KŌDAI MATH. SEM. REP.
21 (1969), 226—232

ON CONFORMALLY FLAT SPACES WITH DEFINITE RICCI CURVATURE

BY SAMUEL I. GOLDBERG[1]

1. Introduction. It is well-known that there are no harmonic p-forms on a compact conformally flat manifold M of positive Ricci curvature, $0<p<d$, $d=\dim M$, so if M is orientable, it is a rational homology sphere ([5], Theorem 4.1). By applying a technique to obtain local versions of global results, it can be shown that there are no covariant constant p-forms, $0<p<d$, on a conformally flat manifold with either positive or negative definite Ricci tensor (Proposition 1, Corollary 4). Recently, Tani [4] proved that a compact and orientable Riemannian manifold admitting a conformally flat metric of positive Ricci curvature and constant scalar curvature is a space form that is, a space of constant curvature. The technique referred to above then yields the following local statement.

THEOREM 1. *A conformally flat Riemannian manifold whose Ricci tensor is parallel and either positive or negative definite is a space form.*[2]

In fact, if a conformally flat manifold is reducible it is either flat, a product $M_1 \times M_2$ where M_1 and M_2 have constant curvature of the same magnitude and opposite sign, or $M_1 \times N$ where N is 1-dimensional [2].

Theorem 1 will be used to obtain the following generalization.

THEOREM 2. *Let M be a conformally flat manifold with positive Ricci curvature. If the scalar curvature and trace Q^2, where Q is the Ricci operator, are both constant, then M is a space form.*

COROLLARY. *A conformally flat homogeneous Riemannian manifold of positive Ricci curvature is a space form.*

If the Ricci curvature is only positive semi-definite, then the same method of reasoning gives

THEOREM 3. *Let M be a conformally flat manifold with positive semi-definite Ricci curvature. Then, if the scalar curvature and trace Q^2 are both constant, M is locally symmetric.*

Received October 31, 1968.

1) Research supported by the National Science Foundation.

2) We do not require the metric to be complete. Moreover, the proof we give of this theorem is more informative than the one suggested.

Thus, M is either a space form, a product $M_1 \times M_2$ where M_1 and M_2 have constant curvature of the same magnitude and opposite sign or the product $M_1 \times N$ where dim $N=1$.

COROLLARY. *A conformally flat homogeneous Riemannian manifold with positive semi-definite Ricci curvature is symmetric.*

On the other hand, an application of the generalized Gauss-Bonnet theorem yields the following global statement hinted at by Kurita [2].

THEOREM 4. *A compact and oriented conformally flat Riemannian manifold of even dimension* $2n$ *whose sectional curvatures are all nonnegative has nonnegative Euler-Poincaré characteristic* χ, *and if the sectional curvatures are all nonpositive* $(-1)^n \chi \geqq 0$.

Moreover, from the formula for the Pontrjagin classes, we obtain

THEOREM 5. *The Pontrjagin classes of a compact and oriented conformally flat Riemannian manifold all vanish.*

2. Definitions and formulae. Let (M, g) be a Riemannian manifold with metric tensor g. The curvature transformation $R(X, Y)$ ($X, Y \in M_m$——the tangent space at $m \in M$) and the metric g are related by

$$R(X, Y)=\nabla_{[X,Y]}-[\nabla_X, \nabla_Y]$$

where ∇_X is the operation of covariant differentiation with respect to X and

$$2g(X, \nabla_Z Y)=Zg(X, Y)-Xg(Y, Z)+Yg(X, Z)$$
$$+g(Z, [X, Y])-g(X, [Y, Z])+g(Y, [X, Z]).$$

In terms of a basis $\{X_1, \cdots, X_d\}$ of M_m we set

$$R_{ijkh}=g(R(X_i, X_j)X_k, X_h),$$
$$R_{ij}=\mathrm{tr}(X_k \rightarrow R(X_i, X_k)X_j),$$
$$t_{i_1 \cdots i_p}=t(X_{i_1}, \cdots, X_{i_p}),$$
$$\nabla_i t_{i_1 \cdots i_p}=(\nabla_{X_i} t)(X_{i_1}, \cdots, X_{i_p}).$$

