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On Constrained Steady-State Regulation: Dynamic KKT Controllers

Andrej Jokić, Mircea Lazar, and Paul P. J. van den Bosch

Abstract—This technical note presents a solution to the problem of regulating a general nonlinear dynamical system to an economically optimal operating point. The system is characterized by a set of exogenous inputs as an abstraction of time-varying loads and disturbances. The economically optimal operating point is implicitly defined as a solution to a given constrained convex optimization problem, which is related to steady-state operation. The system outputs and the exogenous inputs represent respectively the decision variables and the parameters in the optimization problem. The proposed solution is based on a specific dynamic extension of the Karush–Kuhn–Tucker optimality conditions for the steady-state related optimization problem, which is conceptually related to the continuous-time Arrow–Hurwicz–Uzawa algorithm. Furthermore, it can be interpreted as a generalization of the standard output regulation problem with respect to a constant reference signal.

Index Terms—Complementarity systems, constraints, convex optimization, optimal control, steady-state.

I. INTRODUCTION

In many production facilities, the optimization problem reflecting economical benefits of production is associated with *steady-state operation* of the system. The control action is required to maintain the production in an optimal regime in spite of various disturbances, and to efficiently and rapidly respond to changes in demand. Furthermore, it is desirable that the system settles in a steady-state that is optimal for novel operating conditions. The vast majority of control literature is focused on regulation and tracking with respect to known setpoints or trajectories, while coping with different types of uncertainties and disturbances in both the plant and its environment. Typically, setpoints are determined off-line by solving an appropriate optimization problem and they are updated in an open-loop manner. The increase of the frequency with which the economically optimal setpoints are updated can

result in a significant increase of economic benefits accumulated in time. If the time-scale on which economic optimization is performed approaches the time-scale of the underlying physical system, i.e., of the plant dynamics, dynamic interaction in between the two has to be considered. Economic optimization then becomes a challenging control problem, especially since it has to cope with inequality constraints that reflect the physical and security limits of the plant [1].

In this technical note, we consider the problem of regulating a general nonlinear dynamical system to an implicitly defined economically optimal operating point. The considered dynamical system is characterized by a set of exogenous inputs as an abstraction of time-varying loads and disturbances acting on the system. Economic optimality is defined through a convex constrained optimization problem with system outputs as decision variables, and with the values of exogenous inputs as parameters in the optimization problem. A similar problem has already been considered in [1], see also the references therein, where the authors propose a solution that uses penalty and barrier functions to deal with inequality constraints. We propose a novel solution based on a specific dynamic extension of the Karush–Kuhn–Tucker (KKT) optimality conditions, which is conceptually related to the continuous-time Arrow–Hurwicz–Uzawa algorithm [2]. The proposed feedback controller belongs to the class of complementarity systems (CS), which was formally introduced in 1996 by Van der Schaft and Schumacher [3] (see also [4] and [5]) and have become an extensive topic of research in the hybrid systems community.

Nomenclature: For a matrix $A \in \mathbb{R}^{m \times n}$, $[A]_{ij}$ denotes the element in the ith row and jth column of A. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the ith element of x. A vector $x \in \mathbb{R}^n$ is said to be nonnegative (nonpositive) if $[x]_i \geq 0 ([x]_i \leq 0)$ for all $i \in \{1, \ldots n\}$, and in that case we write $x \geq 0 (x \leq 0)$. The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n | x \geq 0\}$. The operator $\mathrm{col}(\cdot, \ldots, \cdot)$ stacks its operands into a column vector, and $\mathrm{diag}(\cdot, \ldots, \cdot)$ denotes a square matrix with its operands on the main diagonal and zeros elsewhere. For $u, v \in \mathbb{R}^k$ we write $u \perp v$ if $u^\top v = 0$. We use the compact notational form $0 \leq u \perp v \geq 0$ to denote the complementarity conditions $u \geq 0, v \geq 0, u \perp v$. The matrix inequality $A \succ B$ means A and B are Hermitian and $A - B$ is positive definite. For a scalar-valued differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ denotes its gradient at $x = \mathrm{col}(x_1, \ldots, x_n)$ and is defined as a *column vector*, i.e., $\nabla f(x) \in \mathbb{R}^n, [\nabla f(x)]_i = (\partial f)/(\partial x_i)$. For a vector-valued differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m, f(x) = \mathrm{col}(f_1(x), \ldots, f_m(x))$, the Jacobian at $x = \mathrm{col}(x_1, \ldots, x_n)$ is the matrix $Df(x) \in \mathbb{R}^{m \times n}$ and is defined by $[Df(x)]_{ij} = (\partial f_i(x))/(\partial x_j)$. For a vector valued function $f : \mathbb{R}^n \to \mathbb{R}^m$, we will use $\nabla f(x)$ to denote the transpose of the Jacobian, i.e., $\nabla f(x) \in \mathbb{R}^{n \times m}, \nabla f(x) \triangleq Df(x)^\top$, which is consistent with the gradient notation ∇f when f is a scalar-valued function. With a slight abuse of notation we will often use the same symbol to denote a signal, i.e., a function of time, as well as possible values that the signal may take at any time instant.

