting (24) with ${b^k}_v$, we have

$$a_{lm}b_{rv}{C^r}_{jih} - a_{jm}b_{rv}{C^r}_{lih} = \frac{1}{n-2}[b_{hv}(R_{il}b_{jm} - R_{ij}b_{lm}) - b_{iv}(R_{hl}b_{jm} - R_{hj}b_{lm})]\,.$$

Comparing the last two results we get

$$(n-4)[b_{hm}(R_{il}b_{jv} - R_{ij}b_{lv}) - b_{im}(R_{hl}b_{jv} - R_{hj}b_{lv})] = 0\,,$$

whence, multiplying by R_{ab}, in virtue of (54), we obtain

$$(n-4)(R_{bl}b_{jv} - R_{bj}b_{lv})(R_{ai}b_{hm} - R_{ah}b_{im}) = 0\,. \tag{63}$$

Thus (24) and (63) imply

$$a_{lm}C_{hijk} + a_{jm}C_{hikl} + a_{km}C_{hilj} = 0\,. \tag{64}$$

Moreover, from (35) it follows that $a_{rm}{C^r}_{ijk} = 0$.

Finally, transvecting (64) with ${C^k}_{pqt}$, we can follow step by step the proof of Lemma 9 to obtain

$$C_{tqlr}{C^r}_{jih} - C_{tqjr}{C^r}_{lih} = 0\,. \tag{65}$$

Now, (57) follows from (29), (65) and (56). This completes the proof.

THEOREM 2. *Under hypothesis* (C) *let the inequalities* $a_{lm}(x) \neq 0$, $R_{ij,l}(x) \neq 0$, $R_{ij,lm}(x) \neq 0$ *hold. Then, in some neighbourhood of* x, *the curvature tensor takes the form*

$$R_{qthj} = k_tk_hS_{qj} - k_tk_jS_{qh} + k_qk_jS_{th} - k_qk_hS_{tj}\,, \tag{66}$$

where $S_{qj} = p^rp^sR_{rqjs}$, $p^rk_r = 1$ *and* $R_{ij} = ek_ik_j$, $|e| = 1$.

Proof. Substituting (54) into (57), then alternating in (h,p,j) and making use of [13], Lemma 4, we get

$$k_pC_{qthj} + k_hC_{qtjp} + k_jC_{qtph} = 0\,. \tag{67}$$

Since the scalar curvature vanishes and (54) is satisfied, from (67), by a direct calculation, we have

$$k_p R_{qthj} + k_h R_{qtjp} + k_j R_{qtph} = 0\,.$$

Now, with the help of the last result, we can follow step by step a proof of Walker ([15], p. 45 and [14], p. 155) to obtain (66).

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INSTITUTE OF MATHEMATICS
SZCZECIN TECHNICAL UNIVERSITY
AL. PIASTÓW 17
70-310 SZCZECIN
POLAND

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