Nonlinear Functional Analysis and Applications
Vol. 21, No. 3 (2016), pp. 513-522

http://nfaa.kyungnam.ac.kr/journal-nfaa

ON CONSTRUCTION OF FIXED POINT THEORY UNDER IMPLICIT RELATION IN HILBERT SPACES

M. Pitchaimani[1] **and D. Ramesh Kumar**[2]

[1]Ramanujan Institute For Advanced Study In Mathematics
University of Madras, Chennai, India
e-mail: mpitchaimani@yahoo.com

[2]Ramanujan Institute For Advanced Study In Mathematics
University of Madras, Chennai, India
e-mail: rameshkumard14@gmail.com

Abstract. The aim of this paper is to study existence and uniqueness of common fixed point for a family of self-mappings satisfying implicit relation in a Hilbert space. We also prove well-posedness of a common fixed point problem.

1. Introduction

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. This principal is constructive in nature and is one of the most useful techniques in the study of nonlinear equations. Different generalizations of the Banach's contraction mapping principle were studied by many authors in metric spaces and Banach spaces, see [1]-[8], [12, 13, 15, 16, 19] and references given therein. These generalizations were made either by using the contractive condition or by imposing some additional conditions on an ambient space. Koparde and Waghmode [11], Pandhare [14], Veerapandi and Anil Kumar [20] investigated the properties of fixed points

[0]Received January 25, 2016. Revised April 1, 2016.
[0]2010 Mathematics Subject Classification: 46C15, 47H10, 58C30.
[0]Keywords: Implicit relation, common fixed point, well-posedness, Hilbert space.

of sequence of mappings under contraction condition in Hilbert spaces. In [2, 3, 4], the authors studied implicit relation in metric spaces.

In spite of the above work, study of contraction condition through implicit relation in Hilbert spaces needs more investigation. In continuation of these results, in this paper, we study common fixed point theorems for a family of self mappings in Hilbert space by applying implicit relation of three dimension and then well-posed problem.

2. Preliminaries

Definition 2.1. A sequence $\{x_n\}$ in a Hilbert space X is said to be asymptotically T-regular if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Definition 2.2. Let X be a Hilbert space and T be a self mapping on X. Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.3. (Implicit Relation)
Let Φ be the class of real valued continuous functions $\phi : \mathbb{R}_+^3 \to \mathbb{R}_+$ nondecreasing in the first argument and satisfying the following condition: for $x, y > 0$,

(i) $x \leq \phi\left(y, \dfrac{x+y}{2}, x+y\right)$,

or

(ii) $x \leq \phi(x, 0, x)$,

there exists a real number $0 < h < 1$ such that $x \leq hy$.

The following are examples of the implicit relation defined above.

Example 2.4. Let $\phi(t_1, t_2, t_3) = t_1 - \alpha \max(t_2, t_3) + (2+\alpha)t_3$, where $\alpha \geq 0$.

Example 2.5. Let $\phi(t_1, t_2, t_3) = t_1 + a \min\left(t_2, \dfrac{t_3}{2}\right)$, where $a \geq 2$.

Example 2.6. Let $\phi(t_1, t_2, t_3) = t_1 + c \max(t_2, t_3)$, where $\alpha \geq 1$.

Definition 2.7. Let X be a nonempty set and $S, T : X \to X$ be two self mappings. An element $x \in X$ is called a common fixed point of S and T, if $x = S(x) = T(x)$.

3. Main results

Saluja [19] studied the concept of implicit relation for obtaining common fixed point for two continuous self mappings in 2-Banach spaces. We formulate it in Hilbert spaces and further we extend the result as follows.

Theorem 3.1. *Let C be a closed subset of a Hilbert space X and $S, T : C \to C$ be two continuous self mappings such that*

$$\|Sx - Ty\|^2 \leq \phi\Big(\|x-y\|^2, \frac{\|x-Sx\|^2 + \|y-Ty\|^2}{2}, \frac{\|x-Ty\|^2 + \|y-Sx\|^2}{2}\Big), \tag{3.1}$$

for all $x, y \in C$. Then S and T have a unique common fixed point in C.

Proof. Let $x_0 \in C$ and we choose $x_1, x_2 \in C$ in such a way that $x_1 = Sx_0$ and $x_2 = Tx_1$. In general, we get a sequence in C such that

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, \ldots.$$

By using (3.1) and parallelogram law, we get

$$\begin{aligned}\|x_{2n+1} - x_{2n}\|^2 &= \|Sx_{2n} - Tx_{2n-1}\|^2 \\ &\leq \phi\Big(\|x_{2n} - x_{2n-1}\|^2, \frac{\|x_{2n} - Sx_{2n}\|^2 + \|x_{2n-1} - Tx_{2n-1}\|^2}{2}, \\ &\qquad \frac{\|x_{2n} - Tx_{2n-1}\|^2 + \|x_{2n-1} - Sx_{2n}\|^2}{2}\Big) \\ &\leq \phi\Big(\|x_{2n} - x_{2n-1}\|^2, \frac{\|x_{2n} - x_{2n+1}\|^2 + \|x_{2n-1} - x_{2n}\|^2}{2}, \\ &\qquad \|x_{2n} - x_{2n}\|^2 + \|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2\Big) \\ &= \phi\Big(\|x_{2n} - x_{2n-1}\|^2, \frac{\|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2}{2}, \\ &\qquad \|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2\Big).\end{aligned}$$

Hence from Definition 2.3 (i), there exists $0 < h < 1$ such that

$$\|x_{2n+1} - x_{2n}\|^2 \leq h\|x_{2n} - x_{2n-1}\|^2.$$

Proceeding in this way, finally we obtain

$$\|x_{2n+1} - x_{2n}\|^2 \leq h^n\|x_0 - x_1\|^2, \; n = 1, 2, \ldots.$$

For any positive integer p, we get

$$\begin{aligned}\|x_n - x_{n+p}\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \cdots + \|x_{n+p-1} - x_{n+p}\| \\ &\leq \left(h^n + h^{n+1} + \cdots + h^{n+p-1}\right)\|x_0 - x_1\| \\ &\leq \frac{h^n}{1-h}\|x_0 - x_1\|.\end{aligned}$$

Since $0 < h < 1$, $\dfrac{h^n}{1-h} \to 0$ as $n \to \infty$. Therefore, $\|x_n - x_{n+p}\| \to 0$ as $n \to \infty$. Since C is closed, there exists an element $u \in C$ such that $x_n \to u$ as $n \to \infty$. By the continuity of S and T, it is clear that $Su = Tu = u$. Hence u is a common fixed point of S and T.

For the uniqueness, let $v \in C$ be another common fixed point of S and T where $v \neq u$. Then

$$\begin{aligned}\|u - v\|^2 &= \|Su - Tu\|^2 \\ &\leq \phi\left(\|u - v\|^2, \frac{\|u - Su\|^2 + \|v - Tv\|^2}{2}, \frac{\|u - Tv\|^2 + \|v - Su\|^2}{2}\right) \\ &= \phi\left(\|u - v\|^2, 0, \|u - v\|^2\right).\end{aligned}$$

Now, by the Definition 2.3 (ii), we have

$$\|u - v\|^2 \leq h\|u - v\|^2, \text{ where } 0 < h < 1.$$

Thus $u = v$ which shows that u is a unique common fixed point of S and T. □

The following corollary is a particular case of Theorem 3.1, when $S = T$.

Corollary 3.2. *Let C be a closed subset of a Hilbert space X and $T : C \to C$ be a continuous self mapping such that*

$$\|Tx - Ty\|^2 \leq \phi\left(\|x - y\|^2, \frac{\|x - Tx\|^2 + \|y - Ty\|^2}{2}, \frac{\|x - Ty\|^2 + \|y - Tx\|^2}{2}\right), \tag{3.2}$$

for all $x, y \in C$. Then T has a unique fixed point in C.

Proof. The proof follows from the Theorem 3.1, if we take $S = T$. □

Theorem 3.3. *Let C be a closed subset of a Hilbert space X and $T : C \to C$ be a self mapping satisfying* (3.2). *Then T has a unique fixed point in C and*

$\{z_n\}$ is asymptotically T-regular if and only if T is continuous at the fixed point of T.

Proof. Let $v \in C$ be a fixed point of T and $z_n \to v$ as $n \to \infty$. Now

$$\begin{aligned}&\|Tz_n - Tv\|^2\\ &\leq \phi\bigg(\|z_n - v\|^2, \frac{\|z_n - Tz_n\|^2 + \|v - Tv\|^2}{2}, \frac{\|z_n - Tv\|^2 + \|v - Tz_n\|^2}{2}\bigg).\end{aligned}$$

Since $Tv = v$ and $\{z_n\}$ is asymptotically T-regular, we get

$$\|Tz_n - Tv\|^2 \quad \leq \quad \phi\Big(\|Tz_n - Tv\|^2, 0, \|Tz_n - Tv\|^2\Big).$$

By the Definition 2.3 (ii), there exists $0 < h < 1$ such that

$$\|Tz_n - Tv\|^2 \leq h\|Tz_n - Tv\|^2$$

which yields that

$$Tz_n \to Tv \quad \text{as} \quad n \to \infty.$$

Hence T is continuous at $v \in X$. Conversely, assume that T is continuous at $v \in X$. Note that, by Corollary 3.2, T has a unique fixed point $v \in C$ and

$$z_n \to v \ \Rightarrow \ Tz_n \to Tv \quad \text{as} \quad n \to \infty$$

which shows that

$$\|z_n - Tz_n\|^2 \quad \to \quad \|v - Tv\|^2 = 0,$$

since $Tv = v$. This completes the proof. □

Now we extend the Theorem 3.1 to the case of pair of mappings S^p and T^q where p and q are some positive integers.

Theorem 3.4. *Let C be a closed subset of a Hilbert space X and $S, T : C \to C$ be two continuous self mappings such that*

$$\begin{aligned}\|S^p x - T^q y\|^2 \quad \leq \quad \phi\bigg(\|x - y\|^2, \frac{\|x - S^p x\|^2 + \|y - T^q y\|^2}{2},\\ \frac{\|x - T^q y\|^2 + \|y - S^p x\|^2}{2}\bigg),\end{aligned} \tag{3.3}$$

for all $x, y \in X$ where p and q are some positive integers. Then S and T have a unique common fixed point.

Proof. Since S^p and T^q satisfy the conditions of Theorem 3.1, S^p and T^q have a unique common fixed point. Let v be the common fixed point. Now

$$\begin{aligned}S^p v = v \quad &\Rightarrow \quad S(S^p v) = Sv,\\ S^p(Sv) \quad &= \quad Sv.\end{aligned}$$

If $Sv = x_0$ then $S^p(x_0) = x_0$. So, Sv is a fixed point of S^p. Similarly, $T^q(Tv) = Tv$. Now, we have

$$\begin{aligned} \|v - Tv\|^2 &= \|S^p v - T^q(Tv)\|^2 \\ &\leq \phi\bigg(\|v - Tv\|^2, \frac{\|v - S^p v\|^2 + \|Tv - T^q(Tv)\|^2}{2}, \\ &\quad \frac{\|v - T^q(Tv)\|^2 + \|Tv - S^p v\|^2}{2} \bigg) \\ &= \phi\bigg(\|v - Tv\|^2, 0, \|v - Tv\|^2 \bigg). \end{aligned}$$

Hence by the Definition 2.3 (ii), we obtain

$$\|v - Tv\|^2 \leq 0.$$

Thus $v = Tv$ for $v \in C$. Similarly, $v = Sv$.

Finally, for the uniqueness of v, let $w \neq v$ be another common fixed point of S and T. Then clearly w is also a common fixed point of S^p and T^q which implies $w = v$. Hence S and T have a unique common fixed point $v \in X$. □

Hence we have proved that if x_0 is a unique common fixed point of S^p and T^q, for some positive integers p, q then x_0 is a unique common fixed point of S and T. Further, we extend Theorem 3.1 to the case of family of mappings satisfying the condition (3.1).

Theorem 3.5. *Let C be a closed subset of a Hilbert space X and $\{F_\alpha\}$ be a family of continuous self mappings on C satisfying*

$$\begin{aligned} &\|F_\alpha x - F_\beta y\|^2 \\ &\leq \phi\bigg(\|x - y\|^2, \frac{\|x - F_\alpha x\|^2 + \|y - F_\beta y\|^2}{2}, \frac{\|x - F_\beta y\|^2 + \|y - F_\alpha x\|^2}{2} \bigg), \end{aligned}$$

for $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ and $x, y \in C$. Then there exists a unique $v \in C$ satisfying $F_\alpha v = v$ for all $\alpha \in \Lambda$.

Proof. Let us take F_α and F_β in place of S and T respectively in Theorem 3.1, an application of which gives a unique $v \in C$ to satisfy $F_\alpha v = F_\beta v = v$. For any other member F_γ, uniqueness of v gives $F_\gamma v = v$ and this completes the proof. □

Theorem 3.6. *Let C be a closed subset of a Hilbert space X and $\{F_n\}$ be a sequence of self mappings on C such that $\{F_n\}$ converging pointwise to a self* *mapping F and*

$$\|F_nx - F_ny\|^2 \leq \phi\Big(\|x-y\|^2, \frac{\|x-F_nx\|^2 + \|y-F_ny\|^2}{2}, \frac{\|x-F_ny\|^2 + \|y-F_nx\|^2}{2}\Big),$$

for all $x, y \in X$. If F_n has a fixed point v_n and F has a fixed point v. Then the sequence $\{v_n\}$ converges to v.

Proof. Note that $F_nv_n = v_n$ and $Fv = v$. Now consider

$$\begin{aligned}\|v-v_n\|^2 &= \|Fv - F_nv_n\|^2 \\ &\leq \|Fv - F_nv\|^2 + \|F_nv - F_nv_n\|^2 + 2Re\langle Fv - F_nv, F_nv - F_nv_n\rangle \\ &\leq \|Fv - F_nv\|^2 + \phi\Big(\|v-v_n\|^2, \frac{\|v-F_nv\|^2 + \|v_n - F_nv_n\|^2}{2}, \\ &\quad \frac{\|v-F_nv_n\|^2 + \|v_n - F_nv\|^2}{2}\Big) + 2Re\langle Fv - F_nv, F_nv - F_nv_n\rangle.\end{aligned}$$

By the fact that $F_nv \to Fv$ as $n \to \infty$, we get

$$\|v-v_n\|^2 \leq \phi\Big(\|v-v_n\|^2, 0, \|v-v_n\|^2\Big).$$

Hence, by Definition 2.3 (ii), we obtain

$$\|v-v_n\|^2 \leq 0,$$

which implies that $v_n \to v$ as $n \to \infty$. □

Remark 3.7. Theorem 3.5 generalizes the concept of common fixed point for continuous self mappings under implicit relation in Hilbert spaces. In Theorem 3.6, convergence of sequence of self mappings to another self mapping implies convergence of corresponding sequence of fixed points. Note that, continuity of mappings is not necessary in Theorem 3.6.

4. Well-Posedness

The study of well-posedness of a fixed point problem has been active area of research and several mathematicians investigated this interesting concept, see [1, 2, 9, 17, 18]. In this section, we prove well-posedness of common fixed point problem of mappings in Theorem 3.1.

Definition 4.1. Let X be a Hilbert space and $f : X \to X$ be a self mapping. The fixed point problem of f is said to be well-posed if

(i) f has a unique fixed point $x_0 \in X$,

(ii) for any sequence $\{x_n\} \subset X$ and $\lim_{n\to\infty} \|x_n - fx_n\| = 0$, we have $\lim_{n\to\infty} \|x_n - x_0\| = 0$.

Let $CFP(T, f, X)$ denote a common fixed point problem of self mappings T and f on X and $CF(T, f)$ denote the set of all common fixed points of T and f.

Definition 4.2. $CFP(T, f, X)$ is called well-posed if $CF(T, f)$ is singleton and for any sequence $\{x_n\}$ in X with

$$\tilde{x} \in CF(T, f) \quad \text{and} \quad \lim_{n\to\infty} \|x_n - fx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

implies $\tilde{x} = \lim_{n\to\infty} x_n$.

Theorem 4.3. *Let C be a closed subset of a Hilbert space X, T and f be continuous self mappings on C as in* Theorem 3.1. *Then the common fixed problem of f and T is well posed.*

Proof. From Theorem 3.1, the mappings T and f have a unique common fixed point, say $v \in X$. Let $\{x_n\}$ be a sequence in X and $\lim_{n\to\infty} \|fx_n - x_n\| = \lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Without loss of generality, assume that $v \neq x_n$ for any non-negative integer n. Using $fv = Tv = v$, we get

$$\begin{aligned}
\|v - x_n\|^2 &= \|Tv - Tx_n + Tx_n - x_n\|^2 \\
&\leq \|Tv - Tx_n\|^2 + \|Tx_n - x_n\|^2 + 2Re\langle Tv - Tx_n, Tx_n - x_n\rangle \\
&\leq \|Tx_n - x_n\|^2 + \phi\bigg(\|v - x_n\|^2, \frac{\|v - Tv\|^2 + \|x_n - Tx_n\|^2}{2}, \\
&\qquad \frac{\|v - Tx_n\|^2 + \|x_n - Tv\|^2}{2}\bigg) + 2Re\langle Tv - Tx_n, Tx_n - x_n\rangle \\
&= \|Tx_n - x_n\|^2 + \phi(\|v - x_n\|^2, 0, \|v - x_n\|^2) \\
&\qquad + 2Re\langle Tv - Tx_n, Tx_n - x_n\rangle.
\end{aligned}$$

