

*ON CONFORMALLY RECURRENT
RICCI-RECURRENT MANIFOLDS*

BY

W. ROTER (WROCLAW)

1. Introduction. Let M be a Riemannian manifold with a (possibly indefinite) metric g .

A tensor field $T^{i_1 \dots i_p}_{j_1 \dots j_q}$ of type (p, q) on M will be called *recurrent* if

$$(1) \quad T^{h_1 \dots h_p}_{i_1 \dots i_q} T^{i_1 \dots i_p}_{j_1 \dots j_q, k} = T^{h_1 \dots h_p}_{i_1 \dots i_q, k} T^{i_1 \dots i_p}_{j_1 \dots j_q},$$

where the comma denotes covariant differentiation with respect to g .

Relation (1) states that at any point $x \in M$ such that $T_x \neq 0$ there exists a (unique) covariant vector u (called the *recurrence vector* of T) which satisfies the condition

$$(2) \quad T^{i_1 \dots i_p}_{j_1 \dots j_q, k}(x) = u_k T^{i_1 \dots i_p}_{j_1 \dots j_q}(x).$$

A Riemannian manifold (M, g) will be called *recurrent* [9] (*Ricci-recurrent* [5]) if its curvature tensor (Ricci tensor) is recurrent.

Throughout this paper we assume that the Ricci tensor of a Ricci-recurrent manifold is not parallel.

According to Adati and Miyazawa [1], an n -dimensional ($n \geq 4$) Riemannian manifold (M, g) will be called *conformally recurrent* if its Weyl conformal curvature tensor

$$(3) \quad C_{hijk} = R_{hijk} - \frac{1}{n-2} (g_{ij} R_{hk} - g_{ik} R_{hj} + g_{hk} R_{ij} - g_{hj} R_{ik}) + \\ + \frac{R}{(n-1)(n-2)} (g_{hk} g_{ij} - g_{ik} g_{hj})$$

is recurrent.

If $C_{hijk, l} = 0$ everywhere on M and $\dim M \geq 4$, then (M, g) is said to be *conformally symmetric* [2].

Clearly, the class of conformally recurrent manifolds contains all conformally symmetric as well as all recurrent manifolds of dimension $n \geq 4$.

The existence of essentially conformally recurrent manifolds, i.e., of conformally recurrent manifolds which are neither conformally symmetric nor recurrent can be stated as follows (see [8], Lemmas 2-3):

THEOREM. *Let R^n be endowed with the metric g_{ij} given by*

$$(4) \quad \begin{aligned} g_{ij} dx^i dx^j &= Q (dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2dx^1 dx^n, \\ Q &= (Ap_{\lambda\mu} + k_{\lambda\mu}) x^\lambda x^\mu, \end{aligned}$$

where $i, j = 1, 2, \dots, n$, $\lambda, \mu = 2, 3, \dots, n-1$, $[k_{\lambda\mu}]$ is a symmetric and non-singular matrix, $[p_{\lambda\mu}]$ is a symmetric matrix satisfying $[p_{\lambda\mu}] \neq (n-2)^{-1} [k_{\lambda\mu}]$ and $k^{\lambda\mu} p_{\lambda\mu} = 1$ with $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$, and A is a non-constant function of x^1 only.

Then (R^n, g) is an essentially conformally recurrent manifold which is, moreover, Ricci-recurrent (with a non-parallel Ricci tensor) and satisfies the condition

$$(5) \quad C_{hijk,lm} - C_{hijk,ml} = 0.$$

The purpose of this paper is to obtain a local metric form for a conformally recurrent manifold M_n ($n > 4$) which is simultaneously Ricci-recurrent and satisfies condition (5).

All manifolds under consideration are connected Hausdorff manifolds. The metrics are not assumed to be definite.

