Appendix C

Congruence lattices of lattices

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In the early sixties, we characterized congruence lattices of universal algebras as algebraic lattices; then our interest turned to the characterization of congruence lattices of lattices. Since these lattices are distributive, we figured that this must be an easier job. After more then 35 years, we know that we were wrong.

In this Appendix, we give a brief overview of the results and methods. In Section 1, we deal with finite distributive lattices; their representation as congruence lattices raises many interesting questions and needs specialized techniques. In Section 2, we discuss the general case.

A related topic is the lattice of complete congruences of a complete lattice. In contrast to the previous problem, we can characterize complete congruence lattices, as outlined in Section 3.

1. The Finite Case

The congruence lattice, $\mathrm{Con}\, L$, of a finite lattice L is a finite distributive lattice (N. Funayama and T. Nakayama, see Theorem II.3.11). The converse is a result of R. P. Dilworth (see Theorem II.3.17), first published G. Grätzer and E. T. Schmidt [1962]. Note that the lattice we construct is sectionally complemented.

In Section II.1, we learned that a finite distributive lattice D is determined by the poset $\mathrm{J}(D)$ of its join-irreducible elements, and every finite poset can be represented as $\mathrm{J}(D)$, for some finite distributive lattice D. So for a finite lattice L, we can reduce the characterization problem of finite congruence lattices to the representation of finite posets as $\mathrm{J}(\mathrm{Con}\, L)$.

Planar lattices, small lattices

We start with the following result (G. Grätzer, H. Lakser, and E. T. Schmidt [A118]):

Theorem 1 Let D be a finite distributive lattice with n join-irreducible elements. Then there exists a planar lattice L of $O(n^2)$ elements with $\operatorname{Con} L \cong D$.

The original constructions (R. P. Dilworth's and also our own) produced lattices of size $O(2^{2n})$ and of order dimension $O(2n)$. In G. Grätzer and H. Lakser [C11], this was improved to size $O(n^3)$ and order dimension 2 (planar).

We sketch the construction for Theorem 1.

Let $P = \mathrm{J}(D)$, $P = \{p_1, p_2, \ldots, p_n\}$, and we take a chain

$$C = \{c_0, c_1, \ldots, c_{2n}\}, \quad c_0 \prec c_1 \prec \cdots \prec c_{2n}.$$

To every prime interval $[c_i, c_{i+1}]$, we assign an element of P as its "color", so that each element of P is the color of two adjacent prime intervals: let the color of $[c_0, c_1]$ and $[c_1, c_2]$ be p_1; of $[c_2, c_3]$ and $[c_3, c_4]$ be p_2, and so on, of $[c_{2n-2}, c_{2n-1}]$ and $[c_{2n-1}, c_{2n}]$ be p_n. Follow this on the two examples in Figures 1 and 2; in Figure 1, $P = \{p_1, p_2\}$ and $p_1 < p_2$, while in Figure 2, $P = \{p_1, p_2, p_3\}$ and $p_1 < p_2$, $p_3 < p_2$. The colors are indicated on the diagrams.

In C^2, we fill in a "covering square" C_2^2 with one more element so that we obtain an M_3, if the two sides have the same color, see Figure 3. Moreover, if p, $q \in P$ and $p < q$, then we take the "double covering square" $C_3 \times C_2$, where the longer side has two prime intervals of color q and the shorter side is of color p, and we add one more element, as illustrated in Figure 3, to obtain the sublattice $N_{5,5}$.

It is an easy computation to show that $|L| \leq kn^2$, for some constant k, and that $D \cong \operatorname{Con} L$; this isomorphism is established by assigning to $p \in P$ the congruence of L generated by collapsing any (all) prime intervals of color p.

