On conjugacy classes of maximal subgroups of finite simple groups, and a related zeta function

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Abstract

We prove that the number of conjugacy classes of maximal subgroups of bounded order in a finite group of Lie type of bounded rank is bounded. For exceptional groups this solves a longstanding open problem. The proof uses, among other tools, some methods from Geometric Invariant Theory.

Using this result we provide a sharp bound for the total number of conjugacy classes of maximal subgroups of Lie type groups of fixed rank, drawing conclusions regarding the behaviour of the corresponding 'zeta function' $\zeta_G(s) = \sum_{M \max G} |G : M|^{-s}$, which appears in many probabilistic applications. More specifically, we are able to show that for simple groups G and for any fixed real number $s > 1$, $\zeta_G(s) \to 0$ as $|G| \to \infty$. This confirms a conjecture made in [27].

We also apply these results to prove the conjecture made in [29] that the symmetric group S_n has $n^{o(1)}$ conjugacy classes of primitive maximal subgroups.

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1 Introduction

For a finite group G and a real number s, define

$$\zeta_G(s) = \sum_{M \ max \ G} |G : M|^{-s}.$$

This 'zeta function' was introduced and studied in the case where G is simple in [27], following earlier investigation in [13]. Theorem 2.1 of [27] states that for G a classical or alternating simple group and $s > 1$, we have $\zeta_G(s) \to 0$ as $|G| \to \infty$, and it is conjectured there that this conclusion holds for all simple groups. In this paper we complete the proof of this conjecture.

Theorem 1.1 *If G is a finite simple group, and $s > 1$, then*

$$\zeta_G(s) \to 0 \text{ as } |G| \to \infty.$$

Let $m_n(G)$ denote the number of maximal subgroups of index n in G. Then $\zeta_G(s) = \sum_{n>1} m_n(G) n^{-s}$. It therefore follows from our theorem that for any $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that for all $n > N$ and for all finite simple groups G we have

$$m_n(G) < n^{1+\epsilon}.$$

This confirms a conjecture posed in [33].

Theorem 1.1 has several applications, mainly concerning questions of probabilistic generation. For example, since the probability of generating G with k randomly chosen elements is at least $1 - \zeta_G(k)$, Theorem 1.1 provides a rather quick proof of Dixon's conjecture, originally established in [9, 13, 26], that two randomly chosen elements of a simple group G generate G with probability tending to 1 as $|G| \to \infty$. Some further applications of the theorem are discussed in Section 5.

A key new ingredient in the proof of Theorem 1.1 is the following result, which is the main result of this paper, showing that the number of classes of maximal subgroups of bounded order in a group of Lie type of bounded rank is bounded. This is known for classical groups, but is a longstanding open problem for exceptional groups of Lie type. Indeed, there has been much work on finite subgroups of exceptional groups (see for example [10, 24]); while such subgroups are essentially known up to isomorphism, their conjugacy is far from understood, although there has been some recent progress in this direction, such as [31].

Theorem 1.2 *Let N, R be positive integers, and let G be an almost simple group whose socle is a finite simple group of Lie type of rank at most R. Then the number of conjugacy classes of maximal subgroups of order at most N in G is bounded by a function $f(N, R)$ of N and R only.*

Our proof of this result does not use the classification of finite simple groups (CFSG). The proof is given in Section 2, and involves three main tools. The first is a recent result of the second author [34] (see Proposition 2.1) which shows that the number of conjugacy classes of embeddings of a finite group of order N as a 'strongly reductive' subgroup of a simple algebraic group of rank R over an algebraically closed field is bounded by a function of N and R; this generalizes to arbitrary characteristic a result of Weil (see [39]) proving the conclusion when the field has characteristic zero.

The second tool consists of various results and arguments from Geometric Invariant Theory [14], which we use to prove the key result of the proof, Proposition 2.2. This states that a finite subgroup of a simple algebraic group $\bar{G}$ which is invariant under a group S of automorphisms of $\bar{G}$ is either strongly reductive, or lies in a proper S-invariant parabolic subgroup of $\bar{G}$.

Thirdly, we make use of a number of results from the literature on maximal subgroups of finite and algebraic groups of Lie type [8, 21, 23].

Using further recent work on maximal subgroups of exceptional groups from [25], we shall also deduce the following. For a finite group G, denote by $\mathcal{M}(G)$ the set of conjugacy classes of maximal subgroups of G.

Theorem 1.3 *There is a function $c(r)$ and an absolute constant d such that if G is a finite almost simple group with socle of Lie type of rank r over $\mathbb{F}_q$, then*

$$|\mathcal{M}(G)| < c(r) + dr \log\log q.$$

Note that the $\log\log q$ term comes from the subfield subgroups of the form $G(q^{1/s})$, where s is a prime divisor of $\log_p q$, and hence this bound is essentially best possible for groups of fixed rank. For classical groups, Theorem 1.3 is an improvement of [11, Theorem 2.7], which gives $|\mathcal{M}(G)| < c(r) \cdot (\log q)^{\log r}$. For exceptional groups it is new, apart from large characteristics, and also a few families of rank 1 or 2 for which the maximal subgroups are completely known.

Again our proof of Theorem 1.3 does not use CFSG (although it does use the Larsen-Pink theorem [19] at one point as a 'substitute' for the classification). If one does use the classification, our proof shows that the conclusion of Theorem 1.3 holds with the function $c(r)$ of the order of r^r, and it should certainly be possible to improve on this.

As a final comment on our use of the classification, Theorem 1.1 for groups G of Lie type of bounded rank follows immediately from 1.3 (see Section 4), hence is not dependent on CFSG; however, for groups of unbounded rank we cannot improve on the proof given in [27, 2.1], which does use CFSG.