2. Preliminaries. In the sequel we shall need the following lemmas:

LEMMA 1. *The Weyl conformal curvature tensor satisfies the well-known relations*

$$(6) \quad C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkh i}, \quad C^r_{rij} = C^r_{trk} = C^r_{ijr} = 0,$$

$$(7) \quad C_{hijk} + C_{hjki} + C_{hkij} = 0,$$

$$(8) \quad C^r_{ijk,r} = \frac{n-3}{n-2} \left[(R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)} (R_{,k} g_{ij} - R_{,j} g_{ik}) \right].$$

LEMMA 2 ([1], equation (3.7), and [4], p. 91). *The Weyl conformal curvature tensor satisfies the relation*

$$(9) \quad \begin{aligned} C_{hijk,l} + C_{hikl,j} + C_{hijl,k} &= \frac{1}{n-3} (g_{ik} C^r_{hjl,r} + g_{hj} C^r_{ikl,r} + g_{il} C^r_{hkf,r} + \\ &+ g_{hk} C^r_{ij,r} + g_{ij} C^r_{hkl,r} + g_{hl} C^r_{ijk,r}). \end{aligned}$$

LEMMA 3 (see [10], Lemma 1, and [9], p. 153). *The curvature tensor of an arbitrary Riemannian manifold satisfies the identity*

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0.$$

LEMMA 4. *The Ricci tensor of a Ricci-recurrent manifold satisfies the equations*

$$(10) \quad R_{ab}R_{ij,l} = R_{ab,l}R_{ij},$$

$$(11) \quad R_{,l}R_{ij} = RR_{ij,l}.$$

Proof. Relations (10) and (11) are immediate consequences of (1).

LEMMA 5. *Let M be a conformally recurrent Ricci-recurrent manifold. If condition (5) is satisfied, then the equation*

$$(12) \quad R_{ij,lm} - R_{ij,ml} = R_{,ri}R^r_{jlm} + R_{,rj}R^r_{ilm} = 0$$

holds on M .

Proof. It is sufficient to prove (12) at points where $R_{ij} \neq 0$.

As an immediate consequence of (3) and (5) we get

$$R_{hijk,lm} - R_{hijk,ml} = \frac{1}{n-2} [g_{ij}(R_{hk,lm} - R_{hk,ml}) - g_{ik}(R_{hj,lm} - R_{hj,ml}) + \\ + g_{hk}(R_{ij,lm} - R_{ij,ml}) - g_{hj}(R_{ik,lm} - R_{ik,ml})],$$

whence, in view of (2) and Lemma 3, we obtain

$$A_{lm}D_{hijk} + A_{hi}D_{jklm} + A_{jk}D_{lmhi} = 0,$$

where $D_{hijk} = g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}$, $A_{ij} = a_{i,j} - a_{j,i}$, and a_j denotes the recurrence vector of R_{ij} .

But from the last relation, since $R_{ij} \neq 0$, we get (see [10], Lemma 2) $A_{ij} = 0$. Using now (2) and the Ricci identity, we obtain easily (12). The lemma is thus proved.

Remark 1. Equation (12) follows from the proof of Adati's and Miyazawa's Lemma (2.1) of [1]. Since the definition of recurrency in [1] differs slightly from the ours, we have included the proof of (12) for completeness.

LEMMA 6. *Let M be a Ricci-recurrent manifold satisfying (12). Then the relation*

$$(13) \quad R_{,ri}R^r_j = \frac{1}{2}RR_{ij}$$

holds on M .

Proof. This lemma was proved for the open subset U of M where $R_{hk,l} \neq 0$ (see [6], Lemma 2). But outside of U , $R_{ij,k}$ vanishes and, therefore, the tensor $B_{ij} = R_{,ri}R^r_j - \frac{1}{2}RR_{ij}$ is parallel on M . Since B_{ij} vanishes on U , it vanishes everywhere on M . This completes the proof.

LEMMA 7. *Let M be a conformally recurrent Ricci-recurrent manifold. If condition (5) is satisfied, then the relations*

$$(14) \quad (R_{ab}R_{dpq,l} - R_{ab,l}R_{dpq})R^{rs}C_{rij_s} = 0,$$

$$(15) \quad R(R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt}) = 0$$

hold on M .