For a natural number n, define $\operatorname{cr}(n)$ as the smallest integer such that, for any distributive lattice D with n join-irreducible elements, there exists a finite lattice L satisfying $\operatorname{Con} L \cong D$ and $|L| \leq \operatorname{cr}(n)$. From the construction sketched above, it follows that

$$\operatorname{cr}(n) < 3(n+1)^2.$$

In Section A.1.7, we discussed the lower bound $\frac{n^2}{16 \log_2 n}$ for $\operatorname{cr}(n)$. Here is how it evolved: G. Grätzer, I. Rival, and N. Zaguia [C18] proved that Theorem 1 is "best possible" in the sense that size $O(n^2)$ cannot be replaced by size $O(n^\alpha)$, for any $\alpha < 2$; that is,

$$kn^\alpha < \operatorname{cr}(n),$$

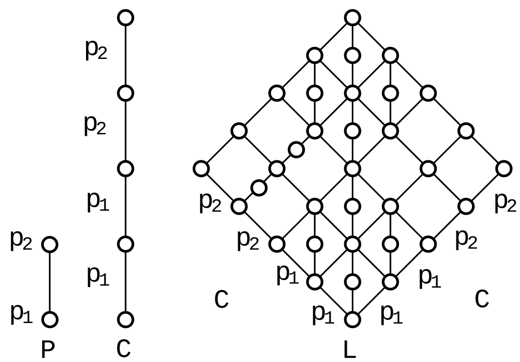

Figure 1

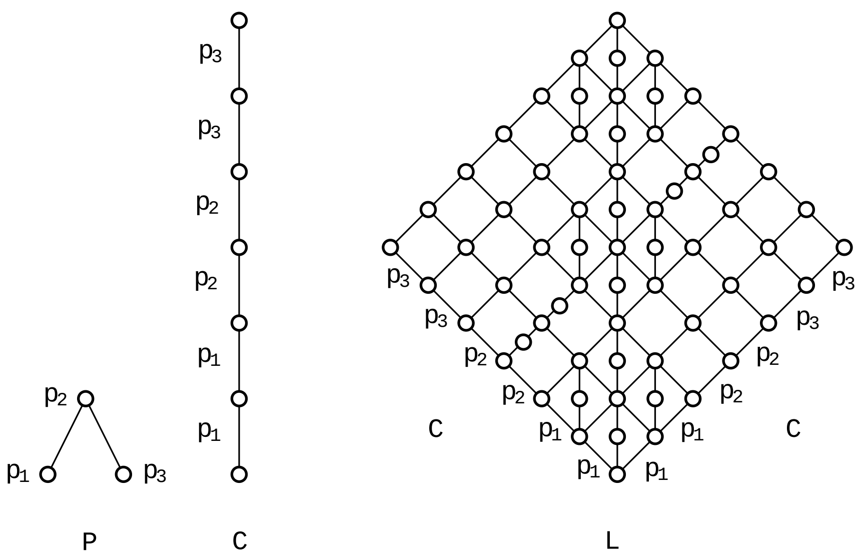

Figure 2

for any constant k, for any $\alpha < 2$, and for any sufficiently large integer n. Y. Zhang [C39] noticed that the proof of this inequality can be improved to obtain the following result: for $n \geq 64$,

$$\frac{1}{64}\frac{n^2}{(\log_2 n)^2} < \operatorname{cr}(n).$$

The lower bound $\frac{n^2}{16 \log_2 n}$ for $\operatorname{cr}(n)$ is, of course, much stronger than the last one.

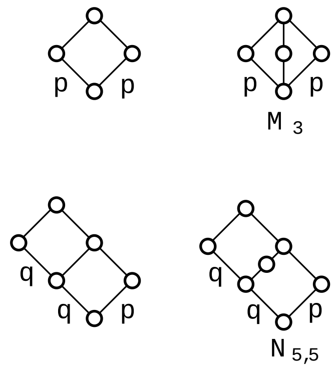

Figure 3

A different kind of lower bound is obtained in R. Freese [C5]; it is shown that if $\mathrm{J}(\mathrm{Con}\, L)$ has e edges ($e > 2$), then

$$\frac{e}{2\log_2 e} \leq |L|.$$

R. Freese also proves that $\mathrm{J}(\mathrm{Con}\, L)$ can be computed in time $O(|L|^2 \log_2 |L|)$.

Consider the optimal *length* of L. E. T. Schmidt [1975] constructs a finite lattice L of length $5m$, where m is the number of dual atoms of D; S.-K. Teo [C36] proves that this result is best possible. (For finite chains, this was done in J. Berman [1972].)