II. PROBLEM FORMULATION

In this section, we formally present the constrained steady-state optimal regulation problem considered in this technical note. Furthermore, we list several standing assumptions, which will be instrumental in the subsequent sections. Consider a dynamical system

$$\dot{x} = f(x, w, u) \tag{1a}$$
$$y = g(x, w) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is an exogenous input, $y(t) \in \mathbb{R}^m$ is the measured output, $f :$

$\mathbb{R}^n \times \mathbb{R}^{n_w} \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^m$ are arbitrary nonlinear functions.

For a constant $w \in W$, with $W \subset \mathbb{R}^{n_w}$ denoting a known bounded set, consider the following convex optimization problem associated with the output y of the dynamical system (1)

$$\min_y \quad J(y) \tag{2a}$$

$$\text{subject to} \quad Ly = h(w) \tag{2b}$$

$$q_i(y) \leq r_i(w), \quad i = 1, \ldots, k \tag{2c}$$

where $J : \mathbb{R}^m \to \mathbb{R}$ is a strictly convex and continuously differentiable function, $L \in \mathbb{R}^{l \times m}$ is a constant matrix, $h : \mathbb{R}^{n_w} \to \mathbb{R}^l$ and $r_i : \mathbb{R}^{n_w} \to \mathbb{R}, i = 1, \ldots, k$ are continuous functions, while $q_i : \mathbb{R}^m \to \mathbb{R}, i = 1, \ldots, k$ are convex, continuously differentiable functions. For the matrix L we require $\operatorname{rank}(L) = l < m$. For a constant exogenous signal $w(t) = w \in W$, the optimization problem (2) implicitly defines the optimal operating point in terms of the steady-state value of the output vector y in (1). The constraints in (2) represent the security-type "soft" constraints for which transient violation may be accepted, but their feasibility is required in steady-state, i.e., at the economically optimal operating point corresponding to each particular, constant value of w. With respect to time, the above requirement implies that constraints are satisfied asymptotically, i.e., as $t \to \infty$. The objective of the control input u is to drive the output y to the optimal steady-state operating point given by (2).

We continue by listing several assumptions concerning the dynamics (1) and the optimization problem (2). Let $\mathcal{I}_l$ denote the set of indices i for which the function q_i in (2c) is linear or affine, and let $\mathcal{I}_n$ denote the set of indices for which the function q_i is nonlinear and non-affine.

Assumption II.1: For each $w \in W$ the set

$$\{y | Ly = h(w), \quad q_i(y) < r_i(w)$$
$$\text{for} \quad i \in \mathcal{I}_n, \quad q_i(y) \leq r_i(w) \quad \text{for} \quad i \in \mathcal{I}_l\}$$

is nonempty. $\qquad \square$

Assumption II.1 states that the convex optimization problem (2) satisfies Slater's constraint qualification [6] for each $w \in W$, implying that strong duality holds for the considered problem. Note also that due to strict convexity of the objective function in (2), the optimization problem has an unique minimizer $\tilde{y}(w)$ for each $w \in W$.

Assumption II.2: For each $w \in W$ the minimum is attained in problem (2). $\qquad \square$

Assumption II.3: For each $w \in W$, there is a unique pair $(\tilde{x}(w), \tilde{u}(w))$ such that $\tilde{y}(w) = g(\tilde{x}(w), w)$ and $0 = f(\tilde{x}(w), w, \tilde{u}(w))$, where $\tilde{y}(w)$ denotes the corresponding minimizer in problem (2). $\qquad \square$

With the definitions and assumptions made so far, we are now ready to formally state the regulation problem considered in this technical note.

