Nihonkai Math. J.
Vol.13(2002), 121-131

ON CONFORMALLY FLAT LP-SASAKIAN MANIFOLDS WITH A COEFFICIENT α

U.C. De, J.B. Jun and Absos Ali Shaikh

Abstract

Recently, the notion of Lorentzian almost paracontact manifolds with a coefficient α has been introduced and studied by De et al [3]. In the present paper we investigate conformally flat LP-Sasakian manifolds with a coefficient α.

0. Introduction

In 1989, Matsumoto [1] introduced the notion of LP-Sasakian manifolds. Then Mihai and Rosca [2] introduced the same notion independently and they obtained several results in this manifold. In a recent paper, De, Shaikh and Sengupta [3] introduced the notion of LP-Sasakian manifolds with a coefficient α which generalizes the notion of LP-Sasakian manifolds. Recently, T.Ikawa and his coauthors [4],[5] studied Sasakian manifolds with Lorentzian metric and obtained several results in this manifold. The object of the present paper is to study an LP-Sasakian manifold with a coefficient α.

Mathematics Subject Classification. (1991), 53C15, 53C40.
Keywords and Phrases. LP-Sasakian manifolds with a coefficient α, concircular vector field, torse-forming vector field.

After preliminaries, in section 2 we study conformally flat LP-Sasakian manifold with a coefficient α and obtain several interesting results. We mainly prove that in a conformally flat LP-Sasakian manifold with a coefficient α the characteristic vector field ξ is a concircular vector field if and only if the manifold is η-Einstein and a conformally flat LP-Sasakian manifold with a constant coefficient α is a manifold of constant curvature if the scalar curvature r is a constant.

1. Preliminaries

Let M^n be an n-dimensional differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor g_p : $T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,+,\dots +)$, where T_pM denotes the tangent vector space of M at p and R is the real number space, which satisfies

$$\eta(\xi) = -1, \quad \phi^2 X = X + \eta(X)\xi, \tag{1.1}$$

$$g(X,\xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{1.2}$$

for all vector fields X and Y. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold M^n with the structure (ϕ, ξ, η, g) is called *Lorentzian almost paracontact manifold* [1]. In the Lorentzian almost paracontact manifold M^n, the following relations hold good [1] :

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{1.3}$$

$$\Omega(X,Y) = \Omega(Y,X), \text{ where } \Omega(X,Y) = g(X,\phi Y). \tag{1.4}$$

In the Lorentzian almost paracontact manifold M^n, if the relations

$$\begin{aligned}(\nabla_Z \Omega)(X,Y) &= \alpha\left[\{g(X,Z) + \eta(X)\eta(Z)\}\,\eta(Y)\right. \\ &\quad \left. + \{g(Y,Z) + \eta(Y)\eta(Z)\}\,\eta(X)\right], \ (\alpha \neq 0)\end{aligned} \tag{1.5}$$

$$\Omega(X,Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y), \tag{1.6}$$

hold where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g, then M^n is called an *LP-Sasakian manifold with a coefficient* α [3]. An LP-Sasakian manifold with coefficient 1 is an *LP-Sasakian manifold* [1].

If a vector field V satisfies the equation of the following form :

$$\nabla_X V = \beta X + T(X)V,$$

where β is a non-zero scalar function and T is a covariant vector field, then V is called a *torse-forming vector field* [6].

In a Lorentzian manifold M^n, if we assume that ξ is a unit torse-forming vector field, then we have the equation :

$$(\nabla_X \eta)(Y) = \alpha \left[g(X,Y) + \eta(X)\eta(Y) \right], \tag{1.7}$$

where α is a non-zero scalar function. Hence the manifold admitting a unit torse-forming vector field satisfying (1.7) is an LP-Sasakian manifold with a coefficient α. Especially, if η satisfies

$$(\nabla_X \eta)(Y) = \epsilon \left[g(X,Y) + \eta(X)\eta(Y) \right], \epsilon^2 = 1 \tag{1.8}$$

then M^n is called an *LSP-Sasakian manifold* [1]. In particular, if α satisfies (1.7) and the equation of the following form :

$$\alpha(X) = p\eta(X), \; \alpha(X) = \nabla_X \alpha, \tag{1.9}$$

where p is a scalar function, then ξ is called a *concircular vector field.*

Let us consider an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$ with a coefficient α. Then we have the following relations [3] :

$$\begin{aligned} \eta(R(X,Y)Z) &= -\alpha(X)\Omega(Y,Z) + \alpha(Y)\Omega(X,Z) \\ &\quad + \alpha^2 \left\{ g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \right\}, \end{aligned} \tag{1.10}$$

$$S(X,\xi) = -\psi\alpha(X) + (n-1)\alpha^2\eta(X) + \alpha(\phi X), \tag{1.11}$$

where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold and $\psi = \mathrm{Trace}(\phi)$.

We now state the following results which will be needed in the later section.

Lemma 1.1. ([3]) *In an LP-Sasakian manifold M^n with a non-constant coefficient α, one of the following cases occur:*

i) $\psi^2 = (n-1)^2$

ii) $\alpha(Y) = -p\eta(Y)$, where $p = \alpha(\xi)$.

Lemma 1.2. ([3]) *In a Lorentzian almost paracontact manifold $M^n(\phi,\xi,\eta,g)$ with its structure (ϕ,ξ,η,g) satisfying $\Omega(X,Y) = \frac{1}{\alpha}(\nabla_X\eta)(Y)$, where α is a non-zero scalar function, the vector field ξ is torse-forming if and only if the relation $\psi^2 = (n-1)^2$ holds good.*

2. Conformally flat LP-Sasakian manifold with a coefficient α

Let us consider a conformally flat LP-Sasakian manifold $M^n (n > 3)$ with a coefficient α. First we suppose that α is not constant. Then since the conformal curvature tensor C vanishes, the curvature tensor $'R$ satisfies

$$\begin{aligned} {}'R(X,Y,Z,W) = \frac{1}{n-2}&[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &+S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ &-\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned} \tag{2.1}$$

where r is the scalar curvature of the manifold. Putting $W = \xi$ in (2.1) and then using (1.10) and (1.11), we get

$$
\begin{aligned}
-\alpha(X)\Omega(Y,Z)+\alpha(Y)\Omega(X,Z)+\alpha^2\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right]& \\
=\ & \frac{1}{n-2}\Big[g(Y,Z)\Big\{-\psi\alpha(X)+(n-1)\alpha^2\eta(X)+\alpha(\phi X)\Big\} \\
& -g(X,Z)\Big\{-\psi\alpha(Y)+(n-1)\alpha^2\eta(Y)+\alpha(\phi Y)\Big\} \\
& +S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\Big] \\
& -\frac{r}{(n-1)(n-2)}\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right].
\end{aligned} \tag{2.2}
$$

Again if we put $X=\xi$ in (2.2) and using (1.3) and (1.11) we obtain by straightforward calculations

$$
\begin{aligned}
S(Y,Z) =\ & \left[\frac{r}{n-1}-\alpha^2-\psi p\right]g(Y,Z)+\left[\frac{r}{n-1}-n\alpha^2\right]\eta(Y)\eta(Z) \\
& +\{\psi\alpha(Z)-\alpha(\phi Z)\}\eta(Y)+\{\psi\alpha(Y)-\alpha(\phi Y)\}\eta(Z) \\
& +p(n-2)\Omega(Y,Z),
\end{aligned} \tag{2.3}
$$

where $p=\alpha(\xi)$.

We now suppose that M^n is η-Einstein. If an LP-Sasakian manifold M^n with the coefficient α satisfies the relation

$$
S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),
$$

where a,b are the associated functions on the manifold, then the manifold M^n is called an *η-Einstein manifold.* Then we have [3]

$$
\begin{aligned}
S(X,Y) =\ & \left[\frac{r}{n-1}-\alpha^2-\frac{\psi p}{n-1}\right]g(X,Y) \\
& +\left[\frac{r}{n-1}-n\alpha^2-\frac{n\psi p}{n-1}\right]\eta(X)\eta(Y).
\end{aligned} \tag{2.4}
$$

By virtue of (2.4) and (2.3) we get

$$
\begin{aligned}
&\frac{(n-2)\psi p}{n-1}g(Y,Z)-\frac{n\psi p}{n-1}\eta(Y)\eta(Z)-\{\psi\alpha(Z)-\alpha(\phi Z)\}\eta(Y) \\
&-\{\psi\alpha(Y)-\alpha(\phi Y)\}\eta(Z)-p(n-2)\Omega(Y,Z)=0.
\end{aligned} \tag{2.5}
$$

Putting $Z = \xi$ in (2.5) we obtain

$$\psi\alpha(Y) - \alpha(\phi Y) = -\psi p\eta(Y), \text{ for all } Y. \tag{2.6}$$

Using (2.6) in (2.5) we get by simplification

$$p\left\{\frac{\psi}{n-1}[g(Y,Z) + \eta(Y)\eta(Z)] - \Omega(Y,Z)\right\} = 0. \tag{2.7}$$

If $p = 0$, then from (2.6) we have $\alpha(\phi Y) = \psi\alpha(Y)$. Thus since ψ is an eigenvalue of the matrix (ϕ), ψ is equal to ± 1. Hence, by virtue of Lemma 1.1, we get $\alpha(Y) = 0$ for all Y and hence α is constant, which contradicts to our assumption.