Special classes of lattices

Modular and semimodular lattices

E. T. Schmidt [1974] proves that *Every finite distributive lattice D can be represented as the congruence lattice of a* modular *lattice M.* It follows from Theorem III.4.9 that the congruence lattice of a finite modular lattice is a Boolean lattice, therefore, we cannot expect M to be finite. For a short proof of this result, see [C35].

A much deeper result was proved in E. T. Schmidt [C33]:

Theorem 2 Every finite distributive lattice can be represented as the congruence lattice of a *complemented modular lattice.* —

It was pointed out by F. Wehrung that the ring theoretic and ordered group-theoretic results of G. A. Elliott [C4], P. A. Grillet [C28], E. G. Effros, D. E. Handelman, and Chao Liang Shen [C3] (along with some elementary results in F. Wehrung [A274]) contain this theorem; see K. R. Goodearl and F. Wehrung [C7] for more detail.

In G. Grätzer, H. Lakser, and E. T. Schmidt [A119], we constructed a *finite semimodular* lattice L:

Theorem 3 Every finite distributive lattice D can be represented as the congruence lattice of a *finite semimodular* lattice S. In fact, S can be constructed as a *planar* lattice of size $O(n^3)$, where n is the number of join-irreducible elements of D. —

Congruence-preserving extensions

In Section A.1.7, we discussed the concept of congruence-preserving extensions and some of the major result concerning it. Many of the older results use the following construction, see E. T. Schmidt [1968]. Let D be a bounded distributive lattice; let $M_3[D]$ (the extension of M_3 by D) consist of all triples $\langle x, y, z\rangle \in D^3$ satisfying $x \wedge y = x \wedge z = y \wedge z$. Then $M_3[D]$ is a (modular) lattice, $x \mapsto \langle x, 0, 0\rangle$, $x \in D$, is an embedding of D into $M_3[D]$ and by identifying x with $\langle x, 0, 0\rangle$, we obtain that $M_3[D]$ is a congruence-preserving extension of D. (See a variant of this construction in Section 2.)

In G. Grätzer, H. Lakser, and R. W. Quackenbush [C12], it is proved that the $M_3[D]$ construction is a special case of *tensor products*. If A and B are lattices with zero, then $A \otimes B$, the tensor product of A and B, is the join-semilattice freely generated by the poset $(A - \{0\}) \times (B - \{0\})$ subject to the relations: $\langle a_0, b\rangle \vee \langle a_1, b\rangle = \langle a_0 \vee a_1, b\rangle$ and $\langle a, b_0\rangle \vee \langle a, b_1\rangle = \langle a, b_0 \vee b_1\rangle$ (a, a_0, $a_1 \in A$, b, b_0, $b_1 \in B$).

Let A and B be finite lattices. Then $A \otimes B$ is obviously a lattice, and the following isomorphism holds (see [C12]):

Theorem 4

$$\operatorname{Con} A \otimes \operatorname{Con} B \cong \operatorname{Con}(A \otimes B). \qquad —$$

For any finite simple lattice S, this isomorphism implies that $\operatorname{Con} A \cong \operatorname{Con}(A \otimes S)$. The case $A = D$, $S = M_3$ is the $M_3[D]$ result.

Theorem 4 has been extended to wide classes of infinite lattices (substituting Con_c for Con) in G. Grätzer and F. Wehrung [C26] and to arbitrary lattices with zero using "box products" (a variant of tensor products) in G. Grätzer and F. Wehrung [C27].