The layout of the paper is as follows. In Sections 2 and 3 we prove Theorems 1.2 and 1.3 respectively, and Section 4 contains the very quick

deduction of Theorem 1.1 from these. In the final Section 5 we discuss the impact of Theorem 1.1 in probabilistic group theory, and prove a conjecture made in [29] regarding maximal subgroups of symmetric groups.

2 Proof of Theorem 1.2

Let p be a prime and let $\bar{G}$ be a simple adjoint algebraic group over $\bar{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$. The proof will be based on two results (Propositions 2.1 and 2.2 below) concerning *strongly reductive* subgroups of $\bar{G}$: following Richardson [36], we define a closed subgroup H of $\bar{G}$ to be strongly reductive in $\bar{G}$ if H is not contained in any proper parabolic subgroup of $C_{\bar{G}}(T)$, where T is a maximal torus of $C_{\bar{G}}(H)$.

Proposition 2.1 *Let N be a positive integer, and let $R = \mathrm{rank}(\bar{G})$. The number of conjugacy classes of strongly reductive subgroups of $\bar{G}$ of order at most N is bounded above by a function $g(N, R)$ of N and R alone.*

Proof By [34, Theorem 1.2], the number of classes of strongly reductive subgroups of order n in $\bar{G}(\bar{\mathbb{F}}_p)$ is finite; call it $g(n, \bar{G}, p)$. For p coprime to n, this number is constant, equal to the corresponding number $g(n, \bar{G}, 0)$ of classes in $\bar{G}(\mathbb{C})$, by a result of Larsen [18, Theorem A.12]. Setting

$$g(N, R) = \sum_{n \leq N,\, \mathrm{rank}(\bar{G}) \leq R} \mathrm{max}_{p|n}(g(n, \bar{G}, p), g(n, \bar{G}, 0)),$$

we have the conclusion. ■

Now define $\mathrm{Aut}^+(\bar{G})$ to be the group generated by inner automorphisms of $\bar{G}$, together with p^i-power field morphisms ($i \geq 1$), and also graph automorphisms when $\bar{G}$ is of type A_r, D_r or E_6.

The next result lies at the heart of the proof of Theorem 1.2, and is really the key result of the paper.

Proposition 2.2 *Let F be a finite subgroup of $\bar{G}$, and suppose F is invariant under a subgroup S of $\mathrm{Aut}^+(\bar{G})$. Then one of the following holds:*

(i) *F is strongly reductive in $\bar{G}$;*

(ii) *F is contained in a proper S-invariant parabolic subgroup of $\bar{G}$.*

Notice that it is immediate from the definition that if F is not strongly reductive then it lies in a proper parabolic subgroup; it is the S-invariance of this parabolic that is the point of the proposition.

In the proof of Proposition 2.2 we shall use the theory of optimal destabilising one-parameter subgroups and their associated parabolic subgroups, developed by Kempf [14].

We recall the parts of Kempf's theory that we need. A *length function* on the space of one-parameter subgroups $Y(\bar{G})$ of $\bar{G}$ is a conjugation-invariant function $\|\cdot\|$ from $Y(\bar{G})$ to the non-negative real numbers with the following property: for every maximal torus T of $\bar{G}$, there exists a positive definite W-invariant $\mathbb{Z}$-valued bilinear form $\langle\cdot,\cdot\rangle$ on $Y(T)$ such that $\|\lambda\| = \sqrt{\langle\lambda,\lambda\rangle}$ for all $\lambda \in Y(T)$, where W denotes the Weyl group $N_{\bar{G}}(T)/T$. Our first task is to construct a length function that is invariant under $\mathrm{Aut}^+(\bar{G})$ in an appropriate sense.

Fix a maximal torus T of $\bar{G}$. A description of the automorphisms of $\bar{G}$ can be found in [42, Section 10]. From this it follows that we may choose a p-power field morphism F_p of $\bar{G}$ which acts on T as $t \to t^p$, and a group Γ of graph automorphisms of $\bar{G}$ which fixes T and commutes with F_p, where $\Gamma = C_2$ if $\bar{G} = A_r, E_6$ or D_r $(r \neq 4)$, $\Gamma = S_3$ if $\bar{G} = D_4$ and $\Gamma = 1$ otherwise. Write Δ for the cyclic group generated by F_p, and let $\phi_p\colon \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$ be the operation of raising to the pth power. Then $\mathrm{Aut}^+(\bar{G})$ is generated by inner automorphisms of $\bar{G}$, together with Γ and Δ.

We define an action of $\mathrm{Aut}^+(\bar{G})$ on $Y(\bar{G})$ as follows. For $\gamma \in \Gamma$, $g \in \bar{G}$, $m \in \mathbb{Z}$, $\lambda \in Y(\bar{G})$ and $x \in \bar{\mathbb{F}}_p$, we set

$$\begin{aligned}(\gamma.\lambda)(x) &= \gamma(\lambda(x)),\\ (g.\lambda)(x) &= g\lambda(x)g^{-1},\\ (F_p^m.\lambda)(x) &= F_p^m(\lambda(\phi_p^{-m}(x)))\end{aligned}$$

(compare [14, Section 4]). It is readily checked that this does indeed define an action of $\mathrm{Aut}^+(\bar{G})$.

Let C be the finite group $N_{\bar{G}\Gamma}(T)/T \cong W\Gamma$. Pick a positive definite $\mathbb{Z}$-valued bilinear form $\langle\cdot,\cdot\rangle$ on $Y(T)$. By summing over C, we obtain a C-invariant positive definite $\mathbb{Z}$-valued bilinear form $\langle\cdot,\cdot\rangle_1$ on $Y(T)$. Since $\langle\cdot,\cdot\rangle_1$ is W-invariant, we can now define a length function as follows (see [14, Lemma 2.1]): for $\lambda \in Y(\bar{G})$, set $\|\lambda\|_1 = \sqrt{\langle g.\lambda, g.\lambda\rangle_1}$, for any $g \in \bar{G}$ such that $g.\lambda \in Y(T)$.

