TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 203, 1975

ON CONSTRUCTING LEAST SQUARES SOLUTIONS TO TWO-POINT BOUNDARY VALUE PROBLEMS

BY

JOHN LOCKER

ABSTRACT. For an nth order linear boundary value problem $Lf = g_0$ in the Hilbert space $L^2[a, b]$, a sequence of approximate solutions is constructed which converges to the unique least squares solution of minimal norm. The method is practical from a computational viewpoint, and it does not require knowing the null spaces of the differential operator L or its adjoint L^*.

1. **Introduction.** For a closed interval $[a, b]$ let S be the real Hilbert space $L^2[a, b]$ with the standard inner product (f, g) and norm $\|f\|$. We denote convergence in S by $f_i \to f$ and denote the domain, range, and null space of any operator L by $\mathcal{D}(L)$, $\mathcal{R}(L)$, and $\mathcal{N}(L)$, respectively.

Given an nth order formal differential operator

$$\tau = \sum_{i=0}^{n} a_i(t)\left(\frac{d}{dt}\right)^i,$$

where the coefficients $a_i(t)$ belong to $C^\infty[a, b]$ and $a_n(t) \neq 0$ on $[a, b]$, and given k linearly independent boundary values

$$B_i(f) = \sum_{j=0}^{n-1} \alpha_{ij} f^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{ij} f^{(j)}(b), \qquad i = 1, \cdots, k,$$

we define a differential operator L in S as follows: Let $H^n[a, b]$ be the subspace of S consisting of all functions f in $C^{n-1}[a, b]$ with $f^{(n-1)}$ absolutely continuous on $[a, b]$ and $f^{(n)}$ in S, and let $\mathcal{D}(L) = \{f \in H^n[a, b] | B_i(f) = 0, i = 1, \cdots, k\}$, $Lf = \tau f$. For a fixed function g_0 in S we consider the boundary value problem

$$Lf = g_0. \tag{1}$$

In a previous paper [3] we used the method of least squares to construct approximate solutions to equation (1). A careful examination of the approximation

Received by the editors January 2, 1974.

AMS (MOS) subject classifications (1970). Primary 34B05, 47E05; Secondary 34B25, 65L10.

Key words and phrases. Least squares solution, boundary value problem, approximation scheme, generalized inverse.

scheme shows that the null spaces $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$ must be known in order to apply the method. In most practical problems it is impossible to calculate these null spaces exactly, and hence, this approach appears to be of limited applicability.

The purpose of this paper is to give a new least squares development which is independent of these null spaces and which is computationally feasible. The method yields approximate solutions which converge to the unique least squares solution of (1) of minimal norm, and it can be used whether (1) is solvable or not.

In §2 we introduce the generalized inverse $L^\dagger$ of the differential operator L and discuss its properties which are relevant to least squares solutions of equation (1). In §3 the selfadjoint differential operator LL^* is studied. This operator plays an important role in our approximation scheme, and in a future paper we will describe its relationship to the generalized Green's function for L. The approximation scheme, including error estimates, is developed in §4. For the special case in which the eigenfunctions of LL^* are used, the scheme has a particularly simple form.

2. **The generalized inverse of L.** The restriction of L to the subspace $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$ is a 1-1 closed operator, and its inverse

$$H = [L|\mathcal{D}(L) \cap \mathcal{N}(L)^\perp]^{-1}$$

is a 1-1 bounded linear operator with domain $\mathcal{R}(L)$ and range $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$. This operator is examined in [2].

Let P and Q denote the L^2-orthogonal projections from S onto $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$, respectively. We observe that $I - P$ and $I - Q$ are the L^2-orthogonal projections from S onto the closed subspaces $\mathcal{R}(L^*)$ and $\mathcal{R}(L)$, respectively. Also, $LHf = f$ for all $f \in \mathcal{R}(L)$ and $HLf = f - Pf$ for all $f \in \mathcal{D}(L)$.