Many of the newer results utilize a new technique in G. Grätzer and E. T. Schmidt [A123] (applied also in G. Grätzer and E. T. Schmidt [A124] and [C25]).

The *rectangular extension* $\mathbb{R}(K)$ of a finite lattice K is defined as the direct product of *all* subdirect factors of K, that is,

$$\mathbb{R}(K) = \prod(\, K/\Phi \mid \Phi \in \mathrm{M}(\operatorname{Con} K)\,),$$

where $\mathrm{M}(\operatorname{Con} K)$ is the set of all meet-irreducible congruences of K.

K has a natural embedding into $\mathbb{R}(K)$ by

$$\psi\colon a \mapsto a^{\mathbb{R}} = \langle [a]\Phi \mid \Phi \in \mathrm{M}(\operatorname{Con} K)\rangle.$$

Let $K\psi = K^{\mathbb{R}}$.

Theorem 5 Let K be a finite lattice. Then $K^{\mathbb{R}}$ has the Congruence Extension Property in $\mathbb{R}(K)$ (that is, every congruence of $K^{\mathbb{R}}$ can be extended to $\mathbb{R}(K)$).

p-algebras

T. Katrinák [C31] characterized the congruence lattices of finite p-algebras (that is, lattices with pseudocomplementation, see Section II.6):

Theorem 6 Every finite distributive lattice is isomorphic to the congruence lattice of a finite p-algebra.

In [C22], we give a more elementary proof of this theorem. In fact, we prove the following generalization:

Theorem 7 Let D be an algebraic distributive lattice in which the unit element of D is compact and every compact element of D is a finite join of join-irreducible compact elements. Then D can be represented as the congruence lattice of a p-algebra P.

Simultaneous representation

It is well-known that, given a lattice L and a convex sublattice K, the restriction map $\operatorname{Con} L \to \operatorname{Con} K$ is a $\{0,1\}$-preserving lattice homomorphism. In G. Grätzer and H. Lakser [C9], see also E. T. Schmidt [C34], the converse is proved: any $\{0,1\}$-preserving homomorphism of finite distributive lattices can be realized as such a restriction and, indeed, as a restriction to an ideal of a finite lattice.

If the sublattice K is not a convex sublattice, then the restriction map $\operatorname{Con} L \to \operatorname{Con} K$ need not preserve join, but it still preserves meet, 0, and 1.

Similarly, we can *extend* congruences from the sublattice K to L by minimal extension. This extension map need not preserve meet, but it does preserve join and 0. Furthermore, it *separates* 0, that is, nonzero congruences extend to nonzero congruences. We now formalize this.

Let K and L be lattices, and let φ be a homomorphism of K into L. Then φ induces a map $\operatorname{ext}\varphi$ of $\operatorname{Con} K$ into $\operatorname{Con} L$: for a congruence relation Θ of K, let the image Θ under $\operatorname{ext}\varphi$ be the congruence relation of L generated by the set $\Theta\varphi = \{\, \langle a\varphi, b\varphi \rangle \mid a \equiv b \ (\Theta) \,\}$.

The following result was proved by A. P. Huhn in [C29] in the special case when ψ is an embedding and was proved for arbitrary ψ in G. Grätzer, H. Lakser, and E. T. Schmidt [C14]:

Theorem 8 Let D and E be finite distributive lattices, and let

$$\psi\colon D \to E$$

be a $\{0, \vee\}$-homomorphism. Then there are finite lattices K and L, a lattice homomorphism $\varphi\colon K \to L$, and isomorphisms

$$\alpha\colon D \to \operatorname{Con} K,$$
$$\beta\colon E \to \operatorname{Con} L$$

with

$$\psi\beta = \alpha(\operatorname{ext}\varphi).$$

Furthermore, φ is an embedding iff ψ separates 0. —

Theorem 8 concludes that the following diagram is commutative:

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\alpha", "\cong"'] & E \arrow[d, "\beta", "\cong"'] \\ \operatorname{Con} K \arrow[r, "\operatorname{ext}\varphi"] & \operatorname{Con} L \end{tikzcd}$$

In G. Grätzer, H. Lakser, and E. T. Schmidt [C15] the following stronger version is proved (see G. Grätzer, H. Lakser, and E. T. Schmidt [C13] for a short proof):

Theorem 9 Let K be a finite lattice, let E be a finite distributive lattice, and let $\psi\colon \operatorname{Con} K \to E$ be a $\{0, \vee\}$-homomorphism. Then there is a finite lattice L, a lattice homomorphism $\varphi\colon K \to L$, and an isomorphism $\beta\colon E \to \operatorname{Con} L$ with $\operatorname{ext}\varphi = \psi\beta$. Furthermore, φ is an embedding iff ψ separates 0. —

If L is a lattice and L_1, L_2 are sublattices of L, then there is a map

$$\operatorname{Con} L_1 \to \operatorname{Con} L_2$$

obtained by first extending each congruence relation of L_1 to L and then restricting the resulting congruence relation to L_2. All we can say about this map is that it is isotone and that it preserves 0. The main result of G. Grätzer, H. Lakser, and E. T. Schmidt [C14] (see G. Grätzer, H. Lakser, and E. T. Schmidt [C13] for a short proof) is that this is, in fact, a characterization of 0-preserving isotone maps between *finite* distributive lattices:

Theorem 10 Let D_1 and D_2 be finite distributive lattices, and let

$$\psi\colon D_1 \to D_2$$

be an isotone map that preserves 0. Then there is a finite lattice L with sublattices L_1 and L_2 and there are isomorphisms

$$\alpha_1\colon D_1 \to \operatorname{Con} L_1,$$
$$\alpha_2\colon D_2 \to \operatorname{Con} L_2$$

such that the diagram

$$\begin{tikzcd}
D_1 \arrow[rr, "\psi"] \arrow[d, "\alpha_1", "\cong"'] & & D_2 \arrow[d, "\alpha_2", "\cong"'] \\
\operatorname{Con} L_1 \arrow[r, "\text{extension}"] & \operatorname{Con} L \arrow[r, "\text{restriction}"] & \operatorname{Con} L_2
\end{tikzcd}$$

is commutative. —

G. Grätzer, H. Lakser, and E. T. Schmidt [C16] and [C17] prove that if ψ is a 0-preserving join-homomorphism of D into E (D and E are finite distributive lattices) and $n = \max(|\mathrm{J}(D)|, |\mathrm{J}(E)|)$, then we can construct the finite lattices K and L and the lattice homomorphism $\varphi\colon K \to L$ that represent ψ so that the size of K and L is $O(n^5)$ and the order dimensions of K and L are 3. We conjecture that this result is the best.

The D-relation

In a finite lattice L, every element a is determined by the set of join-irreducible elements contained in a. Therefore, it is quite natural that we can characterize congruences in terms of the join-irreducible elements. This idea was developed in the papers B. Jónsson and J. B. Nation [C30], A. Day [A42], and R. Wille [C38]. On the set $\mathrm{J}(L)$, a binary relation D is defined as follows: for p, $q \in \mathrm{J}(L)$, let $p \mathrel{D} q$ iff there exists an $x \in L$ such that $p \leq q \vee x$ and $p \nleq q_* \vee x$, where q_* denotes the lower cover of q. This relation D defines a closure operator $\mathcal{D}$ as follows: for $A \subseteq \mathrm{J}(L)$, let A be $\mathcal{D}$-closed iff $p \mathrel{D} q$ and $q \in A$ implies that $p \in A$. The following result describes congruence lattices.

Theorem 11 The congruence lattice of a finite lattice L is isomorphic to the lattice of $\mathcal{D}$-closed subsets of $\mathrm{J}(L)$. This isomorphism is given by

$$\Theta \mapsto \{\, p \in \mathrm{J}(L) \mid \Theta(p_*, p) \leq \Theta \,\}.$$ —

This topic is treated in depth in the book R. Freese, J. Ježek, and J. B. Nation [A65]; see also M. Tischendorf [A267].

2. The General Case

In the general case, there are two approaches to try to solve the characterization problem of congruence lattices of lattices. The first is due to the second author and the second was suggested by P. Pudlák [C32].

The first approach: distributive join-homomorphisms

Let $\mathrm{Con_c}\, L$ denote the semilattice of compact congruences of the lattice L. Let us call a semilattice S *representable* iff there is a lattice L with $\mathrm{Con_c}\, L \cong S$.

Let D be a finite distributive lattice, and take the Boolean lattice B generated by D (see Section II.4). Then for every $d \in B$, there exists a smallest element $s(d) \in D$ such that $d \leq s(d)$. The map $s\colon B \to D$ is a closure operator (see Definition IV.3.8(i)). We construct a lattice K, whose congruence lattice is D with a new unit element adjoined. Take $B \times C_2$ and its meet-subsemilattice K, consisting of all elements of the form $\langle b, 0\rangle$, $b \in B$ and $\langle d, 1\rangle$, $d \in D$. Then K is a lattice and B can be identified with an ideal of K under the map $b \mapsto \langle b, 0\rangle$, $b \in B$. Consider a congruence $\langle x, 0\rangle \equiv \langle 0, 0\rangle$ in B. Joining both sides with $\langle 0, 1\rangle$, we get $\langle s(x), 1\rangle = \langle x, 0\rangle \vee \langle 0, 1\rangle \equiv \langle 0, 0\rangle \vee \langle 0, 1\rangle = \langle 0, 1\rangle$. Thus $\langle s(x), 0\rangle \equiv \langle 0, 0\rangle$. It is easy to see now that the congruence relation $\Theta(\langle s(x), 0\rangle, \langle 0, 0\rangle)$ of B can be extended to K and $\Theta(\langle 0, 1\rangle, \langle 0, 0\rangle)$ is the unit congruence. Now we have to identify the new unit element with the unit of $\mathrm{Con}\, K$. The following construction solves this problem (see E. T. Schmidt [1968]).

Let L consist of all triples $\langle x, y, z\rangle \in B^3$ satisfying $x \wedge y = x \wedge z = y \wedge z$ and $x \in D$. Then L is a lattice and $D \cong \mathrm{Con}\, L$.

Note that the elements $\langle 0, 0, z\rangle$, $z \in B$, and $\langle x, 1, x\rangle$, $x \in D$, form a sublattice of L isomorphic to K.

This construction also works for certain infinite distributive lattices, namely, for dually relatively pseudocomplemented lattices; indeed, for every $d \in D$, then there exists a smallest $s(d) \in B$ such that $d \leq s(d)$.

Let B be a generalized Boolean lattice and let $h\colon B \to S$ be an onto join-homomorphism. We will say that h is *weakly distributive* iff for all x, y, $z \in B$ with $x \vee y \leq z$ and $h(x \vee y) = h(z)$, there are x_1, $y_1 \in B$ such that $x_1 \vee y_1 = z$, $h(x) = h(x_1)$, and $h(y) = h(y_1)$. If B is the Boolean lattice generated by D and $s(x)$ exists for every $x \in D$, then the map $h\colon x \mapsto s(x)$ is a weakly distributive join-homomorphism.

We call h *distributive* iff $\mathrm{Ker}\, h = \bigvee(\, \mathrm{Ker}\, s_i \mid i \in I\,)$, where each s_i, $i \in I$, is a closure operator on B.

Theorem 12 Let S be a semilattice with zero. If there is a generalized Boolean lattice B and a distributive join-homomorphism $h\colon B \to S$, then S is representable.