Problem II.4: Constrained Steady-State Optimal Regulation: Suppose that assumptions II.1, II.2 and II.3 hold. For a dynamical system given by (1), design a feedback controller that has $y, Ly - h(w)$ and $q(y) - r(w)$ as input signals and u as output signal, such that the following objective is met for any constant-valued exogenous signal $w(t) = w \in W$: the closed-loop system state globally converges to an equilibrium point with $y = \tilde{y}(w)$, where $\tilde{y}(w)$ denotes the corresponding minimizer of the optimization problem (2). $\qquad \square$

Remark II.5: Problem II.4 includes the standard output regulation problem, see e.g., Chapter 12 in [7], as a special case. More precisely, if the constraints (2b) and (2c) are removed from (2) and $J(y) := (1/2)(y - w)^\top (y - w)$, then Problem II.4 reduces to the problem of regulating the output y to the constant reference signal w. Furthermore, note that Assumption II.3 is a necessary condition for Problem II.4

to have a solution, and as such is also present in the standard output regulation problem [7]. $\qquad \square$

In Problem II.4 we have assumed that constraint violations, i.e., signals $Ly - h(w)$ and $q(y) - r(w)$, are directly measurable and as such they can be used for control purposes. However, this assumption can be relaxed to also handle some of the cases when direct measurement of constraints violation is not possible, as illustrated in the example presented in Section V.

III. DYNAMIC KKT CONTROLLERS

Assumption II.1 implies that for each $w \in W$, the first order Karush-Kuhn-Tucker (KKT) conditions are *necessary and sufficient* conditions for optimality [6]. For the optimization problem (2) these conditions are given by the following set of equalities and inequalities:

$$\nabla J(y) + L^\top \lambda + \nabla q(y)\mu = 0 \tag{3a}$$

$$Ly - h(w) = 0 \tag{3b}$$

$$0 \leq -q(y) + r(w) \perp \mu \geq 0 \tag{3c}$$

where $q(y) := \operatorname{col}(q_1(y), \ldots, q_k(y)), r(w) := \operatorname{col}(r_1(w), \ldots, r_k(w))$ and $\lambda \in \mathbb{R}^l$, $\mu \in \mathbb{R}^k$ are Lagrange multipliers. In what follows, based on an appropriate dynamic extension of the above presented Karush–Kuhn–Tucker optimality conditions, we present two controllers that both guarantee that for each $w \in W$ the closed-loop system has an equilibrium point where $y = \tilde{y}(w)$ as described in Problem II.4. Later in this section, it will be shown that there are certain insightful differences as well as similarities between these controllers.

Max-Based KKT Controller: Let $K_\lambda \in \mathbb{R}^{l \times l}, K_\mu \in \mathbb{R}^{k \times k}, K_c \in \mathbb{R}^{m \times m}, K_o \in \mathbb{R}^{k \times k}$ be diagonal matrices with non-zero elements on the diagonal and $K_\mu \succ 0, K_o \succ 0$. Consider the dynamic controller

$$\dot{x}_\lambda = K_\lambda(Ly - h(w)) \tag{4a}$$

$$\dot{x}_\mu = K_\mu(q(y) - r(w)) + v \tag{4b}$$

$$\dot{x}_c = K_c(L^\top x_\lambda + \nabla q(y)x_\mu + \nabla J(y)) \tag{4c}$$

$$0 \leq v \perp K_o x_\mu + K_\mu(q(y) - r(w)) + v \geq 0 \tag{4d}$$

$$u = x_c \tag{4e}$$

where x_λ, x_μ and x_c denote the controller states and the matrices K_λ, K_μ, K_c and K_o represent the controller gains. Note that the input vector $v(t) \in \mathbb{R}^k$ in (4b) is at any time instant required to be a solution to a finite-dimensional linear complementarity problem (4d). $\qquad \square$