We denote the scalar curvature by r, that is $r=\mathrm{tr}\, Q$ where $Q=(R^i{}_j)$. The manifold (M, g) is *conformally flat* if g is conformally related to a locally flat metric. It follows that the *Weyl conformal curvature tensor* defined by

$$C^i{}_{jkh}=R^i{}_{jhk}-\frac{1}{d-2}(R_{jk}\delta^i_h-R_{jh}\delta^i_k+g_{jk}R^i{}_h-g_{jh}R^i{}_k)+\frac{r}{(d-1)(d-2)}(g_{jk}\delta^i_h-g_{jh}\delta^i_k) \tag{2.1}$$

vanishes, so if (M, g) is conformally flat

$$R^{i}{}_{jhk}=\frac{1}{d-2}(R_{jk}\delta^{i}_{h}-R_{jh}\delta^{i}_{k}+g_{jk}R^{i}{}_{h}-g_{jh}R^{i}{}_{k})-\frac{r}{(d-1)(d-2)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k}). \tag{2.2}$$

From (2.1) and the second Bianchi identity $\nabla_{i}C^{i}{}_{jkh}=(d-3)C_{jkh}$ where

$$C_{jkh}=\frac{1}{d-2}(\nabla_{h}R_{jk}-\nabla_{k}R_{jh})-\frac{1}{2(d-1)(d-2)}(g_{jk}\nabla_{h}r-g_{jh}\nabla_{k}r). \tag{2.3}$$

For $d=3$ it can be shown that if (M, g) is conformally flat then C_{ijk} vanishes.

3. Harmonic forms of constant length. A p-form will be called *harmonic* if it is annihilated by the differential and codifferential operators. Observe that compactness is not required in the following result.

PROPOSITION 1. *A harmonic p-form ξ of constant length on a Riemannian manifold has vanishing covariant derivative if and only if the quadratic form*

$$F(\xi)=R_{ij}\xi^{ii_2\cdots i_p}\xi^{j}{}_{i_2\cdots i_p}-\frac{p-1}{2}R_{ijkh}\xi^{iji_3\cdots i_p}\xi^{kh}{}_{i_3\cdots i_p}$$

is nonnegative.

For,

$$\frac{1}{2}p!\,\Delta|\xi|^2=pF(\xi)+g^{jk}\nabla_{k}\xi^{i_1\cdots i_p}\nabla_{j}\xi_{i_1\cdots i_p}$$

where Δ is the Laplacian and $|\xi|$ denotes the length of ξ.

We list some interesting consequences of this proposition which are easily obtained.

COROLLARY 1. *Let ξ be a harmonic p-form of constant length on the Riemannian manifold (M, g). Then if $F(\xi)$ is nonnegative it must vanish and $\nabla\xi=0$. Thus*

$$R_{ij}\xi^{ii_2\cdots i_p}\xi^{j}{}_{i_2\cdots i_p}=\frac{p-1}{2}R_{ijkh}\xi^{iji_3\cdots i_p}\xi^{kh}{}_{i_3\cdots i_p}.$$

If g is an Einstein metric, then

$$\frac{p-1}{2}R_{ijkh}\xi^{iji_3\cdots i_p}\xi^{kh}{}_{i_3\cdots i_p}=\frac{rp!}{d}|\xi|^2,$$

so, for $d>2$

$$R_{ijkh}\xi^{iji_3\cdots i_p}\xi^{kh}{}_{i_3\cdots i_p}=\text{const}.$$

The curvature tensor defines a symmetric linear transformation of the space of bivectors. Thus, if it is negative definite there are no harmonic p-forms of constant length on M, $0<p<d$,

COROLLARY 2. *Let M be a Riemannian manifold whose Ricci curvature tensor is positive semi-definite. Then, a harmonic* 1-*form of constant length is covariant constant. If the Ricci curvature is positive definite there are no harmonic vector fields of constant length. In particular, there are no parallel vector fields if the Ricci curvature is definite.*

COROLLARY 3. *There are no covariant constant p-forms, $0<p<d$ on a nonflat manifold of constant curvature.*

It is well-known that there are no harmonic p-forms on a compact conformally flat manifold of positive Ricci curvature, $0<p<d$. We state a corresponding local result.

COROLLARY 4. *There are no covariant constant forms on a conformally flat manifold whose Ricci curvature tensor is definite.*

This generalizes Corollary 3, for, a nonflat manifold of constant curvature is conformally flat with definite Ricci tensor.