Proof. By a direct calculation, in view of (12) and (13), we get

$$(16) \quad R_{rm}C^r_{ijk} + R_{rt}C^r_{mjk} = \frac{3-n}{2(n-1)(n-2)} R(g_{ij}R_{mk} - g_{ik}R_{mj} + g_{mj}R_{ik} - g_{mk}R_{ij}),$$

whence, by contraction with g^{mk} ,

$$(17) \quad R^{rs}C_{rijs} = \frac{3-n}{2(n-1)(n-2)} R(Rg_{ij} - nR_{ij}).$$

Differentiating (17) covariantly and making use of (10), (11), and (17), we obtain

$$R_{ab}R^{rs}C_{rijs,l} = R^{rs}C_{rijs}R_{ab,l},$$

whence, because of (1),

$$(18) \quad (R_{ab}C_{dpqt,l} - R_{ab,l}C_{dpqt})R^{rs}C_{rijs} = 0.$$

But as an immediate consequence of (3), (10), and (11) we get

$$(19) \quad R_{ab}C_{dpqt,l} - R_{ab,l}C_{dpqt} = R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt}.$$

The last result, together with (18), yields (14).

Now, in view of (14), equation (17) implies

$$R(Rg_{ij} - nR_{ij})(R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt}) = 0,$$

whence, transvecting with R^i_k and making use of (13), we obtain

$$RR_{kj}(R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt}) = 0,$$

which, evidently, completes the proof.

LEMMA 8. *Let M be a conformally recurrent manifold. If M is Ricci-recurrent and satisfies (5), then the relation*

$$(20) \quad (R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt})T_{hlmijk} = 0$$

holds on M , where

$$\begin{aligned} T_{hlmijk} = & g_{hl}R_{rm}C^r_{ijk} - g_{hm}R_{rl}C^r_{ijk} - g_{il}R_{rm}C^r_{hjk} + \\ & + g_{im}R_{rl}C^r_{hjk} + g_{jl}R_{rm}C^r_{khi} - g_{jm}R_{rl}C^r_{khi} + \\ & + g_{kl}R_{rm}C^r_{jih} - g_{km}R_{rl}C^r_{jih} + R_{hl}C_{mijk} - R_{hm}C_{lijk} + \\ & + R_{il}C_{hmjk} - R_{im}C_{hljk} + R_{jl}C_{himk} - R_{jm}C_{hilk} + \\ & + R_{kl}C_{htjm} - R_{km}C_{htjl}. \end{aligned}$$

Proof. Applying the Ricci identity to (5), we get

$$(21) \quad C_{rijk}R^r_{hlm} + C_{hrjk}R^r_{ilm} + C_{hirk}R^r_{jlm} + C_{hijr}R^r_{klm} = 0,$$

whence, by covariant differentiation, the use of (1), (21), (3), (6), and Lemma 4, we obtain

$$(22) \quad R_{qt}C_{abcd,p}(C_{rijk}C^r_{hlm} + C_{hrjk}C^r_{ilm} + C_{hirk}C^r_{jlm} + C_{hijr}C^r_{klm}) + \\ + \frac{1}{n-2}C_{abcd}\left(T_{hlmijs} - \frac{R}{n-1}D_{hlmijs}\right)R_{qt,p} = 0,$$

where

$$D_{hlmijs} = g_{hl}C_{mijk} - g_{hm}C_{lijk} + g_{il}C_{hmjk} - g_{im}C_{hljk} + \\ + g_{jl}C_{himk} - g_{jm}C_{hilk} + g_{kl}C_{hijm} - g_{km}C_{hijl}.$$

On the other hand, substituting into (21) the expression

$$R^h_{ijk} = C^h_{ijk} + \frac{1}{n-2}(g_{ij}R^h_k - g_{ik}R^h_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}) - \\ - \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

differentiating covariantly, and using (1) and Lemma 4, we get

$$2R_{qt}(C_{rijk}C^r_{hlm} + C_{hrjk}C^r_{ilm} + C_{hirk}C^r_{jlm} + C_{hijr}C^r_{klm})C_{abcd,p} + \\ + \frac{1}{n-2}(R_{qt}C_{abcd,p} + R_{qt,p}C_{abcd})\left(T_{hlmijs} - \frac{R}{n-1}D_{hlmijs}\right) = 0.$$

But the last relation, together with (22), implies

$$(R_{qt}C_{abcd,p} - R_{qt,p}C_{abcd})\left(T_{hlmijs} - \frac{1}{n-1}RD_{hlmijs}\right) = 0,$$

which, in view of (19) and (15), leads immediately to our assertion. The lemma is proved.

LEMMA 9. *Let M be a conformally recurrent manifold. If M is Ricci-recurrent and condition (5) is satisfied, then the relations*

$$(23) \quad (R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt})\dot{R}_{rm}C^r_{ijk} = 0,$$

$$(24) \quad (R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt})(R_{hl}C_{mijk} - R_{hm}C_{lijk} + R_{il}C_{hmjk} - R_{im}C_{hljk} + \\ + R_{jl}C_{himk} - R_{jm}C_{hilk} + R_{kl}C_{hijm} - R_{km}C_{hijl}) = 0,$$

hold on M .

Proof. Contracting (20) with g^{hl} and using (6), (7), (14), and (15), we get

$$(25) \quad (R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt})[(n-2)R_{rm}C^r_{ijk} + R_{rj}C^r_{imk} + \\ + R_{rk}C^r_{ijm} + R_{ri}C^r_{mjk}] = 0.$$

On the other hand, as a consequence of (15) and (16), we have

$$(R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt})R_{rm}C^r_{ijk} = -(R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt})R_{ri}C^r_{mjk}.$$

The last result, together with (25) and (7), leads immediately to (23). Relation (24) follows now from (20) and (23). The lemma is thus proved.

LEMMA 10. *Let M be a Riemannian manifold whose Ricci tensor satisfies*

$$(26) \quad R_{nl}C_{mijk} - R_{hm}C_{lijk} + R_{il}C_{hmjk} - R_{im}C_{hljk} + \\ + R_{jl}C_{himk} - R_{jm}C_{hilk} + R_{kl}C_{hijm} - R_{km}C_{hijl} = 0.$$

Suppose, moreover, that $x \in M$ and $\text{rank } R_{ij}(x) > 1$.

Then the Weyl conformal curvature tensor on some neighbourhood of x is of the form

$$(27) \quad C_{hijk} = S(R_{ij}R_{hk} - R_{hj}R_{ik}).$$

Proof. This lemma was proved for conformally symmetric manifolds (see [3], Theorem 3). But its proof, as one can easily verify, requires only relations (6), (7), and (26). Therefore, (27) remains true for an arbitrary Riemannian manifold satisfying (26).

LEMMA 11. *Let M_n ($n > 4$) be a conformally recurrent Ricci-recurrent manifold whose Weyl conformal curvature tensor satisfies (5). Suppose that at $x \in M_n$ the conditions*

$$(28) \quad R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt} \neq 0,$$

$$(29) \quad \text{rank } R_{ij} > 1$$

hold.

Then $R_{ij,l}(x) = 0$.

Proof. By (24), (27), and (23), we obtain on some neighbourhood U of x

$$(30) \quad C_{rijk}C^r_{hlm} = 0.$$

Since (28) holds, in each neighbourhood of x there exists a point y such that $C_{hijk}(y) \neq 0$. For otherwise, C_{hijk} as well as $C_{hijk,l}$ would vanish on some neighbourhood W of x and, consequently, the recurrence condition (1) would be satisfied for R_{hijk} on W , a contradiction.

Now, in view of (2) and (9), we get at y

$$b_i C_{hijk} + b_j C_{hikl} + b_k C_{hilj} = \frac{1}{n-3} (g_{ik} b_r C^r_{hjl} + g_{hj} b_r C^r_{ikl} + g_{il} b_r C^r_{hkj} + \\ + g_{hk} b_r C^r_{ij} + g_{ij} b_r C^r_{hkk} + g_{hl} b_r C^r_{ijk}),$$

where b_j denotes the recurrence vector of C_{hijk} .

Transvecting the last relation with C^l_{ptm} and using (30), we obtain for $y \in U$

$$b_r C^r_{ptm} C_{hijk} = \frac{1}{n-3} (b_r C^r_{hki} C_{iptm} + b_r C^r_{ijk} C_{hptm}),$$

which, by a further transvection with b^h , yields

$$b_r C^r_{ptm} b_s C^s_{ijk} = \frac{1}{n-3} b_s C^s_{ijk} b_r C^r_{ptm}.$$

Hence $b_r C^r_{ijk} = 0 = C^r_{ijk,r}$. But the last result, in view of (15), reduces (8) to the form $R_{ij,k}(y) = R_{ik,j}(y)$. If now $R_{ij,k}(x)$ were non-zero, then, by (2), we would have $\text{rank } R_{ij}(y) = 1$ for some set of y 's from U , a contradiction. The lemma is thus proved.

In the sequel, we shall often assume the following hypothesis:

(31) (M, g) is an n -dimensional ($n > 4$) conformally recurrent Ricci-recurrent manifold satisfying (5), and x is a point of M such that the conditions

$$(32) \quad R_{ij} \neq 0, \quad R_{ij,k} \neq 0$$

and (28) hold at x .

Remark 2. For the metric (4), as one can easily verify [8], conditions (28) and (32) are equivalent to: $A \neq 2 - n, \partial_1 A \neq 0$.

PROPOSITION 1. Under hypothesis (31), $\text{rank } R_{ij}(x) = 1$ and M admits a non-trivial null parallel vector field on some neighbourhood of x .

Proof. The first part of our assertion follows immediately from (32) and Lemma 11.

Since (2) and

$$(33) \quad R_{ij} = e c_i c_j, \quad |e| = 1$$

hold, we have

$$(c_{i,k} - \frac{1}{2} a_k c_i) c_j + (c_{j,k} - \frac{1}{2} a_k c_j) c_i = 0.$$

Hence

$$(34) \quad c_{j,k} = \frac{1}{2} a_k c_j.$$

On the other hand, as a consequence of (2) and (12), we get $a_{i,j} = a_{j,i}$. Therefore, there exists a function a such that $a_{,j} = a_j$. Now, in view of (34), it is easy to verify that the vector field $B_j = \exp(-\frac{1}{2}a)c_j$ is parallel. Moreover, because of (15) and (33), B_j is null. This completes the proof.

LEMMA 12. *Under hypothesis (31), the equation*

$$(35) \quad R_{hl}R_{mijk} - R_{hm}R_{lijk} + R_{il}R_{hmjk} - R_{im}R_{hljk} + \\ + R_{jl}R_{himk} - R_{jm}R_{hilk} + R_{kl}R_{hijm} - R_{km}R_{hijl} = 0$$

holds on some neighbourhood of x .

Proof. As a consequence of (15), (20), and (22), we get

$$C_{abcd,p}(C_{rijk}C^r_{hlm} + C_{hrjk}C^r_{ilm} + C_{hirk}C^r_{jlm} + C_{hijr}C^r_{klm}) = 0$$

on some neighbourhood $U \ni x$. For the open subset of U (if it exists) where C_{hijk} is parallel, (35) follows immediately from (24) of [7].

Thus, we can assume that we have on U

$$(36) \quad C_{rijk}C^r_{hlm} + C_{hrjk}C^r_{ilm} + C_{hirk}C^r_{jlm} + C_{hijr}C^r_{klm} = 0.$$

On the other hand, using (12) and (33), we obtain

$$c_i c_r R^r_{jlm} + c_j c_r R^r_{ilm} = 0.$$

Hence $c_r R^r_{jlm} = 0$ and, consequently, $R_{ri}R^r_{jlm} = 0$.

But the last result, in view of (33), (23), and

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}),$$

yields

$$(37) \quad C_{rijk}C^r_{hlm} = R_{rijk}R^r_{hlm} - \frac{1}{n-2}(R_{ij}R_{khlm} - R_{ik}R_{jhlm} + \\ + R_{hl}R_{mijk} - R_{hm}R_{lijk}) + \frac{1}{(n-2)^2}(g_{km}R_{ij}R_{hl} - g_{jm}R_{ik}R_{hl} + g_{jl}R_{ik}R_{hm} - \\ - g_{lk}R_{ij}R_{hm}).$$

Since

$$0 = R_{hijk,lm} - R_{hijk,ml} = R_{rijk}R^r_{hlm} + R_{hrjk}R^r_{ilm} + R_{hirk}R^r_{jlm} + R_{hijr}R^r_{klm},$$

which follows easily from (3), (5), and (12), relation (36), together with (37) and (33), leads to (35). The last remark completes the proof.

PROPOSITION 2. *Under hypothesis (31), the curvature tensor takes on some neighbourhood of x the form*

$$(38) \quad R_{jkhm} = c_h c_k S_{mj} - c_h c_j S_{mk} + c_m c_j S_{hk} - c_m c_k S_{hj},$$

where $S_{ij} = S_{ji} = v^r v^s R_{rij s}$ and $v^r c_r = 1$.

Proof. Since equations (33) and (35) hold, we have (see [7], Lemma 4)

$$c_i R_{jkhm} + c_h R_{jkmi} + c_m R_{jklh} = 0.$$

Now, with the help of the last result, we can follow step by step a proof of Walker (see [10], p. 45, and [9], p. 155) to obtain (38). This completes the proof.

3. Local structure theorem. We are now in a position to prove our main result. In this section each Latin index runs over $1, 2, \dots, n$, and each Greek index over $2, 3, \dots, n-1$.

THEOREM. (i) *Let (M, g) be an n -dimensional ($n > 4$) conformally recurrent Ricci-recurrent manifold satisfying (5). Suppose, moreover, that at $x \in M$ conditions (28), (32), and $C_{hijk} \neq 0$ hold. Then there exists a coordinate system x^1, x^2, \dots, x^n on a neighbourhood of x such that the metric of M takes the form*

$$(39) \quad g_{ij} dx^i dx^j = Q(dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2 dx^1 dx^n,$$

$$(40) \quad \partial_\lambda \partial_\mu Q = Ak_{\lambda\mu} + Bc_{\lambda\mu},$$

where $[k_{\lambda\mu}]$ is a symmetric and non-singular matrix, $[c_{\lambda\mu}]$ is a symmetric and non-zero matrix satisfying $k^{\alpha\beta} c_{\alpha\beta} = 0$ with $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$, and A, B are functions not depending on x^n such that (40) as well as

$$(41) \quad A \neq 0 \neq B, \quad \partial_j A \neq 0, \quad A \partial_j B - B \partial_j A \neq 0$$

are satisfied.

(ii) *Let R^n ($n \geq 4$) be endowed with the metric g_{ij} given by (39), where $[k_{\lambda\mu}]$ and $[c_{\lambda\mu}]$ are as above, and A, B are non-constant functions satisfying (40), independent of x^n , and such that $A \partial_j B - B \partial_j A$ does not indentially vanish. Then (R^n, g) is an essentially conformally recurrent Ricci-recurrent manifold whose Weyl conformal curvature tensor satisfies (5).*

Proof. From Walker's considerations it follows ([9], p. 176-179, and [10], p. 51-54) that if a Riemannian manifold admits on a neighbourhood U of a point x a non-trivial null parallel vector field $B_j = f e_j$ and on U the curvature tensor is of the form (38), then one can choose coordinates so that the metric can be written as

$$(42) \quad g_{ij} dx^i dx^j = Q(dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2 dx^1 dx^n,$$

where $k_{\lambda\mu} = k_{\mu\lambda}$ are constants, $\det[k_{\lambda\mu}] \neq 0$, and Q is a function independent of x^n .

As one can easily verify,

$$[g^{ij}] = \begin{bmatrix} \bar{0} & 0 & \dots & 0 & \bar{1} \\ 0 & & & & 0 \\ \vdots & & k^{\lambda\mu} & & \vdots \\ 0 & & & & 0 \\ \underline{1} & 0 & \dots & 0 & \underline{-Q} \end{bmatrix}$$

and, in the metric (42), the only Christoffel symbols not identically zero are

$$\left\{ \begin{matrix} \lambda \\ 11 \end{matrix} \right\} = -\frac{1}{2} k^{\lambda\omega} Q_{.\omega}, \quad \left\{ \begin{matrix} n \\ 11 \end{matrix} \right\} = \frac{1}{2} Q_{.1}, \quad \left\{ \begin{matrix} n \\ 1\lambda \end{matrix} \right\} = \frac{1}{2} Q_{.1},$$

where the dot denotes partial differentiation with respect to coordinates.

Moreover, in view of the formula

$$R_{hijk} = \frac{1}{2} (g_{hk,ij} + g_{ij,hk} - g_{hj,ik} - g_{ik,hj}) + g_{pq} \left\{ \begin{matrix} p \\ hk \end{matrix} \right\} \left\{ \begin{matrix} q \\ ij \end{matrix} \right\} - g_{pq} \left\{ \begin{matrix} p \\ hj \end{matrix} \right\} \left\{ \begin{matrix} q \\ ik \end{matrix} \right\},$$

it follows ([9], p. 179) that the only components R_{hijk} not identically zero are those related to $R_{1\lambda\mu 1} = \frac{1}{2} Q_{.\lambda\mu}$.

It can be also found that

$$R_{11} = \frac{1}{2} k^{\alpha\beta} Q_{.\alpha\beta}, \quad R_{11,1} = \frac{1}{2} k^{\alpha\beta} Q_{.\alpha\beta 1}, \quad R_{11,\gamma} = \frac{1}{2} k^{\alpha\beta} Q_{.\alpha\beta\gamma},$$

$$R_{1\lambda\mu 1,1} = \frac{1}{2} Q_{.\lambda\mu 1}, \quad R_{1\lambda\mu 1,\gamma} = \frac{1}{2} Q_{.\lambda\mu\gamma},$$

$$(43) \quad C_{1\lambda\mu 1} = \frac{1}{2} \left[Q_{.\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{.\alpha\beta}) \right],$$

$$C_{1\lambda\mu 1,1} = \frac{1}{2} \left[Q_{.\lambda\mu 1} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{.\alpha\beta 1}) \right],$$

$$C_{1\lambda\mu 1,\gamma} = \frac{1}{2} \left[Q_{.\lambda\mu\gamma} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{.\alpha\beta\gamma}) \right],$$

and that all other components are identically zero.

Since M is conformally recurrent and conditions (5) and (2) hold, $b_{i,j} = b_{j,i}$ on some neighbourhood of x . The recurrence vector b_j is therefore a gradient.

Hence, in view of (43),

$$\left[Q_{\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{\alpha\beta}) \right]_{,j} = C_{.j} \left[Q_{\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{\alpha\beta}) \right]$$

for some function C .

Thus

$$Q_{\lambda\mu} = Ak_{\lambda\mu} + Bc_{\lambda\mu},$$

where $[c_{\lambda\mu}]$ is a symmetric and non-zero matrix satisfying $k^{\alpha\beta} c_{\alpha\beta} = 0$.

Moreover, because of (40) and (43), we have

$$R_{11} = \frac{n-2}{2} A, \quad R_{11,j} = \frac{n-2}{2} A_{.j}, \quad C_{1\lambda\mu 1} = \frac{1}{2} Bc_{\lambda\mu}, \quad C_{1\lambda\mu 1,j} = \frac{1}{2} B_{.j} c_{\lambda\mu},$$

(44)

$$R_{1\lambda\mu 1} = \frac{1}{2} (Ak_{\lambda\mu} + Bc_{\lambda\mu}), \quad R_{1\lambda\mu 1,j} = \frac{1}{2} (A_{.j} k_{\lambda\mu} + B_{.j} c_{\lambda\mu}).$$

Since conditions (28), (32), and $C_{hijk} \neq 0$ are satisfied by assumption, A and B , as one can easily verify, satisfy (41).

Suppose now that the conditions of (ii) hold. Then, as follows easily from (44) and (1), (R^n, g) is a conformally recurrent Ricci-recurrent manifold. Since $B \neq \text{const}$ and $AB_{.j} - BA_{.j}$ does not identically vanish, (R^n, g) is neither conformally symmetric nor recurrent. Moreover, because of $C_{1\lambda\mu 1,ij} - C_{1\lambda\mu 1,ji} = 0$, condition (5) is satisfied. This completes the proof.

Remark 3. If Q in (39) is of the form $Q = (Ak_{\lambda\mu} + Bc_{\lambda\mu})x^\lambda x^\mu$, where A and B are functions of x^1 only, satisfying assumptions of (ii) and $A \neq 0 \neq B$, then (R^n, g) is an essentially conformally recurrent Ricci-recurrent manifold such that the recurrence vectors of R_{ij} as well as of C_{hijk} are both null at each point of R^n .

Now we shall prove the existence of essentially conformally recurrent Ricci-recurrent manifolds whose recurrence vectors are non-null everywhere.

Suppose that R^n is endowed with the metric (39), where

$$[k_{\lambda\mu}] = \begin{bmatrix} e_2 & & & \\ & \ddots & & 0 \\ 0 & & & \\ & & & e_{n-1} \end{bmatrix}, \quad |e_\lambda| = 1,$$

and

$$Q = \frac{1}{2} \sum_{i=2}^{n-1} e_i (x^i)^2 + (n-2) e_{n-1} \exp(x^{n-1}).$$

Then, as one can easily verify, we have

$$Q_{. \lambda \mu} = (1 + \exp(x^{n-1}))k_{\lambda \mu} + c_{\lambda \mu} \exp(x^{n-1}),$$

where

$$[c_{\lambda \mu}] = \begin{bmatrix} -e_2 & & & & & & & & & \\ & -e_3 & & & & & & & & 0 \\ & & \ddots & & & & & & & \\ & & & \ddots & & & & & & \\ & 0 & & & -e_{n-2} & & & & & \\ & & & & & & & & & (n-3)e_{n-1} \end{bmatrix}.$$

Since we have $k^{\alpha\beta}c_{\alpha\beta} = 0$ and the functions $A = 1 + \exp(x^{n-1})$ and $B = \exp(x^{n-1})$ satisfy the condition $AB_{.j} - BA_{.j} \neq 0$, (R^n, g) is an essentially conformally recurrent Ricci-recurrent manifold.

Taking now into account (44), we obtain

$$a_j = \partial_j \log(1 + \exp(x^{n-1})) \quad \text{and} \quad b_j = \delta_j^{n-1},$$

where a_j and b_j denote the recurrence vectors of R_{ij} and C_{hijk} , respectively. Moreover, as one can easily verify, $a^r a_r \neq 0 \neq b^r b_r$ everywhere on R^n .

Remark 4. The metric (4) can be obtained from (40) as follows: Let

$$A = \frac{2}{n-2} \bar{A} + 2 \quad \text{and} \quad Bc_{\lambda\mu} = 2\bar{A} \left(p_{\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu} \right),$$

where $p_{\lambda\mu} = p_{\mu\lambda} = \text{const}$ and \bar{A} is a non-constant function of x^1 only. Then $k^{\alpha\beta}p_{\alpha\beta} = 1$ and $Q_{. \lambda \mu} = 2(\bar{A}p_{\lambda\mu} + k_{\lambda\mu})$. Hence

$$Q = (\bar{A}p_{\lambda\mu} + k_{\lambda\mu})x^\lambda x^\mu + \varrho_\alpha x^\alpha + \eta,$$

where ϱ_α, η are functions of x^1 only. Taking $\varrho_\alpha = \eta = 0$, we obtain (4).

Remark 5. The subset of points satisfying (28), (32), and $C_{hijk} \neq 0$ is dense if (M, g) is an analytic non-recurrent Ricci-recurrent manifold (with non-parallel Ricci tensor).

REFERENCES

[1] T. Adati and T. Miyazawa, *On a Riemannian space with recurrent conformal curvature*, Tensor, New Series, 18 (1967), p. 348-354.
 [2] M. C. Chaki and B. Gupta, *On conformally symmetric spaces*, Indian Journal of Mathematics 5 (1963), p. 113-122.
 [3] A. Derdziński and W. Roter, *Some properties of conformally symmetric manifolds which are not Ricci-recurrent*, Tensor, New Series, 34 (1980), p. 11-20.

-
- [4] L. P. Eisenhart, *Riemannian geometry*, Princeton 1966.
- [5] E. M. Patterson, *Some theorems on Ricci-recurrent spaces*, The Journal of the London Mathematical Society 27 (1952), p. 287-295.
- [6] W. Roter, *Some remarks on infinitesimal projective transformations in recurrent and Ricci-recurrent spaces*, Colloquium Mathematicum 15 (1966), p. 121-127.
- [7] — *On conformally symmetric Ricci-recurrent spaces*, ibidem 31 (1974), p. 87-96.
- [8] — *On the existence of conformally recurrent Ricci-recurrent spaces*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 24 (1976), p. 973-979.
- [9] H. S. Ruse, A. G. Walker and T. J. Willmore, *Harmonic spaces*, Roma 1961.
- [10] A. G. Walker, *On Ruse's spaces of recurrent curvature*, Proceedings of the London Mathematical Society 52 (1950), p. 36-64.

Reçu par la Rédaction le 5. 4. 1979