Saturation-Based KKT Controller: Let $K_\lambda \in \mathbb{R}^{l \times l}, K_\mu \in \mathbb{R}^{k \times k}, K_c \in \mathbb{R}^{m \times m}$ be diagonal matrices with non-zero elements on the diagonal and $K_\mu \succ 0$. Consider the dynamic controller

$$\dot{x}_\lambda = K_\lambda(Ly - h(w)) \tag{5a}$$

$$\dot{x}_\mu = K_\mu(q(y) - r(w)) + v \tag{5b}$$

$$\dot{x}_c = K_c(L^\top x_\lambda + \nabla q(y)x_\mu + \nabla J(y)) \tag{5c}$$

$$0 \leq v \perp x_\mu \geq 0 \tag{5d}$$

$$u = x_c \tag{5e}$$

$$x_\mu(0) \geq 0 \tag{5f}$$

where x_λ, x_μ and x_c denote the controller states and the matrices K_λ, K_μ and K_c represent the controller gains. The input $v(t) \in \mathbb{R}^k$ in (5b) is at any time instant required to be a solution to a finite-dimensional linear complementarity problem (5d). The initial state constraint (5f) is required as a necessary condition for well-posedness via the complementarity condition (5d). $\qquad \square$

The choice of names *max-based KKT controller* and *saturation-based KKT controller* will become clear later in this section. Notice that both controllers belong to the class of CS [3], [4].

Theorem III.1: Let $w(t) = w \in W$ be a constant-valued signal, and suppose that Assumption II.1 and Assumption II.3 hold. Then the closed-loop system, i.e., the system obtained from the system (1) connected with controller (4) or (5) in a feedback loop, has an equilibrium point with $y = \tilde{y}(w)$, where $\tilde{y}(w)$ denotes the corresponding minimizer of the optimization problem (2).

Proof: We first consider the closed-loop system with the max-based KKT controller, i.e., controller (4). By setting the time derivatives of the closed-loop system states to zero and by exploiting the non-singularity of the matrices K_λ, and K_c, we obtain the following complementarity problem:

$$0 = f(x, w, x_c) \tag{6a}$$
$$y = g(x, w) \tag{6b}$$
$$0 = Ly - h(w) \tag{6c}$$
$$0 = K_\mu(q(y) - r(w)) + v \tag{6d}$$
$$0 = L^\top x_\lambda + \nabla q(y) x_\mu + \nabla J(y) \tag{6e}$$
$$0 \leq v \perp K_o x_\mu + K_\mu(q(y) - r(w)) + v \geq 0 \tag{6f}$$

with the closed-loop system state vector $x_{cl} := \mathrm{col}(x, x_\lambda, x_\mu, x_c)$ and the vector v as variables. Any solution x_{cl} to (6) is an equilibrium point of the closed-loop system. By substituting $v = -K_\mu(q(y) - r(w))$ from (6d), utilizing $K_\mu \succ 0, K_o \succ 0$ and the fact that K_μ and K_o are diagonal, the complementarity condition (6f) becomes $0 \leq -q(y) + r(w) \perp x_\mu \geq 0$. With $\lambda := x_\lambda$ and $\mu := x_\mu$, the conditions (6c), (6d), (6e), (6f) therefore correspond to the KKT conditions (3) and, under Assumption II.1, these conditions necessarily have a solution in (y, x_λ, x_μ, v). Furthermore, for any solution (y, x_λ, x_μ, v) to (6c), (6d), (6e), (6f), it necessarily holds that $y = \tilde{y}(w)$. It remains to show that (6a), (6b) admit a solution in (x, x_c) for $y = \tilde{y}(w)$. This is, however, the hypothesis of Assumption II.3. Moreover, Assumption II.3 implies uniqueness of x and x_c at an equilibrium. Now, consider the closed-loop system with the saturation-based KKT controller, i.e., controller (5). The difference in this case comes only through (5d). It is therefore sufficient to show that (5d) implyies $0 \leq -q(y) + r(w) \perp x_\mu \geq 0$. This implication is obvious since $K_\mu \succ 0$. $\square$

Remark III.2: Theorem III.1 states that for any constant-valued exogenous signal $w(t) \in W$, the closed-loop system necessarily has an equilibrium. Furthermore, from the proof of this theorem it follows that for *all* corresponding equilibrium points (i.e., each equilibrium corresponds to a constant $w \in W$) the values of the state vectors (x, x_c) are *unique*. For a given $w(t) = w \in W$, the necessary and sufficient condition for uniqueness of the remaining closed-loop system state vectors (x_λ, x_μ), and therefore a necessary and sufficient condition for uniqueness of the closed-loop system equilibrium, corresponds to the condition for uniqueness of the Lagrange multipliers in (3). This condition is known as the *strict Mangasarian-Fromovitz constraint qualification* (SMFCQ) and is presented in [8]. $\square$

Note that when the optimization problem (2) is such that it defines standard regulation problem, see Remark II.5, then both KKT controllers reduce to the standard integral controllers, see e.g., Chapter 12 in [7], i.e., they reduce to $\dot{x}_c = K_c(y - w), u = x_c$.

A. Complementarity Integrators

The main distinguishing feature between the max-based KKT controller (4) and the saturation-based KKT controller (5) is in the way the steady-state complementarity slackness condition (3c) is enforced. In the following two paragraphs, our attention is on the (4b), (4d) and (5b), (5d), and the goal is to show the following:
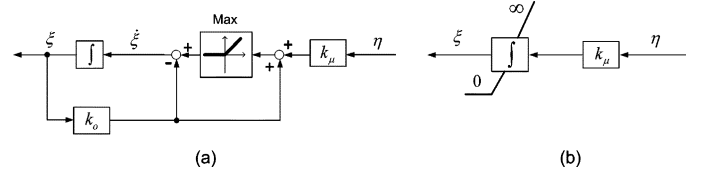

Fig. 1. Complementarity integrators: (a) Max-based CI; (b) Saturation-based CI.

- the *max-based KKT controller* (4) can be represented as a dynamical system in which certain variables are coupled by means of static, continuous, piecewise linear characteristics;
- the *saturation-based KKT controller* (5) can be represented as a dynamical system with state saturations.

Max-Based Complementarity Integrator: Let $\eta = [q(y) - r(w)]_i, \xi = [x_\mu]_i, \nu = [v]_i, k_o = [K_o]_{ii}$ and $k_\mu = [K_\mu]_{ii}$, for some $i \in \{1, \dots, k\}$. Then the ith row in (4b) and (4d) is given by

$$\dot{\xi} = k_\mu \eta + \nu \tag{7a}$$
$$0 \leq \nu \perp k_o \xi + k_\mu \eta + \nu \geq 0 \tag{7b}$$

respectively, where $k_o > 0$ and $k_\mu > 0$.

Let a, b and c be real scalars related through the complementarity condition $0 \leq c \perp a + b + c \geq 0$. It is easily verified, e.g., by checking all possible combinations, that this complementarity condition is equivalent to $b + c = \max(a + b, 0) - a$.

Now, by taking $c = \nu, a = k_o \xi$ and $b = k_\mu \eta$, it follows that (7) can be equivalently described by

$$\dot{\xi} = \max(k_o \xi + k_\mu \eta, 0) - k_o \xi. \tag{8}$$

Fig. 1(a) presents a block diagram representation of (8). The block labeled "Max" in the figure, represents the scalar max relation as a static piecewise linear characteristic. $\square$

Saturation-Based Complementarity Integrator: Let $\eta = [q(y) - r(w)]_i, \xi = [x_\mu]_i, \nu = [v]_i$ and $k_\mu = [K_\mu]_{ii}$, for some $i \in \{1, \dots, k\}$. Then the ith row in (5b), (5d), and (5f) is given by

$$\dot{\xi} = k_\mu \eta + \nu \tag{9a}$$
$$0 \leq \nu \perp \xi \geq 0 \tag{9b}$$
$$\xi(0) \geq 0 \tag{9c}$$

respectively, where $k_\mu > 0$. The dynamical system (9) can equivalently be described by

$$\dot{\xi} = \phi_{\mathrm{SCI}}(\xi, \eta) := \begin{cases} 0 & \text{if } \xi = 0 \text{ and } k_\mu \eta < 0, \\ k_\mu \eta & \text{if } \xi = 0 \text{ and } k_\mu \eta \geq 0, \\ k_\mu \eta & \text{if } \xi > 0. \end{cases} \tag{10}$$