We claim that the length function $\|\cdot\|_1$ is invariant under the action of $\mathrm{Aut}^+(\bar{G})$. Invariance under the $\bar{G}$-action is part of the definition of length function, and invariance under the action of Γ follows from the construction. Since Δ acts trivially on $Y(T)$, invariance under the Δ-action also follows easily, proving the claim.

Now suppose that $\bar{G}$ acts on an affine variety V. Given $v \in V$ and $\lambda \in Y(\bar{G})$, we say that $\lim_{x\to 0} \lambda(x).v$ exists and equals w if there exists a morphism $M_\lambda\colon \bar{\mathbb{F}}_p \to V$ (necessarily unique) such that for all $x \neq 0$, $M_\lambda(x) = \lambda(x).v$ and $M_\lambda(0) = w$. In the important special case that $V = \bar{G}$ and $\bar{G}$ acts by conjugation, the subset $P_\lambda := \{g \in \bar{G} \mid \lim_{x\to 0} \lambda(x).g \text{ exists}\}$ is a parabolic subgroup of $\bar{G}$, and all parabolic subgroups arise in this way. If $g \in P_\lambda$ then $\lim_{x\to 0} \lambda(x).g$ belongs to the subgroup $L_\lambda := C_{\bar{G}}(\lambda(k^*))$ of P_λ; L_λ is a Levi subgroup of P_λ. The unipotent radical $R_u(P_\lambda)$ equals the set

$\{g \in G \mid \lim_{x\to 0} \lambda(x).g = 1\}$. The Hilbert-Mumford Theorem states that if w belongs to the closure of the orbit $\bar{G}.v$ then $g.w = \lim_{x\to 0} \lambda(x).v$ for some λ and some $g \in \bar{G}$. Kempf's main theorem (see Theorem 2.3 below) states that λ can be chosen in a more or less canonical way.

For any closed $\bar{G}$-stable subset D of V, we denote by $|V, v|_D$ the set of indivisible one-parameter subgroups λ such that $\lim_{x\to 0} \lambda(x).v$ exists and belongs to D. If $v \notin D$ then, given $\lambda \in |V, v|_D$, $M_\lambda^{-1}(D)$ is a divisor supported inside the set $x = 0$; we define $\alpha_{D,v}(\lambda)$ to be the degree of this divisor.

Theorem 2.3 *(*[14, Theorem 3.4]*) Fix a length function $\|\cdot\|$ on $Y(\bar{G})$. Let v, V, D be as above, and assume that $v \notin D$ and that $|V, v|_D$ is nonempty. Then the function $\lambda \mapsto \alpha_{D,v}(\lambda)/\|\lambda\|$ defined on $|V, v|_D - \{0\}$ attains a maximum value, and for any two elements $\lambda, \mu \in |V, v|_D - \{0\}$ such that this maximum value is attained, we have $P_\lambda = P_\mu$.*

We will write $P_{D,v}$ for the parabolic subgroup of the theorem. Note that if $\lambda \in |V, v|_D$ then λ cannot be central, and it follows that $P_{D,v}$ is proper (see [40, 8.4.5]).

Proof of Proposition 2.2

Fix $N \in \mathbb{N}$. We consider the special case of the above theory where $V = \bar{G}^N$ and $\bar{G}$ acts by simultaneous conjugation, and work with the length function $\|\cdot\|_1$ constructed above. The symmetric group S_N acts on $\bar{G}^N$ in the obvious way, and this action commutes with the $\bar{G}$-action. The connection with strongly reductive subgroups of $\bar{G}$ is given by the following result of Richardson [36, 16.4]: for $\mathbf{g} = (g_1, \ldots, g_N) \in \bar{G}^N$, the closed subgroup generated by $g_1, \ldots, g_N$ is a strongly reductive subgroup of $\bar{G}$ if and only if the orbit $\bar{G}.\mathbf{g}$ is closed.

Let $D'(\mathbf{g})$ denote the unique closed orbit in the closure of $\bar{G}.\mathbf{g}$. Set $D(\mathbf{g}) = \bigcup_{\pi\in S_N} \pi.D'(\mathbf{g})$. Suppose that $\bar{G}.\mathbf{g}$ is not closed. Then $\mathbf{g} \notin D'(\mathbf{g})$ but $D'(\mathbf{g})$ meets the closure of $\bar{G}.\mathbf{g}$. Clearly $D(\mathbf{g})$ is a finite union of closed $\bar{G}$-orbits, so $\mathbf{g} \notin D(\mathbf{g})$. Thus $\mathbf{g}$ and $D(\mathbf{g})$ satisfy the hypotheses of Theorem 2.3, so there exists $\lambda \in Y(\bar{G})$ such that $\lim_{x\to 0} \lambda(x).\mathbf{g}$ exists and $P_\lambda = P_{D(\mathbf{g}),\mathbf{g}}$. Since $\lim_{x\to 0} \lambda(x).\mathbf{g}$ exists, $\lim_{x\to 0} \lambda(x).g_i$ exists for each i, whence $g_i \in P_\lambda$ for each i. This implies that the closed group generated by $g_1, \ldots, g_N$ is contained in the proper parabolic subgroup $P_{D(\mathbf{g}),\mathbf{g}}$. Since $D(\mathbf{g})$ is S_N-invariant, equation (4) of [35, p.672] gives

$$P_{D(\mathbf{g}),\mathbf{g}} = P_{D(\mathbf{g}),\pi.\mathbf{g}} \tag{1}$$

for any $\pi \in S_N$.