Consequently, we have $p \neq 0$ and hence from (2.7) we get

$$\frac{\psi}{n-1}[g(Y,Z) + \eta(Y)\eta(Z)] - \Omega(Y,Z) = 0. \tag{2.8}$$

Putting $Y = \phi Y$ in (2.8) we have by virtue of (1.3)

$$\frac{\psi}{n-1}\Omega(Y,Z) - \{g(Y,Z) + \eta(Y)\eta(Z)\} = 0. \tag{2.9}$$

Combining (2.8) and (2.9) we get

$$\left\{\psi^2 - (n-1)^2\right\}[g(Y,Z) + \eta(Y)\eta(Z)] = 0,$$

which gives by virtue of $n > 3$

$$\psi^2 = (n-1)^2. \tag{2.10}$$

Hence Lemma 1.2 proves that ξ is torse-forming. We have that

$$(\nabla_X\eta)(Y) = \beta\ \{g(X,Y) + \eta(X)\eta(Y)\}.$$

Then from (1.6) we get

$$\begin{aligned}\Omega(X,Y) &= \frac{\beta}{\alpha}\{g(X,Y) + \eta(X)\eta(Y)\} \\ &= g\left(\frac{\beta}{\alpha}(X + \eta(X)\xi, Y\right)\end{aligned}$$

and $\Omega(X, Y) = g(\phi X, Y)$.

Since g is non-singular, we have

$$\phi(X) = \frac{\beta}{\alpha}(X + \eta(X)\xi)$$

and

$$\phi^2(X) = \left(\frac{\beta}{\alpha}\right)^2 (X + \eta(X)\xi).$$

It follows from (1.1) that $\left(\frac{\beta}{\alpha}\right)^2 = 1$ and hence, $\alpha = \pm\beta$. Thus we have

$$\phi(X) = \pm(X + \eta(X)\xi).$$

By virtue of (2.6) we see

$$\alpha(Y) = -p\eta(Y),$$

where $p = \alpha(\xi)$. Thus, we conclude that ξ is a concircular vector field.

Conversely, we suppose that ξ is a concircular vector field. Then we have the equation of the following form :

$$(\nabla_X \eta)(Y) = \beta \{g(X, Y) + \eta(X)\eta(Y)\},$$

where β is a certain function and $\nabla_X \beta = q\eta(X)$ for a certain scalar function q. Hence by virtue of (1.6) we have $\alpha = \pm\beta$. Thus

$$\Omega(X, Y) = \epsilon \{g(X, Y) + \eta(X)\eta(Y)\}, \; \epsilon^2 = 1,$$

$$\psi = \epsilon(n-1), \nabla_X \alpha = \alpha(X) = p\eta(X), \; p = \epsilon q.$$

Using these relations in (2.3) and (2.6), it can be easily seen that M^n is η-Einstein.

Thus we can state the following :

Theorem 2.1. *In a conformally flat LP-Sasakian manifold $M^n (n > 3)$ with a non-constant coefficient α, the characteristic vector field ξ is a concircular vector field if and only if M^n is η-Einstein.*

For $n = 3$, it is clear that the following theorem holds good:

Theorem 2.2. *In a 3-dimensional LP-Sasakian manifold with a non-constant coefficient α, the characteristic vector field ξ is a concircular vector field if and only if the manifold is η-Einstein.*

Next we consider the case when the coefficient α is constant. In this case the following relations hold good :

$$\eta(R(X,Y)Z) = \alpha^2 \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.11}$$

$$S(X,\xi) = (n-1)\alpha^2\eta(X). \tag{2.12}$$

We now consider a conformally flat LP-Sasakian manifold $M^n (n > 3)$ with a constant coefficient α. Then we have the relation (2.1). Putting $W = \xi$ in (2.1) and then using (2.11) and (2.12), we get

$$\begin{aligned} &\alpha^2 [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &\qquad = \frac{1}{n-2}\big[(n-1)\alpha^2 \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \\ &\qquad\quad + S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] \\ &\qquad\quad - \frac{r}{(n-1)(n-2)} [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \end{aligned} \tag{2.13}$$

Again putting $X = \xi$ in (2.13) we get by virtue of (2.12) that

$$S(Y,Z) = \left\{\frac{r}{n-1} - \alpha^2\right\} g(Y,Z) + \left\{\frac{r}{n-1} - n\alpha^2\right\} \eta(Y)\eta(Z). \tag{2.14}$$

Hence we can state the following :

Theorem 2.3. *A conformally flat LP-Sasakian manifold $M^n (n > 3)$ with a constant coefficient α is an η-Einstein manifold.*