Since a holomorphic form on a Kaehler manifold is harmonic we obtain (cf. [5], Theorem 8. 10)

COROLLARY 5. *A holomorphic p-form, $0<p\leqq d/2$, of constant length on a Kaehler manifold with positive semi-definite Ricci curvature is covariant constant. If the Ricci curvature is positive definite, there are no holomorphic p-forms of constant length.*

4. Proofs of theorems. The proof of the following proposition due to Kurita [4] is essentially due to R. L. Bishop. It is required in the proof of Theorem 1.

PROPOSITION 2. *If M is reducible and conformally flat, then M is one of three types, all of which have parallel curvature*:

1. *Flat*;
2. *$M_1\times M_2$ where M_1 and M_2 have constant curvature of the same magnitude and opposite sign*;
3. *$M_1\times N$ where M_1 has constant curvature and N is* 1-*dimensional.*

Conversely, these three types are conformally flat.

Proof. In terms of a basis which diagonalizes R_{ij} and is adapted to the product structure the only nonzero components of the curvature tensor are the sectional curvatures

$$K_{ij}=R_{ijij}=\frac{1}{d-2}\left(R_{ii}+R_{jj}-\frac{r}{d-1}\right).$$

If i is fixed and j runs through the indices of the product factors not the same as those of i, all of these are 0, so R_{jj} is constant. Hence if there are 3 or more factors we can play them one against the others, showing that M is an Einstein space. But then $r=dR_{11}$ and $K_{ij}=R_{11}/(d-1)$. Since some of these are 0, M is flat.

Thus we may assume that there are two factors, and we have shown that each is an Einstein space.

Let the two values of R_{ii} be A and B. There are two cases:

(a) Neither factor has dimension 1. Then for i, j belonging to different factors

$$K_{ij}=0=A+B-\frac{r}{d-1}.$$

For k belonging to the same factor as i and l belonging to the same factor as j:

$$K_{ik}=\frac{1}{d-2}\left(2A-\frac{r}{d-1}\right),$$

$$K_{jl}=\frac{1}{d-2}\left(2B-\frac{r}{d-1}\right),$$

so

$$K_{ik}+K_{jl}=\frac{2}{d-2}\left(A+B-\frac{r}{d-1}\right)=0.$$

Thus the two factors have constant and opposite curvature.

(b) If one factor has dimension 1, then, say, $B=0$ and the formulas for curvature show that the other factor has constant curvature.

Proof of Theorem 1. Since (M, g) is conformally flat and its Ricci tensor is parallel, (M, g) is locally symmetric. Hence, if its holonomy group is irreducible, g is an Einstein metric. Thus, a conformally flat irreducible manifold with parallel Ricci tensor has constant curvature. Observing that none of the types in Proposition 2 has definite Ricci tensor we obtain the desired conclusion.

Proof of Theorem 2. We apply the following formula analogous to one due to Lichnerowicz [3], p. 9 (see also [5], p. 170):

$$\frac{1}{2}\Delta \operatorname{tr} Q^2=g^{ab}\nabla_a R^{ij}\nabla_b R_{ij}+R^{ij}g^{ab}\nabla_a(\nabla_b R_{ij}-\nabla_i R_{bj})+\frac{1}{2}R^{ij}\nabla_i\nabla_j r+K \tag{4.1}$$

where

$$\operatorname{tr} Q^2=R^{ij}R_{ij}$$

and

$$K=R^{ik}(R^j{}_i R_{jk}+R^{hj}R_{ijhk}). \tag{4.2}$$

Now, since $\operatorname{tr} Q^2$ and r are constants, (4. 1) becomes

$$g^{ab}\nabla_a R^{ij}\nabla_b R_{ij}+R^{ij}g^{ab}\nabla_a(\nabla_b R_{ij}-\nabla_i R_{bj})+K=0. \tag{4.3}$$

But M is conformally flat, so that from (2. 3) formula (4. 3) becomes

$$g^{ab}\nabla_a R^{ij}\nabla_b R_{ij}+K=0. \tag{4.4}$$

Thus, $K\leqq 0$. Substituting (2.2) into the right hand side of (4.2) we obtain

$$(d-1)(d-2)K=d(d-1)\operatorname{tr}Q^3-(2d-1)r\operatorname{tr}Q^2+r^3. \tag{4.5}$$