Define an action of $\mathrm{Aut}^+(\bar{G})$ on $\bar{G}^N$ by $\beta.(g_1, \ldots, g_N) = (\beta(g_1), \ldots, \beta(g_N))$ for $\beta \in \mathrm{Aut}^+(\bar{G})$. This action permutes $\bar{G}$-orbits in $\bar{G}^N$. Since $\mathrm{Aut}^+(\bar{G})$

acts on $\bar{G}^N$ by homeomorphisms and commutes with the S_N-action, we have $\beta.D(\mathbf{g}) = D(\beta.\mathbf{g})$ for every $\beta \in \mathrm{Aut}^+(\bar{G})$ and every $\mathbf{g} \in \bar{G}^N$.

Now let F be a finite subgroup of $\bar{G}$, invariant under a subgroup S of $\mathrm{Aut}^+(\bar{G})$ as in the hypothesis of the proposition. Assume that F is not strongly reductive. Label the elements of F as $f_1, \ldots, f_N$, and set $\mathbf{f} = (f_1, \ldots, f_N)$. We have $F \leq P_{D(\mathbf{f}),\mathbf{f}}$. We shall show that the parabolic $P_{D(\mathbf{f}),\mathbf{f}}$ is S-invariant, proving the proposition.

If $\beta \in S$ then β permutes $f_1, \ldots, f_N$, so $\beta.\mathbf{f} = \pi.\mathbf{f}$ for some $\pi \in S_N$, whence

$$P_{D(\beta.\mathbf{f}),\beta.\mathbf{f}} = P_{D(\pi.\mathbf{f}),\pi.\mathbf{f}} = P_{D(\mathbf{f}),\mathbf{f}}, \tag{2}$$

where the last equality follows from (1).

We next claim that for every $\beta \in \mathrm{Aut}^+(\bar{G})$ and every $\mathbf{g} \in \bar{G}^N$ such that $\bar{G}.\mathbf{g}$ is not closed, we have

$$\beta.P_{D(\mathbf{g}),\mathbf{g}} = P_{\beta.D(\mathbf{g}),\beta.\mathbf{g}}. \tag{3}$$

For β inner, this follows from [14, Corollary 3.5(a)] and for $\beta \in \Delta$ it follows from the rationality argument of [14, Lemma 4.1] (note that $\bar{G}^N$ and the $\bar{G}$-action are defined over the field $\mathbb{F}_p$). Now suppose that $\beta \in \Gamma$. Since β is an automorphism of algebraic groups, we have $\beta.|\bar{G}^N, \mathbf{g}|_{D(\mathbf{g})} = |\bar{G}^N, \beta.\mathbf{g}|_{\beta.D(\mathbf{g})}$ and moreover, for any $\lambda \in |\bar{G}^N, \mathbf{g}|_{D(\mathbf{g})}$, $M_{\beta.\lambda} = \beta \circ M_\lambda$ and $\alpha_{\beta.D(\mathbf{g}),\beta.\mathbf{g}}(\beta.\lambda) = \alpha_{D(\mathbf{g}),\mathbf{g}}(\lambda)$. The claim (3) now follows from the Γ-invariance of $\|\cdot\|_1$ and the uniqueness of $P_{\beta.D(\mathbf{g}),\beta.\mathbf{g}}$.

By (3) and (2), we have $\beta.P_{D(\mathbf{f}),\mathbf{f}} = P_{D(\mathbf{f}),\mathbf{f}}$ for all $\beta \in S$. Hence the parabolic subgroup $P_{D(\mathbf{f}),\mathbf{f}}$ is S-invariant and contains F. This completes the proof of the proposition. ∎

Note that in case (ii) of Proposition 2.2, F is not contained in any Levi subgroup of $P = P_{D(\mathbf{f}),\mathbf{f}}$. For we can write $P_{D(\mathbf{f}),\mathbf{f}} = P_\lambda$, where the $\bar{G}$-orbit of $\lim_{x\to 0} \lambda(x).\mathbf{f}$ is closed. Assume that F is contained in a Levi subgroup of P_λ. Then $u.\mathbf{f} \in L_\lambda^N$ for some $u \in R_u(P_\lambda)$. If $g \in P_\lambda$ with ugu^{-1} in L_λ then $\lambda(k^*)$ centralises ugu^{-1}, so

$$\begin{aligned} ugu^{-1} &= \lim_{x\to 0} \lambda(x)ugu^{-1}\lambda(x)^{-1} \\ &= \lim_{x\to 0} \lambda(x)u\lambda(x)^{-1}\lambda(x)g\lambda(x)^{-1}\lambda(x)u^{-1}\lambda(x)^{-1} \\ &= (\lim_{x\to 0} \lambda(x)u\lambda(x)^{-1})(\lim_{x\to 0} \lambda(x)g\lambda(x)^{-1})(\lim_{x\to 0} \lambda(x)u^{-1}\lambda(x)^{-1}) \\ &= \lim_{x\to 0} \lambda(x)g\lambda(x)^{-1}. \end{aligned}$$

It follows that $\lim_{x\to 0} \lambda(x).\mathbf{f} = u.\mathbf{f}$, which is impossible because $\bar{G}.\mathbf{f}$ is not closed.

Remark 2.4 Proposition 2.2 can be extended to include the case where $\bar{G} = B_2\,(p = 2)$, $F_4\,(p = 2)$ or $G_2\,(p = 3)$ and $\mathrm{Aut}^+(\bar{G})$ is replaced by

$\langle \mathrm{Aut}^+(\bar{G}), \phi \rangle$, where ϕ is a graph morphism of $\bar{G}$ (i.e. a morphism sending $x_r(t) \to x_{\rho(r)}(t^{\lambda(r)})$, where ρ is an involutory symmetry of the root system, and $\lambda(r)$ is 1 if r is a long root, and is p is r is short - see [6, 12.3, 12.4] for example). To see this we argue as follows. Let F be a finite subgroup of $\bar{G}$, invariant under a subgroup S of $\langle \mathrm{Aut}^+(\bar{G}), \phi \rangle$. Note that ϕ^2 is a p-power field morphism of $\bar{G}$, and ϕ normalizes $\mathrm{Aut}^+(\bar{G})$. Write $S_0 = S \cap \mathrm{Aut}^+(\bar{G})$. Then $S = \langle S_0, \sigma \rangle$, where $\sigma^2 \in S_0$. Assume F is not strongly reductive in $\bar{G}$. Then by Proposition 2.2, F lies in a proper S_0-invariant parabolic subgroup P of $\bar{G}$. The intersection $P \cap P^\sigma$ contains F and is S-invariant. If $R_u(P \cap P^\sigma) \neq 1$ then [5] implies that F lies in an S-stable parabolic, as required. So assume $R_u(P \cap P^\sigma) = 1$. Then [7, 2.8.7] implies that $P \cap P^\sigma = L$, a Levi subgroup of $\bar{G}$. However this conflicts with the assertion noted just before this remark.

We shall also need the following technical result concerning maximal subgroups of finite groups of Lie type.

Let σ be a Frobenius morphism of $\bar{G}$ such that $G_0 = (\bar{G}_\sigma)'$ is a finite simple group of Lie type, and let G be an almost simple group with socle G_0. Note that by [42], every automorphism of G_0 extends to an endomorphism of $\bar{G}$, so we may regard $\mathrm{Aut}(G_0)$ as a subgroup of $\mathrm{Aut}^+(\bar{G})$ (of $\langle \mathrm{Aut}^+(\bar{G}), \phi \rangle$ in the exceptional cases considered in Remark 2.4).

Proposition 2.5 *If M is a maximal subgroup of G as above, then $M \cap G_0 \neq 1$, and one of the following holds:*

(i) *M normalizes a proper nontrivial connected σ-stable subgroup $\bar{M}$ of $\bar{G}$;*

(ii) $C_{\bar{G}}(M \cap G_0) = 1$.

Proof The fact the $M \cap G_0 \neq 1$ is elementary and well known, and appears for example in [2]. For completeness we give a brief proof. Suppose $M \cap G_0 = 1$. Then $M \cong G_0 M/G_0 \leq \mathrm{Out}(G_0)$, which is solvable. Let Q be a minimal normal subgroup of M, so Q is an elementary abelian r-group for some prime r. As M normalizes $C_{G_0}(Q)$ and is maximal in G, we must have $C_{G_0}(Q) = 1$, and hence r does not divide $|G_0|$. It follows that Q normalizes a unique Sylow 2-subgroup S of G_0. Then $M = N_G(Q) \leq N_G(S)$, so $M < MS < G$, contradicting the maximality of M.

Thus $M \cap G_0 \neq 1$, the first assertion of the proposition. Write $M_0 = M \cap G_0$.

Now assume that conclusion (i) does not hold - that is, that M normalizes no proper nontrivial connected σ-stable subgroup of $\bar{G}$. Suppose for a contradiction that $C_{\bar{G}}(M_0) = C \neq 1$. Then by the above assumption we have $C^0 = 1$ - that is, $C_{\bar{G}}(M_0)$ is finite. For the same reason, so is $C_{\bar{G}}(C)$.

Consider first the case where $\bar{G}$ and also G_0 are of classical type. Let V be the natural module for $\bar{G}$ (in other words, $\bar{G} = PSL(V), PSp(V)$ or

$PSO(V)$). If either $\bar{G} = PSL(V)$, or $G < P\Gamma L(V)$, then the conclusion follows from [23, Theorem 1′]. So assume that neither of these holds, in which case we have $G_0 = D_4(q)$ or $B_2(q)$ (q even), and G contains a triality or graph automorphism of G_0.

We claim that M_0 must be a 2-group. For if not, pick an element $x \in M_0$ of odd prime order, say s. If x is semisimple, then $C_{\bar{G}}(x)$ is connected and has a nontrivial central torus (see [41, II,4.4]), so $Z(C_{\bar{G}}(x))$ is infinite. However this group lies in $C_{\bar{G}}(C)$, so this is a contradiction. Hence $s = p$ and x is unipotent. As $\bar{G}$ is classical and p is odd, p is a good prime for $\bar{G}$. Now the argument given in the third paragraph of the proof of [21, 1.2] shows that again $Z(C_{\bar{G}}(x))$ is infinite, a contradiction.

Thus M_0 is a 2-group, as claimed. Using [5] we see that $p \neq 2$, so we are in the case where $G_0 = D_4(q)$. The maximality of M implies that M_0 is self-normalizing in G_0, and hence must be a Sylow 2-subgroup of G_0. However an elementary argument given in [15, 4.1.1(ii)] shows that in this case $N_G(M_0)$ cannot be maximal in G, which is a contradiction. This completes the proof for G_0 classical.

Now suppose G_0 is of exceptional Lie type. If M is not almost simple, then the conclusion holds by [21, Theorem 2], so assume M is almost simple. Clearly $F^*(M) \leq M_0$.