Let $L^\dagger: S \longrightarrow S$ be the bounded linear operator defined by $L^\dagger f = H(I - Q)f$ for all $f \in S$. Clearly $L^\dagger|\mathcal{R}(L) = H$, and it can be verified that $L^\dagger$ has the following properties:

(i) $LL^\dagger Lf = Lf$ for all $f \in \mathcal{D}(L)$,
(ii) $L^\dagger LL^\dagger f = L^\dagger f$ for all $f \in S$,
(iii) $LL^\dagger f = f - Qf$ for all $f \in S$,
(iv) $L^\dagger Lf = f - Pf$ for all $f \in \mathcal{D}(L)$.

Therefore, $L^\dagger$ is the Moore-Penrose generalized inverse of L. Our description of $L^\dagger$ is similar to the one given by Loud [4, pp. 196–198].

For the boundary value problem (1) we let $g_0 = h_0 + k_0$, where $h_0 = g_0 - Qg_0$ belongs to $\mathcal{R}(L)$ and $k_0 = Qg_0$ belongs to $\mathcal{N}(L^*)$, and then we set $f_0 = L^\dagger g_0 = Hh_0$. The function f_0 belongs to $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$ and has the following properties:

(i) f_0 is a *least squares solution* to equation (1), i.e., $\|Lf_0 - g_0\|$ is equal to the infimum of the set of numbers $\|Lf - g_0\|$ where f ranges over $\mathcal{D}(L)$.

(ii) The set of all least squares solutions to equation (1) is the set $f_0 + \mathcal{N}(L)$.

(iii) f_0 is the unique least squares solution of equation (1) of minimal norm.

(iv) f_0 is a solution to equation (1) when (1) is solvable.

(v) f_0 is the unique solution in $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$ of the boundary value problem

$$Lf = h_0. \tag{2}$$

(vi) $\|Lf_0 - g_0\| = \|h_0 - g_0\| = \|k_0\|$.

The paper by Nashed [5] has a thorough treatment of generalized inverses and least squares solutions, as well as an extensive list of references. In the next two sections we are going to construct a sequence of functions f_i $(i = 1, 2, \cdots)$ in $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$ which converges to $f_0 = L^\dagger g_0$.

3. **The differential operator** LL^*. Let

$$\tau^* = \sum_{i=0}^{n} b_i(t)\left(\frac{d}{dt}\right)^i$$

be the formal adjoint of τ, and let

$$B_i^*(f) = \sum_{j=0}^{n-1} \alpha_{ij}^* f^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{ij}^* f^{(j)}(b), \qquad i = 1, \cdots, 2n-k,$$

be a set of $2n - k$ linearly independent adjoint boundary values. The adjoint operator L^* is given by

$$\mathcal{D}(L^*) = \{f \in H^n[a, b] | B_i^*(f) = 0, i = 1, \cdots, 2n-k\}, \qquad L^*f = \tau^* f.$$

We are going to work with the space $H^n[a, b]$ under the norm

$$|f|_* = \sum_{i=0}^{n-1} \max_{a<t<b} |f^{(i)}(t)| + \|f^{(n)}\|, \qquad f \in H^n[a, b],$$

which makes it into a Banach space. In addition, we introduce an inner product

$$(f, g)_\tau = (f, g) + (\tau f, \tau g), \qquad f, g \in H^n[a, b],$$

and associated norm $|f|_\tau = (f, f)_\tau^{1/2}$, under which $H^n[a, b]$ becomes a Hilbert space. The norms $|f|_*$ and $|f|_\tau$ are equivalent norms for $H^n[a, b]$, and the topology induced by them is called the *strong topology* for $H^n[a, b]$. Convergence in the strong topology is uniform convergence of the first $n - 1$ derivatives on $[a, b]$ together with L^2-convergence of the nth derivatives, and it is denoted by $f_i \xrightarrow{s_n} f$. The strong topology has been discussed in [3].

We also want to consider the space $H^{2n}[a, b]$ under its strong topology induced by the inner product

$$[f, g] = (f, g) + (f^{(2n)}, g^{(2n)}), \qquad f, g \in H^{2n}[a, b],$$

and associated norm $|f| = [f, f]^{1/2}$. In this case strong convergence is denoted by $f_i \xrightarrow{s_{2n}} f$. This particular inner product is convenient for representing boundary values on $H^{2n}[a, b]$.

LEMMA 1. *The operator* τ^* *maps the space* $H^{2n}[a, b]$ *onto the space* $H^n[a, b]$.

PROOF. It is easy to show that $f \in H^{2n}[a, b]$ implies $\tau^* f \in H^n[a, b]$. The onto property follows from [1, Corollary 4, p. 1283].

LEMMA 2. *The operator* τ^* *is a continuous linear operator between the strong topologies on* $H^{2n}[a, b]$ *and* $H^n[a, b]$.

PROOF. Take a sequence of functions f_i $(i = 1, 2, \cdots)$ in $H^{2n}[a, b]$ and a function $f \in H^{2n}[a, b]$ with $f_i \xrightarrow{s_{2n}} f$. We know that $f_i^{(j)} \longrightarrow f^{(j)}$ as $i \longrightarrow \infty$ for $j = 0, 1, \cdots, 2n$, which certainly implies that $\tau^* f_i \longrightarrow \tau^* f$ and $\tau\tau^* f_i \longrightarrow \tau\tau^* f$. Thus, $|\tau^* f_i - \tau^* f|_\tau \longrightarrow 0$, and the proof is complete.

Consider the linear functionals B_i^+, $i = 1, \cdots, k$, defined on $H^{2n}[a, b]$ by

$$B_i^+(f) = B_i(\tau^* f), \qquad f \in H^{2n}[a, b]. \tag{3}$$

By Lemma 2 each B_i^+ is continuous on $H^{2n}[a, b]$ under the strong topology. If $f(t)$ is any function in $H^{2n}[a, b]$ which is identically zero on neighborhoods of a and b, then $\tau^* f(t)$ has the same property, and hence, $B_i^+(f) = 0$. Therefore, each B_i^+ is a boundary value on $H^{2n}[a, b]$. A direct calculation shows that the classical representation of B_i^+ is

$$\begin{aligned} B_i^+(f) = &\sum_{j=0}^{n-1} \left[\sum_{p=0}^{j} \sum_{q=p}^{n-1} \alpha_{iq} \binom{q}{p} b_{j-p}^{(q-p)}(a) \right] f^{(j)}(a) \\ &+ \sum_{j=n}^{2n-1} \left[\sum_{p=j-n}^{n-1} \sum_{q=p}^{n-1} \alpha_{iq} \binom{q}{p} b_{j-p}^{(q-p)}(a) \right] f^{(j)}(a) \\ &+ \sum_{j=0}^{n-1} \left[\sum_{p=0}^{j} \sum_{q=p}^{n-1} \beta_{iq} \binom{q}{p} b_{j-p}^{(q-p)}(b) \right] f^{(j)}(b) \\ &+ \sum_{j=n}^{2n-1} \left[\sum_{p=j-n}^{n-1} \sum_{q=p}^{n-1} \beta_{iq} \binom{q}{p} b_{j-p}^{(q-p)}(b) \right] f^{(j)}(b) \end{aligned} \tag{4}$$

for $f \in H^{2n}[a, b]$ and for $i = 1, \cdots, k$.

Let T be the $2n$th order differential operator defined by $\mathcal{D}(T) = \{f \in H^{2n}[a, b] | B_i^*(f) = B_j^+(f) = 0, i = 1, \cdots, 2n - k \text{ and } j = 1, \cdots, k\}$, $Tf = \tau\tau^* f$. We are going to study the operator T, establishing its relationship with the operators L and L^*.