With respect to an orthonormal basis such that $R_{ij}=0$, $i\neq j$, formula (4.5) becomes

$$\begin{aligned}(d-1)(d-2)K&=d(d-1)\sum R_{ii}^3-(2d-1)\sum R_{ii}\sum R_{jj}^2+\left(\sum R_{ii}\right)^3\\ &=2\sum_{\substack{i,j,k\\ j<k}} R_{ii}(R_{ii}-R_{jj})(R_{ii}-R_{kk}).\end{aligned} \tag{4.6}$$

The Ricci tensor being positive definite, the right hand side of (4.6) can be expressed as the sum of nonnegative terms (see [4]). Thus $K=0$, so that from (4.4) the Ricci tensor is parallel. Theorem 2 is then a consequence of Theorem 1.

REMARK. The condition that M be orientable in Tani's result may be removed by an examination of the proof of Theorem 2. For, formula 4.1 becomes

$$\frac{1}{2}\Delta\operatorname{tr}Q^2=g^{ab}\nabla_a R^{ij}\nabla_b R_{ij}+K,$$

from which since K is nonnegative, so is $\Delta\operatorname{tr}Q^2$. The manifold M being compact, $\Delta\operatorname{tr}Q^2$ must vanish. Hence, the Ricci tensor is parallel, and so by Theorem 1, M is a space form.

THEOREM 6. *A conformally flat compact manifold with metric of positive Ricci curvature and constant scalar curvature is a space form.*

We now prove Theorem 4. As in [1] the idea is to show that the Gauss-Bonnet integrand is either a nonnegative or a nonpositive multiple of the volume element depending on the case in question. For any basis, the integrand is either a nonnegative or nonpositive multiple of the volume element times the sum

$$\sum \varepsilon_{i_1\cdots i_{2n}}\varepsilon_{j_1\cdots j_{2n}}R_{i_1i_2j_1j_2}\cdots R_{i_{2n-1}i_{2n}j_{2n-1}j_{2n}} \tag{4.7}$$

where $\varepsilon_{i_1\cdots i_{2n}}$ is the Kronecker symbol. If the sectional curvatures are all nonnegative, then by choosing an orthonormal basis of M_m, $m\in M$, as in the proof of Theorem 2, the expression (4.7) is easily seen to be

$$\sum K_{i_1i_2}\cdots K_{i_{2n-1}i_{2n}}\geqq 0,$$

and if they are all nonpositive

$$(-1)^n\chi\geqq 0.$$

The proof of Theorem 5 is almost an immediate consequence of the definition of the Pontrjagin classes by virtue of the normalization of the curvature tensor. The Pontrjagin classes are defined by the closed forms

$$\Phi_k = \frac{2^{2k}(k!)^2}{(2k)!} \sum \delta(i_1, \cdots, i_{2k}; j_1, \cdots, j_{2k}) \Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_{2k} j_{2k}}$$

where the Ω_{ij} are the curvature forms and $\delta(i_1, \cdots, i_{2k}; j_1, \cdots, j_{2k})$ is 0 except when $j_1, \cdots, j_{2k}$ is a permutation of $(i_1, \cdots, i_{2k})$ in which case it is $+1$ or -1 depending on whether the permutation is even or odd, the sum being extended over all indices from $1, \cdots, d$.

REFERENCES

[1] BISHOP, R. L., AND S. I. GOLDBERG, Some implications of the generalized Gauss-Bonnet theorem. Trans. Amer. Math. Soc. **112** (1964), 508–535.

[2] KURITA, M., On the holonomy group of the conformally flat Riemannian manifold. Nagoya Math. J. **9** (1955), 161–171.

[3] LICHNEROWICZ, A., Géométrie des groupes de Transformations. Travaux et Recherches Mathématiques, Dunod, Paris (1958).

[4] TANI, M., On a conformally flat Riemannian space with positive Ricci curvature. Tôhoku Math. J. **19** (1967), 227–231.

[5] YANO, K., AND S. BOCHNER, Curvature and Betti numbers. Annals of Mathematics Studies, No. 32, Princeton University Press, Princeton, New Jersey (1953).

UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS.