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ON CONTACT METRIC MANIFOLDS

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1. Introduction. Blair has proven ([1]) that there are no contact metric manifolds of vanishing curvature and of dimension $\geqq 5$. Generalizing this result we prove in the present paper that any contact metric manifold of constant sectional curvature and of dimension $\geqq 5$ has the sectional curvature equal to 1 and is a Sasakian manifold. Moreover we give some restrictions on the scalar curvature in contact metric manifolds which are conformally flat or of constant ϕ-sectional curvature.

2. Preliminaries. Throughout this paper we use the notations and terminology of [1], [2].

Let M be a $(2n+1)$-dimensional contact metric manifold and (ϕ, ξ, η, g) be its contact metric structure. Thus, we have

$$\phi^2 = -I + \eta \otimes \xi\,, \quad \phi\xi = 0\,, \quad \eta\circ\phi = 0\,, \quad \eta(\xi) = 1\,,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\,, \quad g(X, \xi) = \eta(X)\,,$$
$$\Phi(X, Y) = g(X, \phi Y) = d\eta(X, Y)\,.$$

Define an operator h by $h = (1/2)\mathscr{L}_\xi\phi$, where $\mathscr{L}$ denotes Lie differentiation. The vector field ξ is Killing if and only if h vanishes. Blair has shown ([1], [2]) that h and ϕh are symmetric operators, h anti-commutes with ϕ (i.e., $\phi h + h\phi = 0$), $h\xi = 0$ and $\eta\circ h = 0$. Using a ϕ-basis, i.e., an orthonormal frame $\{E_i, E_{i+n} = \phi E_i, E_{2n+1} = \xi\}$ $(i = 1, \cdots, n)$, one can easily verify that $\operatorname{tr} h = 0$ and $\operatorname{tr}\phi h = 0$. In [1], [2] the following general formulas for a contact metric manifold were obtained

$$\nabla_X\xi = -\phi X - \phi h X\,, \tag{2.1}$$

$$(1/2)(R_{\xi X}\xi - \phi R_{\xi\phi X}\xi) = h^2X + \phi^2X\,, \tag{2.2}$$

$$g(Q\xi, \xi) = 2n - \operatorname{tr} h^2\,, \tag{2.3}$$

where $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature transformation and Q is the Ricci curvature operator. Finally, we note that $d\Phi = d^2\eta = 0$ implies

$$(\nabla_X\Phi)(Y, Z) + (\nabla_Y\Phi)(Z, X) + (\nabla_Z\Phi)(X, Y) = 0\,. \tag{2.4}$$

3. Auxiliary results. First of all, using (2.1) we can easily obtain the following relations

$$(3.1)\qquad (\nabla_X\Phi)(\phi Y, Z) - (\nabla_X\Phi)(Y, \phi Z) = -\eta(Y)g(X + hX, \phi Z) - \eta(Z)g(X + hX, \phi Y)\,,$$

$$(3.2)\qquad (\nabla_X\Phi)(\phi Y, \phi Z) + (\nabla_X\Phi)(Y, Z) = \eta(Y)g(X + hX, Z) - \eta(Z)g(X + hX, Y)\,.$$

LEMMA 3.1. *Any contact metric manifold satisfies the conditions*

$$(3.3)\qquad (\nabla_{\phi X}\phi)\phi Y + (\nabla_X\phi)Y = 2g(X, Y)\xi - \eta(Y)(X + hX + \eta(X)\xi)\,,$$