Take any function $f \in \mathcal{D}(T)$ and set $g = \tau^* f$. Clearly g belongs to $H^n[a, b]$ with $B_i(g) = B_i^+(f) = 0$ for $i = 1, \cdots, k$, so $g \in \mathcal{D}(L)$. Also, $f \in \mathcal{D}(L^*)$, which implies that $g \in \mathcal{R}(L^*) = \mathcal{N}(L)^\perp$. Therefore, the operator τ^* maps $\mathcal{D}(T)$ into $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$. This property is essential for our approximation scheme.

Next, take functions f and g in $\mathcal{D}(T)$. Then f and g belong to $\mathcal{D}(L^*)$, $\tau^* f$ and $\tau^* g$ belong to $\mathcal{D}(L)$, and it follows that $(Tf, g) = (f, Tg)$. Thus, $T \subset T^*$.

Let $N = \dim\langle B_1^*, \cdots, B_{2n-k}^*, B_1^+, \cdots, B_k^+\rangle$. We know that T^* is a $2n$th order differential operator determined by $(\tau\tau^*)^* = \tau\tau^*$ and a set of $m = 4n - N$ linearly independent adjoint boundary conditions $C_i(f) = 0, i = 1, \cdots, m$. Choose functions $g_1^*, \cdots, g_{2n-k}^*, g_1^+, \cdots, g_k^+$, and $h_1, \cdots, h_m$ in $H^{2n}[a, b]$ such that

$$B_i^*(f) = [f, g_i^*], \quad i = 1, \cdots, 2n - k, \tag{5}$$

$$B_i^+(f) = [f, g_i^+], \quad i = 1, \cdots, k, \tag{6}$$

and $C_i(f) = [f, h_i]$, $i = 1, \cdots, m$, for all $f \in H^{2n}[a, b]$. In terms of the inner product $[f, g]$ on $H^{2n}[a, b]$ we have

$$\mathcal{D}(T) = \langle g_1^*, \cdots, g_{2n-k}^*, g_1^+, \cdots, g_k^+\rangle^\perp \subset \mathcal{D}(T^*) = \langle h_1, \cdots, h_m\rangle^\perp,$$

and hence, taking orthogonal complements we get

$$\langle h_1, \cdots, h_m\rangle \subset \langle g_1^*, \cdots, g_{2n-k}^*, g_1^+, \cdots, g_k^+\rangle. \tag{7}$$

But $N \leqslant 2n \leqslant m$, and the inclusion in (7) implies these two subspaces are equal. Therefore, $N = m = 2n$, $\mathcal{D}(T) = \mathcal{D}(T^*)$, $T = T^*$, and the boundary values $B_1^*, \cdots, B_{2n-k}^*, B_1^+, \cdots, B_k^+$ are linearly independent. We summarize these results as a theorem, together with some other elementary properties of T.

THEOREM 1. *The $2n$th order differential operator T is selfadjoint with $\mathcal{N}(T) = \mathcal{N}(L^*)$ and $\mathcal{R}(T) = \mathcal{R}(L)$. Moreover, the operator τ^* maps $\mathcal{D}(T)$ onto $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$.*

REMARK 1. Since L and L^* are closed densely defined linear operators in S, it is well known from functional analysis that LL^* is a positive selfadjoint linear operator in S [1, p. 1245]. The operator LL^* is precisely our differential operator T. We have elected to give a detailed discussion of T for two reasons: (a) the discussion is simple and natural, and (b) it emphasizes the structure of T as a differential operator. Henceforth, the differential operator T is denoted by LL^*.

REMARK 2. The functions $g_1^*, \cdots, g_{2n-k}^*$ and $g_1^+, \cdots, g_k^+$ which represent the boundary values $B_1^*, \cdots, B_{2n-k}^*$ and $B_1^+, \cdots, B_k^+$ in equations (5) and (6) can be explicitly calculated using equation (4) and Theorem 3 [3, p. 62]. This is very important for computational considerations.

4. The approximation scheme. To construct a sequence of functions which converges to the least squares solution $f_0 = L^\dagger g_0$ of equation (1), we work in the spaces $H^{2n}[a, b]$ and $H^n[a, b]$ under their strong topologies. Note that $\mathcal{D}(LL^*)$ and $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$ are closed subspaces in $H^{2n}[a, b]$ and $H^n[a, b]$ under these topologies, respectively.

Clearly the operator L is continuous from the induced strong topology on $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$ to the induced L^2-topology on $\mathcal{R}(L)$, and hence, there exists a constant $\gamma > 0$ such that

$$|Hf|_* \leq \gamma \|f\| \quad \text{for all } f \in \mathcal{R}(L). \tag{8}$$

Utilizing the inner product $[f, g]$ on $H^{2n}[a, b]$, let R be the orthogonal projection from $H^{2n}[a, b]$ onto the subspace $\langle g_1^*, \cdots, g_{2n-k}^*, g_1^+, \cdots, g_k^+\rangle$. The various operators are shown below schematically:

$$H^{2n}[a, b] \xrightarrow{I-R} \mathcal{D}(LL^*) \xrightarrow{\tau^*} \mathcal{D}(L) \cap \mathcal{N}(L)^\perp \xrightarrow{\tau} \mathcal{R}(L).$$

Choose a linearly independent sequence of functions ρ_i $(i = 1, 2, \cdots)$ in $H^{2n}[a, b]$ such that the subspace $\langle \rho_1, \rho_2, \cdots \rangle$ is dense in $H^{2n}[a, b]$ under the strong topology. For example, we can use $\rho_i(t) = t^{i-1}$ for $i = 1, 2, \cdots$ (see [3, pp. 60–61]). Let $\varphi_i = \rho_i - R\rho_i$, $\xi_i = \tau^*\varphi_i$, and $\eta_i = \tau\xi_i$ for $i = 1, 2, \cdots$. Clearly $\varphi_i \in \mathcal{D}(LL^*)$ with $LL^*\varphi_i = \eta_i$, and $\xi_i \in \mathcal{D}(L) \cap \mathcal{N}(L)^\perp$ and $\eta_i \in \mathcal{R}(L)$ with $L\xi_i = \eta_i$ and $H\eta_i = \xi_i$ for $i = 1, 2, \cdots$.

REMARK 3. With no loss of generality we can assume that the sequence ξ_i $(i = 1, 2, \cdots)$ is linearly independent, for otherwise we can pass to an appropriate linearly independent subsequence having the same linear span. The sequence η_i $(i = 1, 2, \cdots)$ is also linearly independent since τ is 1-1 on $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$.

The operator $I - R$ maps $H^{2n}[a, b]$ onto $\mathcal{D}(LL^*)$, and it is continuous under the strong topology on $H^{2n}[a, b]$. Consequently, the subspace $\langle \varphi_1, \varphi_2, \cdots \rangle$ is dense in $\mathcal{D}(LL^*)$ under the induced strong topology from $H^{2n}[a, b]$.

Similarly, the subspace $\langle \xi_1, \xi_2, \cdots \rangle$ is dense in $\mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}$ under the induced strong topology from $H^n[a, b]$, and the subspace $\langle \eta_1, \eta_2, \cdots \rangle$ is dense in $\mathcal{R}(L)$ under the induced L^2-topology from S.

For $i = 1, 2, \cdots$ let P_i be the L^2-orthogonal projection from S onto the subspace $\langle \eta_1, \cdots, \eta_i \rangle$. Clearly $P_i g_0 = P_i h_0$ and $(g_0, \eta_i) = (h_0, \eta_i)$ for $i = 1, 2, \cdots$, and from the above discussion we have

$$h_0 = \lim_i P_i h_0. \tag{9}$$

Now L and H are isomorphisms between the subspaces $\langle \xi_1, \cdots, \xi_i \rangle$ and $\langle \eta_1, \cdots, \eta_i \rangle$, and hence, for $i = 1, 2, \cdots$ the equation $Lf = P_i h_0$ has the unique solution $f_i = HP_i h_0$ belonging to $\langle \xi_1, \cdots, \xi_i \rangle$. Using the continuity of H with equation (9), we conclude that $f_i \xrightarrow{s_n} Hh_0 = f_0$, and in fact, from equation (8) we get the error estimate

$$|f_i - f_0|_* \leqslant \gamma \| P_i h_0 - h_0 \| \quad \text{for } i = 1, 2, \cdots. \tag{10}$$