Fig. 1(b) presents a block diagram representation of (10), which is a saturated integrator with the lower saturation point equal to zero. The equivalence of the dynamics (9) and the saturated integrator defined by (10) directly follows from the equivalence of *gradient-type complementarity systems* (GTCS) ((9) belongs to the GTCS class) and *projected dynamical systems* (PDS) ((10) belongs to the PDS class). For the precise definitions of GTCS and PDS system classes and for the equivalence results see [9] and [10]. $\square$

With $k_\mu > 0$ and $k_\mu > 0$ it is easy to verify that for both the system in Fig. 1(a) and the system in Fig. 1(b) in steady-state the value of the input signal η and the value of its output signal ξ necessarily satisfy complementarity condition $0 \leq \xi \perp -\eta \geq 0$. We will use the term *max-based complementarity integrator* (MCI) to refer to the system (7), i.e., the system with the structure as depicted in Fig. 1(a), and we

will use the term *saturation-based complementarity integrator* (SCI) for the system (9), i.e., the system in Fig. 1(b). Together with a pure integrator, complementarity integrators form the basic building blocks of a KKT controller.

Remark III.3: For the MCI given by (7) the following holds:

a) If $\xi(0) < 0$ then either $\xi(\tau) = 0$ for some $0 < \tau < \infty$, or $\xi(t) \to 0$ as $t \to \infty$. Indeed, for $\xi(t) < 0$ from (7) it follows that $\dot{\xi}(t) > 0$ irrespective of the value of the input signal $\eta(t)$.

b) If $\xi(0) \geq 0$, then $\xi(t) \geq 0$ for all $t \in \mathbb{R}_+$. Indeed, for $\xi(t) = 0$ from (7) it follows that $\dot{\xi}(t) \geq 0$ irrespective of the value of the input signal $\eta(t)$. Therefore, similarly to the behavior of the saturation-based KKT controller, if $x_\mu(0) \geq 0$ in the max-based KKT controller (4), then $x_\mu(t) \geq 0$ for all $t \in \mathbb{R}_+$. $\square$

In what follows, we point out an interesting relation between the dynamical behavior of the two types of complementarity integrators. Consider the MCI (7) and let $\xi(0) \geq 0$. Note that according to Remark III.3 it follows that $\xi(t) \geq 0$ for all $t \in \mathbb{R}_+$. For $\xi(t) \geq 0$, the dynamics (7) can be equivalently represented in a piecewise-linear form as follows:

$$\dot{\xi} = \phi_{\mathrm{MCI}}(\xi, \eta) := \begin{cases} k_\mu \eta & \text{if } \xi \geq -\frac{k_\mu}{k_o}\eta, \\ -k_o \xi & \text{if } \xi < -\frac{k_\mu}{k_o}\eta. \end{cases} \quad (11)$$

Now, suppose that the gain k_μ has the same value in (10) and (11). For a given $\eta(t) < \infty$, we define the set $\mathcal{D} := \{\xi | \xi \geq 0, \phi_{\mathrm{SCI}}(\xi, \eta) \neq \phi_{\mathrm{MCI}}(\xi, \eta)\}$. By inspection it can easily be observed that for any $\eta(t) < \infty$, the Lebesgue measure of the set $\mathcal{D}$ tends to zero as k_o tends to ∞. This implies that the SCI can be considered as a special case of the MCI when the gain k_o is set to infinity. In the same sense, the saturation-based KKT controller can be considered as a special case of the max-based KKT controller.

IV. WELL-POSEDNESS AND STABILITY OF THE CLOSED-LOOP SYSTEM

In this section, we shortly present some results concerning the well-posedness and stability of the closed-loop system, i.e., of the system (1) interconnected with one of the two proposed dynamic KKT controllers in a feedback loop. Note that although the results presented in Section III hold for an arbitrary nonlinear system, to address well-posedness and stability issues one has to focus on specific, relevant subclasses of system (1). For a more detailed treatment of these topics, the interested reader is referred to [11].