$$(3.4)(\mathrm{a})\qquad \sum_{i=1}^{2n+1}(\nabla_{E_i}\phi)E_i = 2n\xi\,,$$

$$(3.4)(\mathrm{b})\qquad \sum_{i=1}^{2n+1}(\nabla_{E_i}\phi)\phi E_i = 0\,,$$

where $\{E_1, \cdots, E_{2n+1}\}$ *is an orthonormal frame.*

PROOF. Making use of the identity (2.4) we can write

$$\begin{aligned}&(\nabla_X\Phi)(Y, Z) + (\nabla_Y\Phi)(Z, X) + (\nabla_Z\Phi)(X, Y)\\ &+ (\nabla_{\phi X}\Phi)(\phi Y, Z) + (\nabla_{\phi Y}\Phi)(Z, \phi X) + (\nabla_Z\Phi)(\phi X, \phi Y)\\ &+ (\nabla_{\phi X}\Phi)(Y, \phi Z) + (\nabla_Y\Phi)(\phi Z, \phi X) + (\nabla_{\phi Z}\Phi)(\phi X, Y)\\ &- (\nabla_X\Phi)(\phi Y, \phi Z) - (\nabla_{\phi Y}\Phi)(\phi Z, X) - (\nabla_{\phi Z}\Phi)(X, \phi Y) = 0\,.\end{aligned}$$

Hence in virtue of (3.1) and (3.2) we obtain

$$\begin{aligned}(\nabla_{\phi X}\Phi)(Z, \phi Y) + (\nabla_X\Phi)(Z, Y) &= 2\eta(Z)g(X, Y)\\ &- \eta(Y)g(X + hX, Z) - \eta(X)\eta(Y)\eta(Z)\,,\end{aligned}$$

which is equivalent to (3.3). Choose a ϕ-basis. As from (3.3) follows $\nabla_\xi\phi = 0$, we can find (3.4) (a), (b) by using (3.3).

LEMMA 3.2. *The curvature tensor of a contact metric manifold satisfies the relations*

$$(3.5)\qquad g(R_{\xi X}Y, Z) = -(\nabla_X\Phi)(Y, Z) - g(X, (\nabla_Y\phi h)Z) + g(X, \nabla_Z\phi h)Y)\,,$$

$$\begin{aligned}(3.6)\qquad &g(R_{\xi X}Y, Z) - g(R_{\xi X}\phi Y, \phi Z) + g(R_{\xi\phi X}Y, \phi Z) + g(R_{\xi\phi X}\phi Y, Z)\\ &= 2(\nabla_{hX}\Phi)(Y, Z) - 2\eta(Y)g(X + hX, Z) + 2\eta(Z)g(X + hX, Y)\,.\end{aligned}$$

PROOF. Let X, Y, Z be tangent vectors at a point $m \in M$. By the same letters we denote their extensions to local vector fields. From (2.1) we have

$$R_{YZ}\xi = -(\nabla_Y\phi)Z + (\nabla_Z\phi)Y - (\nabla_Y\phi h)Z + (\nabla_Z\phi h)Y\,.$$

This yields (3.5) in virtue of (2.4). Denote

$$A(X, Y, Z) = -(\nabla_X \Phi)(Y, Z) + (\nabla_X \Phi)(\phi Y, \phi Z) - (\nabla_{\phi X} \Phi)(Y, \phi Z) - (\nabla_{\phi X} \Phi)(\phi Y, Z) ,$$

$$B(X, Y, Z) = -g(X, (\nabla_Y \phi h)Z) + g(X, (\nabla_{\phi Y} \phi h)\phi Z) - g(\phi X, (\nabla_Y \phi h)\phi Z) - g(\phi X, (\nabla_{\phi Y} \phi h)Z) .$$

In view of (3.5) the left hand side of (3.6) equals $A(X, Y, Z) + B(X, Y, Z) - B(X, Z, Y)$. By (3.2) and (3.3) we obtain

$$A(X, Y, Z) = 2g(X, Y)\eta(Z) - 2g(X, Z)\eta(Y) .$$

Rewrite B in the following form

$$B(X, Y, Z) = -g(X, (\nabla_Y \phi)hZ) + g(X, h(\nabla_Y \phi)Z) + g(X, h\phi(\nabla_{\phi Y} \phi)Z) + g(X, \phi(\nabla_{\phi Y} \phi)hZ) + \eta(X)\eta((\nabla_{\phi Y} h)Z) .$$