Since $C_{\bar{G}}(C_{\bar{G}}(F^*(M))) \leq C_{\bar{G}}(C)$, and these groups are finite by assumption, [21, 1.3] implies that

$$\bar{G} = E_8,\ p > 3,\ F^*(M) = Alt_5 \text{ or } Alt_6.$$

As M is almost simple, we have $C_{G_0}(M_0) = 1$. Hence if $C_\sigma \neq 1$ then C_σ is isomorphic to a subgroup of $\bar{G}_\sigma/G_0$, hence has order 2 or 3. But then $C_{\bar{G}}(C_\sigma)$ has positive dimension and is normalized by M and σ, a contradiction. Consequently $C_\sigma = 1$.

Suppose that the Fitting subgroup $F(C) = 1$. Then $F^*(C) = \prod_{i=1}^{k} S_i$, a direct product of non-abelian simple groups, centralized by M_0, and as usual, $C_{\bar{G}}(C_{\bar{G}}(F^*(C)))$ is finite. Then [21, 1.2] implies that $k = 1$ and $F^*(C) = Alt_5$ or Alt_6. However neither of these possesses a fixed point free automorphism, so this contradicts the fact that $C_\sigma = 1$.

We have now established that $F(C) \neq 1$. Pick a prime r such that $O_r(C) \neq 1$. If $r = p$ then M normalizes a σ-stable parabolic subgroup of $\bar{G}$ by [5], so $r \neq p$. Set

$$E = \Omega_1(Z(O_r(C))),$$

an elementary abelian r-group normalized by M and by σ. As M and σ normalize $N_{\bar{G}}(E)$, the latter subgroup is also finite.

Pick elements $t, u \in M_0$ of order 2,3 respectively. Then $C_{\bar{G}}(t) = D_8$ and $C_{\bar{G}}(u) = A_8$ or A_2E_6, by [21, 1.3]. If $r > 2$ then the elementary abelian r-subgroup E of D_8 lies in a maximal torus of D_8 (by [41, II,5.8]), contradicting

the fact that $N_{\bar{G}}(E)$ is finite. Hence $r = 2$. Further, if $C_{\bar{G}}(u) = A_8$ then E lies in a maximal torus of A_8, a contradiction, so $C_{\bar{G}}(u) = A_2E_6$.

At this point we have the elementary abelian 2-group E lying in A_2E_6. Write E_0 for the projection of E to the E_6 factor. For $1 \neq e \in E_0$ we have $C_{E_6}(e) = T_1D_5$ or A_1A_5. In the first case T_1 lies in $Z(C_{\bar{G}}(e))$, hence in $C_{\bar{G}}(E)$, which is finite, a contradiction. Hence $C_{E_6}(e) = A_1A_5$. If f is a non-central involution in the A_5 factor of $C_{E_6}(e)$, then $C_{A_5}(f) = A_1A_3T_1$, and either e or ef lies in $Z(A_3)$. However, the E_6-centralizer of such an involution is T_1D_5: for the restriction of the 27-dimensional module $V_{27} = V_{E_6}(\lambda_1)$ to A_3 has composition factors $100^2, 001^2, 010, 000^5$ (see [22, Table 8.7]), hence the involution in $Z(A_3)$ acts on V_{27} as $(-1^{16}, 1^{11})$. Thus E_0 cannot contain f, and we deduce that E_0 has rank at most 3. Since any two commuting involutions in E_6 lie in a maximal torus by [41, II,5.1], it follows that E_0 has rank 3. Write $E_0 = \langle e, a, b\rangle$. Then $\langle a, b\rangle$ projects to a quaternion subgroup of the A_5 factor of $C_{E_6}(e)$, acting homogeneously on the natural 6-dimensional module, and hence $C_{A_5}(a, b)$ contains a subgroup A_2. Of course this lies in $C_{\bar{G}}(E)$, contradicting the finiteness of this group. This final contradiction completes the proof. ■

Proof of Theorem 1.2

At this point we can complete the proof of Theorem 1.2. Fix positive integers N, R, fix a type of $\bar{G}$ of rank at most R, and let $G_0 = (\bar{G}_\sigma)' = G(q)$ as above. Note that $|\bar{G}_\sigma : G_0| \leq R + 1$. Choose q sufficiently large so that $|\bar{M}_\sigma| > N(R + 1)$ (hence also $|\bar{M}_\sigma \cap G_0| > N$) for any nontrivial connected σ-stable subgroup $\bar{M}$ of $\bar{G}$.

Define $\mathcal{N}$ to be the set of subgroups M_0 of G_0 satisfying the following:

(i) there is an almost simple group G with $\mathrm{soc}(G) = G_0$, and a maximal subgroup M of G, such that $M_0 = M \cap G_0$, and

(ii) $|M_0| \leq N$.

Let $M_0 \in \mathcal{N}$, and let G, M be as in (i), with $M_0 = M \cap G_0$. By Proposition 2.5 we have $M_0 \neq 1$, and hence $M = N_G(M_0)$ by the maximality of M. Thus, given G with socle G_0, the number of G-classes of maximal subgroups of order at most N is bounded above by the number of G-classes of subgroups in $\mathcal{N}$.

If M normalizes a proper nontrivial connected σ-stable subgroup $\bar{M}$ of $\bar{G}$, then by maximality, M contains $\bar{M}_\sigma \cap G_0$, which has order greater than N by our choice of q above. As $|M_0| \leq N$ this cannot be the case. Hence $C_{\bar{G}}(M_0) = 1$ by Proposition 2.5. Moreover, M_0 does not lie in an $\langle M, \sigma\rangle$-invariant proper parabolic subgroup of $\bar{G}$. Hence M_0 is strongly reductive in $\bar{G}$ by Proposition 2.2 and Remark 2.4.