A. Well-Posedness

Since the function $\max(\cdot, 0)$ is globally Lipschitz continuous, for checking well-posedness of the system in closed loop with the max-based KKT controller one can resort to standard Lipschitz continuity conditions. Notice that the system (1) in closed loop with a saturation-based KKT controller belongs to a specific class of gradient-type complementarity systems for which sufficient conditions for well-posedness have been presented in [9] and [10]. More precisely, it was shown that the *hypermonotonicity* property plays a crucial role in establishing well-posedness, see [9] and [10] for details. It can be easily verified, see [11] for details, that Lipschitz continuity implies hypermonotonicity, and therefore we can state the following unified condition for well-posedness of the system (1) in closed loop with a dynamic KKT controller (irrespective of the KKT controller type):

Proposition IV.1: Suppose that the functions f and g in (1) are globally (locally) Lipschitz. Then if the functions $q, \nabla J$ and all entries in ∇q are globally (locally) Lipschitz, the system (1) in closed loop with a dynamic KKT controller of the form (4) or (5) is globally (locally) well-posed.

B. Stability Analysis

1) Stability Analysis for a Fixed $w \in W$: Since both types of complementarity integrators can be represented in an equivalent piecewise affine form [12], for a given $w(t) = w \in W$ characterized by a *unique equilibrium* (see Remark III.2), one can perform a global asymptotic stability analysis based on: *i)* the analysis procedures from [13], [14] in case (2) is a quadratic program and (1) is a linear system; *ii)* the analysis procedure from [15] in case (2) is given by a (higher order) polynomial objective function and (higher order) polynomial inequality constraints, while (1) is a general polynomial system. In the case when $w(t) = w \in W$ is such that the SMFCQ (see Remark III.2) does not hold, the closed-loop system is characterized by a *set of equilibria* (not a singleton), which is then an invariant set for the closed-loop system. Each equilibrium in this set is characterized by different values of the state vectors (x_λ, x_μ), but unique values of the remaining states. Under an additional generalized Slater constraint qualification, see [16] for details, the set of equilibria is guaranteed to be bounded. For stability analysis with respect to this set, one could invoke a suitable extension of LaSalle's invariance principle [17].

2) Stability Analysis for All $w \in W$: A possibility to perform stability analysis for *all possible* constant values of the exogenous signal $w(t)$, i.e., for all $w(t) = w$ where w is *any* constant in W, is to formulate a corresponding robust stability analysis problem. For instance, consider the max-based KKT control structure, which is particulary suitable for this approach. Let $\mathcal{M}$ denote the set of autonomous systems which contains all the closed-loop systems that correspond to one fixed $w \in W$. Furthermore, suppose that each system in $\mathcal{M}$ has the origin as equilibrium, after an appropriate state transformation. Then, it can be shown that for any closed-loop system in $\mathcal{M}$ the static nonlinearity of the MCI, see Fig. 1(a), fulfills certain sector bound conditions. Therefore, stability of all the closed-loop systems in the set $\mathcal{M}$ can be established using the integral quadratic constraint approach [18]. See [11] for a complete description that also deals with non-unique equilibria.

V. ILLUSTRATIVE EXAMPLE

To illustrate the theory, in this section we present the following example that includes nonlinear constraints on the steady-state operating point. Consider a third-order system of the form (1):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -2.5 & 0 & -5 \\ 0 & -5 & -15 \\ 0.1 & 0.1 & -0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 \\ 0 \\ -0.1 \end{pmatrix} w + \begin{pmatrix} 2.5 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (12a)$$

$$y = \mathrm{col}(x_1, x_2, x_3) \quad (12b)$$

and let $u := \mathrm{col}(u_1, u_2)$ collect the control inputs. With $x_p := \mathrm{col}(x_1, x_2)$, the associated steady-state related optimization problem is defined as