Corollary. *The 3-dimensional LP-Sasakian manifold M^3 with a constant coefficient α is always an η-Einstein manifold.*

Now substituting (2.14) into (2.1) we get

$$
\begin{aligned}
{}'R(X,Y,Z,W) &= \frac{1}{n-2}\Big[\left(\frac{r}{n-1}-2\alpha^2\right)\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\
&\quad + \left(\frac{r}{n-1}-n\alpha^2\right)\{g(Y,Z)\eta(X)\eta(W)+g(X,W)\eta(Y)\eta(Z) \\
&\quad -g(X,Z)\eta(Y)\eta(W)-g(Y,W)\eta(X)\eta(Z)\}\Big]. \qquad (2.15)
\end{aligned}
$$

Again differentiating (2.14) covariantly along X and making use of (1.6), we get

$$
\begin{aligned}
(\nabla_X S)(Y,Z) &= \frac{dr(X)}{n-1}[g(Y,Z)+\eta(Y)\eta(Z)] \\
&\quad +\alpha\left(\frac{r}{n-1}-n\alpha^2\right)[\Omega(X,Y)\eta(Z)+\Omega(X,Z)\eta(Y)]\,,
\end{aligned}
$$

where $dr(X)=\nabla_X r$.

This implies that

$$
\begin{aligned}
(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z) &= \frac{dr(X)}{n-1}[g(Y,Z)+\eta(Y)\eta(Z)] \\
&\quad -\frac{dr(Y)}{n-1}[g(X,Z)+\eta(X)\eta(Z)]+\alpha\left(\frac{r}{n-1}-n\alpha^2\right) \\
&\quad [\Omega(X,Z)\eta(Y)-\Omega(Y,Z)\eta(X)]. \qquad (2.16)
\end{aligned}
$$

On the other hand, in our case , since we have $(\nabla_W C)(X,Y)Z=0$, we get $divC=0$, where $'div'$ denotes the divergence. So for $n>3$, $divC=0$ gives

$$
(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)=\frac{1}{2(n-1)}[g(Y,Z)dr(X)-g(X,Z)dr(Y)]. \quad (2.17)
$$

Remark. When $n=3$, the equation (2.17) is the condition for the manifold to be conformally flat.

It follows from (2.16) and (2.17) that

$$
\begin{aligned}
&\frac{1}{2(n-1)}[g(Y,Z)dr(X)-g(X,Z)dr(Y)]+\frac{1}{n-1}[dr(X)\eta(Y) \\
&-dr(Y)\eta(X)]\eta(Z)+\alpha\left(\frac{r}{n-1}-n\alpha^2\right)[\Omega(X,Z)\eta(Y)-\Omega(Y,Z)\eta(X)]=0. \qquad (2.18)
\end{aligned}
$$

If r is constant, then from (2.18) we obtain

$$
r=n(n-1)\alpha^2.
$$

Hence from (2.15) it follows that

$$'R(X,Y,Z,W) = \alpha^2[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],$$

which shows that the manifold is of constant curvature.

Thus we can state the following :

Theorem 2.4. *In a conformally flat LP-Sasakian manifold M^n $(n > 3)$ with a constant coefficient α, if the scalar curvature r is constant, then M^n is of constant curvature.*

Acknowledgement: The authors are thankful to the referee for his valuable suggestions in the improvement of the paper.

References

[1] K. Matsumoto, On Lorentzian paracontact manifolds, *Bull. of Yamagata Univ., Nat. Sci.* **12** (1989), 151-156.

[2] I. Mihai and R. Rosca, On Lorentzian P-Sasakian manifolds, *Classical Analysis*, World Sci. Publi., Singapore, (1992), 155-169.

[3] U.C. De, A.A. Shaikh and A. Sengupta, On LP-Sasakian Manifolds with a coefficient α, to appear in Kyungpook Math. Jour.,July, 2002.

[4] T. Ikawa and M. Erdogan, Sasakian manifolds with Lorentzian metric, *Kyungpook Math. J.*, **35** (1996), 517-526.

[5] T. Ikawa and J.B. Jun, On sectional curvatures of a normal contact Lorentzian manifold, *Korean J. Math. Sciences*, **4** (1997), 27-33.

[6] K.Yano , On the torse-forming direction in Riemannian spaces, Proc. Imp. Acad. Tokyo, **20** (1944),340-345.

U.C. De and Absos Ali Shaikh
Department of Mathematics,
University of Kalyani,
Kalyani - 741235,
West Bengal, INDIA.
e-mail : ucde@klyuniv.ernet.in

J.B.Jun
Department of Mathematics,
Faculty of Natural Science,
Kookmin University,
Seoul Korea, 136-702.
e-mail: jbjun@kookmin.ac.kr

Received July 13, 2001 Revised May 24, 2002