Given $M_0 \in \mathcal{N}$, the set

$$\{M_0^g \,:\, g \in \bar{G},\, M_0^g \leq \bar{G}_\sigma\}$$

falls into at most $N_{\bar{G}}(M_0)/N_{\bar{G}}(M_0)^0$ classes under the action of $\bar{G}_\sigma$, by Lang's theorem [41, I,2.7]. Since $C_{\bar{G}}(M_0) = 1$, the group $N_{\bar{G}}(M_0)$ is finite, of order bounded by $|\mathrm{Aut}(M_0)|$, hence by a function $f(N)$ of N alone.

It now follows using Proposition 2.1 that the total number of $\bar{G}_\sigma$-classes of subgroups in $\mathcal{N}$ is bounded above by $g(N,R) \cdot f(N)$. Since $|\bar{G}_\sigma : G_0| \leq R+1$, it follows that the number of G_0-classes in $\mathcal{N}$ is bounded by $(R+1) \cdot g(N,R) \cdot f(N)$, and hence for any G with socle G_0, the number of G-classes of subgroups in $\mathcal{N}$ is also bounded by this number. This completes the proof of Theorem 1.2.

3 Proof of Theorem 1.3

Let G be a finite almost simple group with socle G_0 of Lie type of rank r over $\mathbb{F}_q$, and denote by $\mathcal{M}(G)$ the set of conjugacy classes of maximal subgroups of G. Observe that the maximal subgroups containing G_0 correspond to maximal subgroups of G/G_0, and it is easily seen that the outer automorphism group $\mathrm{Out}(G_0)$ has at most $dr \log\log q$ subgroups. Hence we consider from now on only subgroups in $\mathcal{M}(G)$ not containing G_0.

Lemma 3.1 *Theorem* 1.3 *holds when G_0 is of exceptional Lie type.*

Proof In this case we use the following result, taken from [25, Corollary 4]: there are absolute constants c, d such that if M is a maximal subgroup of G, then one of the following holds:

(i) M is a known subgroup, belonging to one of at most $d \log\log q$ conjugacy classes,

(ii) M is almost simple, and $|M| < c$.

(Note that the proof of this result uses the classification of finite simple groups only for the statement that a simple subgroup of $GL_n(\overline{\mathbb{F}}_p)$ either lies in Lie(p), or has order bounded by a function of n. This is proved in [19] without using the classification.)

The conclusion of Theorem 1.3 follows from the above result, together with Theorem 1.2. ■

Lemma 3.2 *Theorem* 1.3 *holds when G_0 is of classical type.*

Proof Suppose G_0 is a classical simple group with natural module V of dimension n over $\mathbb{F}_q$, where $q = p^c$ and p is prime. In this case Theorem 1.3

is an improvement of [11, Theorem 2.7], and we use some of the methods of the proof of that result.

First observe that if $G_0 = P\Omega_8^+(q)$ and G contains a triality automorphism of G_0, then the conclusion follows from [15], where the maximal subgroups of G are completely determined (again, for Theorem 1.3 the use of CFSG in this paper can be substituted by [19]). Exclude this case from consideration from now on. Then a theorem of Aschbacher [1] classifies all maximal subgroups of G into eight families $\mathcal{C}_i$ ($1 \leq i \leq 8$) of well understood subgroups, together with a family $\mathcal{S}$ consisting of almost simple subgroups M whose socle has (projective) representation on V which is absolutely irreducible and is not realised over a proper subfield of $\mathbb{F}_q$.

Denote by $n_{\mathcal{C}}$ the number of G_0-classes of subgroups in the union of the families $\mathcal{C}_i$. Then [11, Lemma 2.1] yields

$$n_{\mathcal{C}} \leq c_1(n) + dn \log\log q, \tag{4}$$

where $c_1(n)$ is a function of n and d an absolute constant.

For $\mathcal{S}$, define $n_{\mathcal{S},p}$ (respectively, $n_{\mathcal{S},p'}$) to be the number of G_0-classes of subgroups in $\mathcal{S}$ whose socle is (respectively, is not) a group of Lie type in characteristic p. Then [11, Lemma 2.3] gives

$$n_{\mathcal{S},p'} \leq c_2(n), \tag{5}$$

where $c_2(n)$ is a function of n. (Once again, a classification-free proof of this is given in [19].)

It remains to bound $n_{\mathcal{S},p}$. Here we need to improve [11, Lemma 2.5]. For convenience of notation, replace G_0 by the corresponding classical group on V (i.e. $SL(V), Sp(V)$, etc.). Let $M(s)$ ($s = p^a$) be a quasisimple group of Lie type over $\mathbb{F}_s$ in characteristic p. By [17, 2.10.4(iii)], the conjugacy class of an absolutely irreducible subgroup in the full isometry group of V is determined by its representation on V up to equivalence. Hence it suffices to bound the number of pairs $(M(s), \rho)$, where $\rho : M(s) \to GL(V)$ is absolutely irreducible and realised over no proper subfield of $\mathbb{F}_q$ (recall that $V = V_n(q)$), and $N_G(M(s)\rho)$ is maximal.

Suppose $(M(s), \rho)$ is such a pair. We apply results from [37, 38]. First, [37, Table 1B] provides a list of subgroups of classical groups, of the form $Cl_y(q^r) < Cl_{y^r}(q)$, embedded via a twisted tensor product representation of the form $W \otimes W^{(q)} \otimes \cdots \otimes W^{(q^{r-1})}$, where $W = V_y(q^r)$. Then [38, Corollary 6] and its proof imply that either $N_G(M(s)\rho)$ is the normalizer of one of these subgroups, or $\mathbb{F}_s$ is a subfield of $\mathbb{F}_q$ of index at most 3, and moreover, the representation $\rho \otimes \bar{\mathbb{F}}_p$ is tensor indecomposable.