$$\min_{x_p} \frac{1}{2} x_p^\top H x_p + a^\top x_p \quad (13a)$$

$$\text{subject to} \quad x_1 + x_2 = w \quad (13b)$$

$$(x_1 - 4.7)^2 + (x_2 - 4)^2 \leq 3.5^2 \quad (13c)$$

where $H = \mathrm{diag}(6, 2), a = \mathrm{col}(-4, -4)$, and the value of the exogenous signal w is limited in the interval $W = [4, 11.5]$. It can be verified that for this W and the constraints (13b) and (13c), Assumption II.1 holds. Furthermore, it can easily be verified that Assumption II.3 holds. From the dynamics of the state x_3, it follows that in steady-state
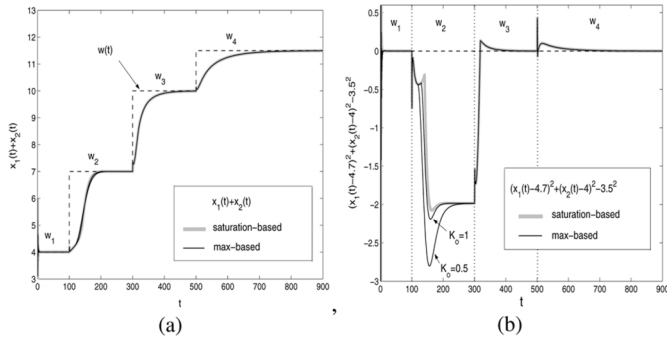
Fig. 2. (a) Values of w and $x_1 + x_2$, i.e., the right hand and the left hand side of the constraint (13b), as a function of time; (b) violation of inequality (13c) as a function of time (when the curves are above zero, the constraint is violated).
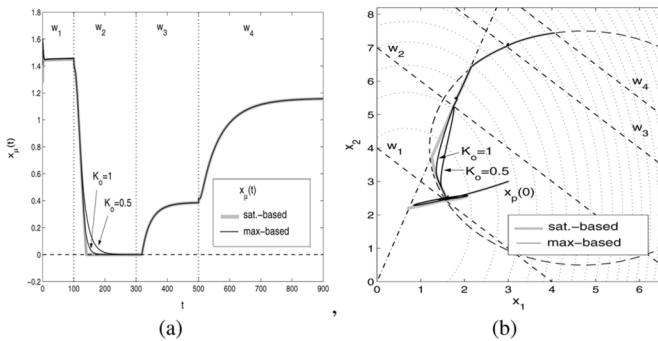

Fig. 3. (a) Simulated trajectory of x_μ. (b) Simulated trajectory of x_p for the closed-loop system.

the equality $x_1 + x_2 - 2x_3 = w$ holds. Therefore, in steady-state, $x_3 = 0$ implies fulfilment of the constraint (13b). This implies that for control purposes we can directly use the value of the state x_3 as a measure of violation of this constraint. Hence, explicit knowledge of w is not required.

Simulations of the closed-loop system response to stepwise changes in the exogenous input $w(t)$, which is presented in Fig. 2(a), have been performed. Figs. 2 and 3 present the results of the simulation when the system is controlled with both a saturation-based and a max-based KKT controller with different values of the gain K_o. Both controllers were implemented with the gains $K_\lambda = 0.15, K_\mu = 0.1, K_c = \mathrm{diag}(-0.7, -0.7)$, and the gain K_o in the max-based controller was set to 0.5 and 1. In each figure, a legend is included to indicate which trajectory belongs to each controller. Fig. 2(a) and (b) clearly illustrate that the controllers continuously drive the closed-loop system towards the steady-state where the constraints (13b), (13c) are satisfied. Figs. 2(b) and 3(a) show fulfilment of the complementarity slackness condition (3c) in steady-state. Finally, Fig. 3(b) illustrates that the controllers drive the system towards the corresponding optimal operating point as defined by (13). In this figure the straight dashed lines labeled $w_i, i = 1, \ldots, 4$, represent the equality constraint $x_1 + x_2 = w_i$, where the values of $w_i, i = 1, \ldots, 4$ are the ones given in Fig. 2(a). The dashed circle represents the inequality constraint (13c), i.e., the steady-state feasible region for x_p is within this circle. Thin dotted lines represent the contour lines of the objective function (13a), while the dash-dot line represents the locus of the optimal point $\tilde{x}_p(w)$ for the whole range of values w in the case when the inequality constraint (13c) would be left out from the optimization problem. From the simulations we can observe that by increasing the gain K_o in the max-based controller, the trajectory of the closed-loop system with the max-based

KKT controller approaches the trajectory of the closed-loop system with saturation-based KKT controller.

VI. CONCLUSION

In this technical note, we have considered the problem of regulating a general nonlinear dynamical system to an economically optimal operating point which is implicitly defined as a solution to a given constrained convex optimization problem. The proposed solution is based on a specific dynamic extension of the Karush–Kuhn–Tucker optimality conditions for the steady-state related optimization problem and can be interpreted as a generalization of the standard output regulation problem with respect to a constant reference signal.

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