We apply the previous construction for every closure operator s_i to obtain the lattices L_i, $i \in I$. We glue these lattices to each other to construct L with

$\mathrm{Con_c}\, L \cong S$.

By construction, every L_i has an ideal B_i isomorphic to B. We define a lattice M_I whose elements are vectors, $\langle \ldots, x_i, \ldots \rangle$, $x_i \in B$, $i \in I$, satisfying the condition that for three distinct indices i, j, and k, $x_i \wedge x_j = x_i \wedge x_k = x_j \wedge x_k$. For every $i \in I$, the elements $\langle x, \ldots, x, 1, x, \ldots, x \rangle$, where the 1 is i-th entry, form a dual ideal B_i' of M_I, which is isomorphic to B and, consequently, to the ideal B_i of L_i. Now we glue together M_I and L_i by identifying B_i and B_i'. This way, we get a meet semilattice. The lattice L is the lattice of all finitely generated ideals of this semilattice.

Theorem 12 is sufficient to obtain most representation theorems:

Theorem 13 Let S be a semilattice with zero. Each one of the following conditions implies that S is representable:

(i) $\mathrm{Id}\, S$ is completely distributive (equivalently, S is isomorphic to the semilattice of all finitely generated hereditary subsets of some partially ordered set).

(ii) S is a lattice.

(iii) S is locally countable (that is, for every $s \in S$, $(s]$ is countable).

(iv) $|S| \leq \aleph_1$. —

(i) was first obtained by R. P. Dilworth. Proofs can be found in G. Grätzer and E. T. Schmidt [1962], H. Dobbertin [C1], and P. Crawley and R. P. Dilworth [1973].

(ii) was proved in E. T. Schmidt [A247]. P. Pudlák [C32] provides another proof, in a more general, categorical context. Another proof can be found in H. Dobbertin [C2].

(iii) was first obtained in A. P. Huhn [C29] under the condition that $|S| \leq \aleph_0$. The general result for locally countable semilattices was obtained in H. Dobbertin [C1]. Dobbertin proved that if B is a locally countable generalized Boolean semilattice and if S is a distributive semilattice, then every weakly distributive homomorphism from B to S is distributive. Further, he proved that every locally countable distributive semilattice with zero is the weakly distributive image of some locally countable generalized Boolean algebra.

(iv) was obtained in A. P. Huhn [C29]. One of the main tools used by Huhn is the notion of *frame* introduced in H. Dobbertin [C1], which is a special sort of lattice with zero used for building transfinite direct limits of direct systems having up to $\aleph_1$ objects.

Some of these results are presented in an axiomatic form in M. Tischendorf [C37].

The second approach

We have seen that the representation is relatively easy for finite distributive lattices. Let D be an arbitrary distributive semilattice with zero; P. Pudlák [C32] proved that for each finite subset F of D, there is a *finite* subsemilattice of D containing F, therefore, D is a direct limit of all the finite distributive lattices contained in it as distributive join-semilattices with zero.

Let S be a distributive semilattice with zero, and let $\mathcal{S}$ be the set of finite subsets of $S - \{0\}$. P. Pudlák's approach (see [C32]) is the following. Choose an order preserving function that assigns to each $F \in \mathcal{S}$ a finite distributive subsemilattice S_F of S containing F; thus S is the direct limit of the S_F, $F \in \mathcal{S}$. For each $F \in \mathcal{S}$, construct a finite atomistic lattice L_F whose congruence lattice is isomorphic to S_F and such that if $F \subseteq G$, then L_F embeds into L_G with the Congruence Extension Property. Then if $L_{\mathcal{S}}$ is the limit of the L_F, $F \in \mathcal{S}$, then the congruence lattice of $L_{\mathcal{S}}$ is isomorphic to $\operatorname{Id} S$.

Applying this method, P. Pudlák [C32] gave a new proof of Theorem 13(ii).

Negative results

There are a number of related negative results.