Proceeding as in [3], we can show that if we write f_i in the form

$$f_i = \sum_{j=1}^{i} a_j^i \xi_j, \tag{11}$$

then the coefficients $a_1^i, \cdots, a_i^i$ form the unique solution of the linear system

$$\sum_{j=1}^{i} (L\xi_j, L\xi_l) a_j^i = (g_0, L\xi_l), \qquad l = 1, \cdots, i. \tag{12}$$

We summarize these results as a theorem and several corollaries.

THEOREM 2. *Let* h_0 *and* k_0 *be the* L^2*-orthogonal projections of* g_0 *on* $\mathcal{R}(L)$ *and* $\mathcal{N}(L^*)$, *respectively, and let the sequence of functions* ξ_i $(i = 1, 2, \cdots)$ *be constructed as above. Then for* $i = 1, 2, \cdots$ *the linear system*

$$\sum_{j=1}^{i} (L\xi_j, L\xi_l) a_j^i = (g_0, L\xi_l), \qquad l = 1, \cdots, i,$$

has a unique solution $a_1^i, \cdots, a_i^i$, *and the sequence of functions* $f_i = \Sigma_{j=1}^i a_j^i \xi_j$ $(i = 1, 2, \cdots)$ *converges in the strong topology on* $H^n[a, b]$ *to the least squares solution* $f_0 = L^\dagger g_0 = Hh_0$ *of the boundary value problem* (1) *having minimal norm. Moreover, the rate of convergence is determined by equation* (10).

COROLLARY 1. *If the boundary value problem* (1) *is solvable, then* $f_0 = L^\dagger g_0$ *is a solution, and* $|f_i - f_0|_* \leqslant \gamma \| P_i g_0 - g_0 \|$ *for* $i = 1, 2, \cdots$ *with* $\| P_i g_0 - g_0 \| \longrightarrow 0$ *as* $i \longrightarrow \infty$.

COROLLARY 2. *If the boundary value problem* (1) *is not solvable, then* $f_0 = L^\dagger g_0$ *is a solution of the boundary value problem* (2), *and* $\|P_i g_0 - g_0\| \geqslant \|k_0\| > 0$ *for* $i = 1, 2, \cdots$.

REMARK 4. Each step needed in determining the functions f_i can actually be computed. Also, the question of the solvability of the boundary value problem (1) can be answered practically by determining whether $\|P_i g_0 - g_0\| \longrightarrow 0$.

Special case. We conclude this paper by looking at the special form of the approximation scheme when the eigenfunctions of the selfadjoint differential operator LL^* are utilized. Let $q = \dim N(LL^*)$, and choose an L^2-orthonormal basis $\omega_{01}, \cdots, \omega_{0q}$ for $N(LL^*) = N(L^*)$.

Consider the operator $H_0 = [LL^* | \mathcal{D}(LL^*) \cap N(LL^*)^\perp]^{-1}$. We know that H_0 is a right inverse for LL^* with domain $R(LL^*) = R(L)$ and range $\mathcal{D}(LL^*) \cap N(LL^*)^\perp$ contained in $R(L)$, that H_0 is selfadjoint and completely continuous on $R(L)$ under its L^2-structure, and that H_0 is continuous from the induced L^2-topology on $R(L)$ to the induced strong topology on $\mathcal{D}(LL^*) \cap N(LL^*)^\perp$ from $H^{2n}[a, b]$.