By the definition of implicit relation and letting $n \to \infty$, we obtain $\|v - x_n\| \to 0$. This completes the proof. □

5. Conclusion and discussion

In this paper, we investigated the behaviour of a family of self mappings under implicit relation of three dimension and well-posedness of common fixed point problem in Hilbert space. In this connection, we obtained a common fixed point theorem for the family of self mappings using the property of continuity, since continuity plays a significant role. Further, this work is a generalization of the study made by Koparde and Waghmode [11], Pandhare [14], Veerapandi and Anil Kumar [20] and Saluja [19]. Now, it makes interesting to consider for future investigation

- Does Theorem 3.5 hold for a family of non-continuous mappings?
- Is it possible to get a point of coincidence for two or family of self mappings under implicit relation in Hilbert space?

References

[1] M. Akkouchi, *Well posedness of the fixed point problem for certain asymptotically regular mappings*, Annalas Mathematicae Silesianae, **23** (2009), 43–52.

[2] M. Akkouchi and V. Popa, *Well posedness of the fixed point problem for mappings satisfying an implicit relations*, Demonstratio Math., **43**(4) (2010), 923–929.

[3] I. Beg and A.R. Butt, *Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces*, Nonlinear Anal., **71**(9) (2009), 3699–3704.

[4] I. Beg and A.R. Butt, *Fixed points for weakly compatible mappings satisfying an implicit relation in partially ordered metric spaces*, Carpathian J. Math., **25**(1) (2009), 1–12.

[5] F.E. Browder and W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl., **20** (1967), 197–228.

[6] Lj.B. Ćirić, *A generalization of Banachs contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273.

[7] Lj.B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Beograd), **12**(26) (1971), 19–26.

[8] Y. Cui and J. Sun, *Fixed point theorems for a class of nonlinear operators in Hilbert spaces and applications*, Positivity, **15** (2011), 455–464.

[9] B.K. Lahiri, *Well-Posedness and certain classes of operators*, Demonstratio Math., **38** (2005), 170–176.

[10] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60** (1968), 71–76.

[11] P.V. Koparde and B.B. Waghmode, *On sequence of mappings in Hilbert space*, The Mathematics Education, **25**(4) (1991), 197–198.

[12] J.J. Nieto, R.L. Pouso and R. Rodríguez-López, *Fixed point theorems in ordered abstract spaces*, Proc. Am. Math. Soc., **135** (2007), 2505–2517.

[13] J.J. Nieto and R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin., **23** (2007), 2205–2212.

[14] D.M. Pandhare, *On the sequence of mappings on Hilbert space*, The Mathematics Education, **32**(2) (1998), 61–63.

[15] M. Pitchaimani and D. Ramesh kumar, *Common and coincidence fixed point theorems for asymptotically regular mappings in 2-Banach Space*, Nonlinear Func. Anal. Appl., **21**(1) (2016), 131–144.

[16] M. Pitchaimani and D. Ramesh kumar, *Some common fixed point theorems using implicit relation in 2-Banach spaces*, Surveys in Mathematics and its Applications, **10** (2015), 159–168.

[17] V. Popa, *Well-Posedness of fixed problem in compact metric space*, Bull. Univ. Petrol-Gaze, Ploicsti, sec. Mat Inform. Fiz., **60**(1) (2008), 1–4.

[18] S. Reich and A.T. Zaslawski, *Well-Posedness of fixed point problems*, Far East J. Math. Sci, Special volume part **III** (2011), 393–401.

[19] G.S. Saluja, *On common fixed point theorems under implicit relation in 2-Banach spaces*, Nonlinear Func. Anal. Appl., **20**(2) (2015), 215–221.

[20] T. Veerapandi and S. Anil Kumar, *Common fixed point theorems of a sequence of mappings on Hilbert space*, Bull.Cal. Math. Soc., **91**(4) (1999), 299–308.