Since the representations of the above subgroups $Cl_y(q^r)$ are determined, they contribute at most $c_3(n)$ classes of maximal subgroups. For other maximal subgroups $N_G(M(s)\rho)$, the rank of $M(s)$ is bounded by that of G_0,

hence is at most $n-1$, and $s \in \{q, q^{1/2}, q^{1/3}\}$, so there are $c_4(n)$ possibilities for $M(s)$. Moreover, the representation of $M(s)$ on V is tensor indecomposable, hence restricted. So its high weight is a sum $\sum c_i\lambda_i$, where λ_i are the fundamental dominant weights and the $0 \leq c_i < p$. Since $\dim V = n$, restriction to subgroups $SL_2(s)$ of $M(s)$ shows that $c_i < n$ for all i, so there are at most n^n possibilities for the high weight $\sum c_i\lambda_i$. It follows that

$$n_{\mathcal{S},p} < c_5(n). \tag{6}$$

The conclusion of the lemma now follows from (4), (5) and (6). ■

This completes the proof of Theorem 1.3.

4 Deduction of Theorem 1.1

The theorem was proved in [27, 2.1] for G alternating or classical, so it remains to prove it for exceptional groups $G = G(q)$. For these groups the rank r is of course bounded, and maximal subgroups have index at least q (this holds for $SL_2(q)$ and ${}^2B_2(q)$, one of which is contained in G). Hence by Theorem 1.3, for $s > 1$ we have

$$\zeta_G(s) = \sum_{M \in \mathcal{M}(G)} |G : M|^{-(s-1)} < (c(r) + dr \log\log q)) \cdot q^{-(s-1)},$$

and hence $\zeta_G(s) \to 0$ as $q \to \infty$. This completes the proof of Theorem 1.1.

5 Applications

The results of this paper, apart from their intrinsic interest, have an impact on many questions concerning probabilistic generation of finite simple groups. Proofs in this field are often harder for exceptional groups of Lie type, and various ad hoc methods have had to be invented in order to compensate for the lack of complete knowledge of their maximal subgroups; see for instance the proofs in [26], [28], [12]. Using Theorem 1.1 one can greatly simplify many of these proofs, and make various arguments used for classical groups of bounded rank applicable to exceptional groups as well. We demonstrate this in Corollary 5.1 below.

Moreover, the theorems in this paper also give rise to new results; in particular they enable us to settle a conjecture concerning symmetric groups (see Theorem 5.2 below).

It was conjectured by Kantor and Lubotzky [13] that a randomly chosen involution and a randomly chosen additional element of a finite simple group G generate G with probability tending to 1 as $|G| \to \infty$. This was proved in

[27] for classical (and alternating) groups, and in [28] for exceptional groups of Lie type. Focusing on exceptional groups and using Theorem 1.1, we can now provide a very short proof of a more general result.

Corollary 5.1 *Let k be a positive integer, and let G be an exceptional simple group of Lie type which has an element of order k. Let $P_{k,*}(G)$ be the probability that a randomly chosen element of order k and a randomly chosen additional element generate G. Then $P_{k,*}(G) \to 1$ as $|G| \to \infty$.*

Proof Let $i_k(H)$ denote the number of elements of order k in a finite group H. Then we easily see that

$$1 - P_{k,*}(G) \leq \sum_{M \max G} \frac{i_k(M)|M|}{i_k(G)|G|}.$$

Let $G = G(q)$ and let C be a non-trivial conjugacy class of G. Then by [20] we have $|M \cap C|/|C| \leq c/q$ for all maximal subgroups M of G, where c is some absolute constant. Summing over conjugacy classes of elements of order k in G this implies

$$i_k(M)/i_k(G) \leq c/q \leq c|G : M|^{-\epsilon}$$

for some fixed $\epsilon > 0$ ($\epsilon = 1/248$ will easily do).

Combining the above inequalities with Theorem 1.1 we conclude that

$$1 - P_{k,*}(G) \leq c\zeta_G(1 + \epsilon) \to 0 \text{ as } |G| \to \infty.$$

■

We note that the above Corollary can in fact be deduced from [12, Theorem 2] (and can be further generalized); however the proof we have given seems to us rather more natural and conceptual.

Theorem 1.1 is also a key tool in some new results on probabilistic generation which will appear in forthcoming work of the first and third authors.

Finally, let us mention that, although this paper deals with groups of Lie type, it gives rise to new results concerning symmetric groups. In 1989 Babai showed that S_n has at most $n^{c \log^3 n}$ conjugacy classes of primitive maximal subgroups [3, 2.5]. This was improved in [29], where it was shown that S_n has at most $n^{6/11+o(1)}$ conjugacy classes of primitive maximal subgroups, where $o(1)$ is a quantity tending to 0 as $n \to \infty$. Here we improve it further, confirming Conjecture 1 of [29].

Theorem 5.2 *The symmetric group S_n has $n^{o(1)}$ conjugacy classes of primitive maximal subgroups.*

Proof The argument is similar to the proof of Theorem 4.4 in [29], except that in parts (ii) and (iii) of Proposition 2.3 we apply Theorems 1.2 and 1.3 to obtain upper bounds of the form $n^{1+o(1)}$. This leads to the required improvement. ∎

Let $d(n)$ denote the number of divisors of n. It is well known that $d(n) = n^{o(1)}$. Note that S_n has $d(n) - 2$ conjugacy classes of transitive imprimitive maximal subgroups (of the form $S_k \wr S_{n/k}$), and $[n/2]$ conjugacy classes of intransitive maximal subgroups (of the form $S_k \times S_{n-k}$). Therefore Theorem 5.2 gives rise to the following.

Corollary 5.3 *The symmetric group S_n has $[n/2] + n^{o(1)}$ conjugacy classes of maximal subgroups.*

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