Choose an L^2-orthonormal basis ω_i $(i = 1, 2, \cdots)$ for $R(L)$ consisting of eigenfunctions for H_0, and let μ_i $(i = 1, 2, \cdots)$ be the corresponding sequence of eigenvalues. Setting $\lambda_i = 1/\mu_i$ for $i = 1, 2, \cdots$, we have that $\omega_i \in \mathcal{D}(LL^*) \cap N(LL^*)^\perp$, $LL^*\omega_i = \lambda_i \omega_i$, and $\lambda_i > 0$ for $i = 1, 2, \cdots$ with $\lambda_i \longrightarrow \infty$. Thus, the sequence of functions $\omega_{01}, \cdots, \omega_{0q}, \omega_1, \omega_2, \cdots$ belongs to $\mathcal{D}(LL^*)$ and forms an L^2-orthonormal basis for S.

If $f \in \mathcal{D}(LL^*)$, then in terms of L^2-convergence we have

$$LL^*f = \sum_{i=1}^{\infty} (LL^*f, \omega_i)\omega_i = \sum_{i=1}^{\infty} \lambda_i (f, \omega_i)\omega_i, \tag{13}$$

and hence, applying H_0 we conclude that

$$f = \sum_{i=1}^{q} (f, \omega_{0i})\omega_{0i} + \sum_{i=1}^{\infty} (f, \omega_i)\omega_i \tag{14}$$

for all $f \in \mathcal{D}(LL^*)$, where the convergence in equation (14) is strong convergence in $H^{2n}[a, b]$. This implies that the subspace $\langle \omega_1, \omega_2, \cdots \rangle$ is dense in $\mathcal{D}(LL^*) \cap N(LL^*)^\perp$ under the induced strong topology from $H^{2n}[a, b]$.

If we let $\xi_i = \tau^* \omega_i$ and $\eta_i = \tau \xi_i = \lambda_i \omega_i$ for $i = 1, 2, \cdots$, then our earlier discussion can be modified to yield a new approximation scheme. In particular, equation (12) takes the simplified form

$$\sum_{j=1}^{i} \lambda_j \lambda_l (\omega_j, \omega_l) a_j^i = \lambda_l (g_0, \omega_l), \qquad l = 1, \cdots, i,$$

so $a_j^i = (1/\lambda_j)(g_0, \omega_j)$ for $j = 1, \cdots, i$. We obtain the following theorem.

THEOREM 3. *Let ω_i $(i = 1, 2, \cdots)$ be a sequence of eigenfunctions for the selfadjoint differential operator LL^* which forms an L^2-orthonormal basis for $R(LL^*) = R(L)$, and let λ_i $(i = 1, 2, \cdots)$ be the corresponding sequence of eigenvalues. Then the sequence of functions*

$$f_i = \sum_{j=1}^{i} \left(\frac{1}{\lambda_j}\right)(g_0, \omega_j)\tau^*\omega_j, \qquad i = 1, 2, \cdots,$$

converges in the strong topology on $H^n[a, b]$ to $f_0 = L^\dagger g_0 = Hh_0$ with

$$|f_i - f_0|_* \leq \gamma \left\| \sum_{j=1}^{i} (h_0, \omega_j)\omega_j - h_0 \right\|, \quad i = 1, 2, \cdots.$$

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators*. I, II, Pure and Appl. Math., vol. 7, Interscience, New York, 1958, 1963. MR 22 #8302; 32 #6181.

2. J. Locker, *An existence analysis for nonlinear boundary value problems*, SIAM J. Appl. Math. 19 (1970), 199–207. MR 42 #578.

3. ———, *The method of least squares for boundary value problems*, Trans. Amer. Math. Soc. 154 (1971), 57–68. MR 43 #7077.

4. W. S. Loud, *Some examples of generalized Green's functions and generalized Green's matrices*, SIAM Rev. 12 (1970), 194–210. MR 41 #3865.

5. M. Z. Nashed, *Generalized inverses, normal solvability, and iteration for singular operator equations*, Nonlinear Functional Anal. and Appl. (Proc. Advanced Sem., Math. Res. Center, Univ. of Wisconsin, Madison, Wis., 1970), Academic Press, New York, 1971, pp. 311–359. MR 43 #1003.

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, COLORADO 80521