H. Dobbertin [C2] constructs a distributive semilattice with zero S and a semilattice homomorphism f of a Boolean algebra B (of size $\aleph_1$) onto S such that f is weakly distributive but not distributive.

F. Wehrung [A273] constructs a bounded distributive semilattice S (of size $\aleph_2$) that is not isomorphic to any weakly distributive image of a generalized Boolean algebra. Note that the $\aleph_2$ size is optimal. This shows that we cannot obtain a positive solution of the congruence lattice characterization problem of lattices by the first approach.

The second approach was also ruled out in F. Wehrung [A274]. The semilattice S of previous paragraph is not isomorphic to $\operatorname{Con_c} L$, for any lattice L which is a direct limit of lattices that are either atomistic or sectionally complemented.

M. Ploščica, J. Tůma, and F. Wehrung [A234] prove that there exists a distributive semilattice S that is representable as $\operatorname{Con_c} L$, for a lattice L, but S cannot be represented using the first or the second approach, that is, S is not a distributive join-homomorphic image of a generalized Boolean lattice nor is S a direct limit of lattices that are either atomistic or sectionally complemented. In fact, one can take $S = \operatorname{Con_c} F$, where F is the free lattice on $\aleph_2$ generators in any nondistributive variety of lattices.

3. Complete Congruences

In Section A.1.8, we briefly mentioned the result (G. Grätzer [A100]):

Theorem 14 Every complete lattice K can be represented as the lattice of complete congruence relations of a complete lattice L.

In a series of papers, much sharper results have been obtained.

G. Grätzer and H. Lakser [A116] had the first published proof of Theorem 14; in fact, it already contained more: L was constructed as a planar lattice.

Let $\mathfrak{m}$ be an infinite regular cardinal, and let K be an $\mathfrak{m}$-complete lattice. Then the lattice $\operatorname{Con}_{\mathfrak{m}} K$ of all $\mathfrak{m}$-complete congruence relations of K is $\mathfrak{m}$-algebraic (this concept is the obvious modification of Definition II.3.12).

G. Grätzer and H. Lakser [C10] proved a partial converse:

Let $\mathfrak{m}$ *be a regular cardinal* $> \aleph_0$*, and let* L *be an* $\mathfrak{m}$-algebraic lattice *with an* $\mathfrak{m}$*-compact unit element. Then* L *is isomorphic to the lattice of* $\mathfrak{m}$*-algebraic congruences of an* $\mathfrak{m}$-algebraic lattice K*.*

A much sharper form of the original result was proved in the paper R. Freese, G. Grätzer, and E. T. Schmidt [C6]:

Every complete lattice L *is isomorphic to the lattice of complete congruence relations of a* complete modular lattice K*.*

The $\mathfrak{m}$-algebraic direction and the modular direction were combined by the present authors in [C19]:

Let $\mathfrak{m}$ *be a regular cardinal* $> \aleph_0$*. Every* $\mathfrak{m}$-algebraic lattice L *is isomorphic to the lattice of* $\mathfrak{m}$*-complete congruence relations of a suitable* $\mathfrak{m}$-complete modular lattice K*.*

G. Grätzer, P. Johnson, and E. T. Schmidt [C8] presents the same construction with a simplified proof.

The sharpest result is the following (G. Grätzer and E. T. Schmidt [C23]):

Theorem 15 Let $\mathfrak{m}$ be a regular cardinal $> \aleph_0$. Every $\mathfrak{m}$*-algebraic lattice* L can be represented as the lattice of $\mathfrak{m}$-complete congruence relations of an $\mathfrak{m}$*-complete distributive lattice* K*.*

In the construction, we use infinite complete-simple complete distributive lattices (a complete lattice is *complete-simple* if it has only the two trivial complete congruences). Such lattices were constructed in G. Grätzer and E. T. Schmidt [C20] and G. Grätzer and E. T. Schmidt [C21]. It can be shown, see G. Grätzer and E. T. Schmidt [C24], that the representation of the three-element chain must contain such a lattice.

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