We have

(3.7) $$\phi(\nabla_{\phi X} \phi)Y = 2\eta(Y)X - g(X + hX, Y)\xi - \eta(X)\eta(Y)\xi + (\nabla_X \phi)Y .$$

In fact

$$\phi(\nabla_{\phi X} \phi)Y = (\nabla_{\phi X} \phi^2)Y - (\nabla_{\phi X} \phi)\phi Y = (\nabla_{\phi X} \eta)(Y)\xi + \eta(Y)\nabla_{\phi X}\xi - (\nabla_{\phi X} \phi)\phi Y ,$$

from which, by (2.1) and (3.3), we find (3.7). Moreover, remembering (2.1), we get

$$\eta((\nabla_{\phi Y} h)Z) = -(\nabla_{\phi Y} \eta)(hZ) = g(-Y + hY, hZ) .$$

Thus, we obtain in virtue of (3.7)

$$B(X, Y, Z) = 2g(hX, (\nabla_Y \phi)Z) + 2\eta(Z)g(hX, Y) - 2\eta(X)g(Y, hZ) .$$

Therefore

$$\begin{aligned} &A(X, Y, Z) + B(X, Y, Z) - B(X, Z, Y) \\ &= -2(\nabla_Y \Phi)(Z, hX) - 2(\nabla_Z \Phi)(hX, Y) - 2\eta(Y)g(X + hX, Z) \\ &\quad + 2\eta(Z)g(X + hX, Y) , \end{aligned}$$

which in view of (2.4) equals the right hand side of (3.6). Thus the proof is complete.

Denote by S the scalar curvature of M and define $S^* = \sum_{i,j=1}^{2n+1} g(R_{E_i E_j}\phi E_j, \phi E_i)$, where $\{E_i\}$ is an orthonormal frame.

LEMMA 3.3. *For any contact metric manifold M we have*

(3.8) $$S^* - S + 4n^2 = \operatorname{tr} h^2 + (1/2)\{\|\nabla\phi\|^2 - 4n\} \geqq 0 .$$

Moreover M is Sasakian if and only if $\|\nabla\phi\|^2 = 4n$ or equivalently $S^* - S + 4n^2 = 0$.

PROOF. Let $\{E_1, \cdots, E_{2n+1}\}$ be a frame in M_m. By the same letters we denote local extension vector fields of this frame which are orthonormal and covariant constant at $m \in M$. Now (3.4) (a) and (2.1) become

$$\sum_{i,j=1}^{2n+1} g((\nabla_{E_j}\nabla_{E_i}\phi)E_i, \phi E_j) = -4n^2 \tag{3.9}$$

because $\operatorname{tr} h = 0$. On the other hand, by (3.4) (b), we obtain

$$-\sum_{i,j=1}^{2n+1} g((\nabla_{E_i}\nabla_{E_j}, \phi)E_i, \phi E_j) = \sum_{i,j=1}^{2n+1} g((\nabla_{E_i}\phi)E_j, (\nabla_{E_j}\phi)E_i) . \tag{3.10}$$

But the right hand side of (3.10) can be transformed as follows

$$\begin{aligned}&\sum_{i,j,k=1}^{2n+1} g((\nabla_{E_i}\phi)E_j, E_k)g((\nabla_{E_j}\phi)E_i, E_k)\\ &\quad = (1/2)\sum_{i,j,k=1}^{2n+1}\{g((\nabla_{E_i}\phi)E_j, E_k)\}^2 = (1/2)\sum_{i,j=1}^{2n+1}\|(\nabla_{E_i}\phi)E_j\|^2 = (1/2)\|\nabla\phi\|^2\end{aligned}$$

in virtue of (2.4). Considering (3.9), (3.10) and the last relation we get

$$\sum_{i,j=1}^{2n+1} g((R_{E_jE_i}\phi)E_i, \phi E_j) = (1/2)\|\nabla\phi\|^2 - 4n^2 .$$

Hence $S^* - S + g(Q\xi, \xi) = (1/2)\|\nabla\phi\|^2 - 4n^2$, which with the help of (2.3) proves the equality part of (3.8).

Recall that M is Sasakian if and only if $(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X$. Consequently, we always have, for a contact metric manifold M,

$$\sum_{i,j=1}^{2n+1}\|(\nabla_{E_i}\phi)E_j - g(E_i, E_j)\xi + \eta(E_j)E_i\|^2 \geqq 0 ,$$

where equality holds if and only if M is Sasakian. By using (3.4) (a) it can be easily shown that the last condition is equivalent to $\|\nabla\phi\|^2 - 4n \geqq 0$, where equality holds only in the Sasakian case. Note that from the symmetry of the operator h it follows that $\operatorname{tr} h^2 \geqq 0$ and $\operatorname{tr} h^2 = 0$ if and only if $h = 0$. Recall also that on a Sasakian manifold the vector field ξ is Killing. Therefore, if M is Sasakian, then h vanishes. The above remarks complete the proof of our lemma.

4. Main results.

THEOREM 4.1. *If a contact metric manifold M is of constant sectional curvature and* $\dim M \geqq 5$, *then the sectional curvature of M is equal to* 1 *and M is Sasakian.*

PROOF. Let c be the constant sectional curvature of M, i.e., $R_{XY}Z = c\{g(Y, Z)X - g(X, Z)Y\}$. Under this assumption, from (2.2) we obtain $h^2X = (c-1)\phi^2X$, hence $\operatorname{tr} h^2 = 2n(1-c)$. Moreover, (3.6) yields

$$(4.1)\qquad (\nabla_{hX}\Phi)(Y, Z) = (1 - c)\{\eta(Y)g(X, Z) - \eta(Z)g(X, Y)\} + \eta(Y)g(hX, Z) - \eta(Z)g(hX, Y)\ .$$

Now let us suppose that $c \neq 1$. Taking in (4.1) hX instead of X, we have

$$(\nabla_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)\ ,$$

as $\nabla_\xi\phi = 0$. From this we find $\|\nabla\phi\|^2 = 4n(2 - c)$. The last relation and equalities $S = 2n(2n + 1)c$, $S^* = 2nc$ applied to (3.8) give $n = 1$, a contradiction. Therefore $c = 1$. But in this case, by Lemma 3.3, the manifold M is Sasakian.

THEOREM 4.2. *Let M be a conformally flat contact metric manifold with* $\dim M \geqq 5$. *Then the scalar curvature S of M satisfies* $S \leqq 2n(2n+1)$. *Moreover, M is Sasakian if and only if* $S = 2n(2n + 1)$.

PROOF. Conformal flatness yields

$$(4.2)\qquad R_{XY}Z = (1/(2n - 1))\Big\{g\Big(QY - \frac{S}{2n}Y, Z\Big)X - g\Big(QX - \frac{S}{2n}X, Z\Big)Y + g(Y, Z)QX - g(X, Z)QY\Big\}\ .$$

Using the relations (2.3) and (4.2) we compute $S^* = (1/(2n - 1))\{S - 2(2n - \operatorname{tr} h^2)\}$. Then, from (3.8) one can obtain

$$(4.3)\qquad (2n - 1)\{\|\nabla\phi\|^2 - 4n\} + 2(2n - 3)\operatorname{tr} h^2 = 2(2n - 2)\{2n(2n + 1) - S\}\ ,$$

which completes the proof.

REMARK 4.1. Any 3-dimensional contact metric manifold M satisfies the condition (4.2) and therefore (4.3), i.e., $\|\nabla\phi\|^2 - 4n = 2\operatorname{tr} h^2$. This shows that M is Sasakian if and only if the vector field ξ is Killing (cf. [6], [3]).

REMARK 4.2. As is known, any conformally flat contact metric manifold with Killing structure vector field ξ is of constant curvature (cf. [5], [4]). Then by Theorem 4.2 any conformally flat contact metric manifold M with $\dim M \geqq 5$ and the scalar curvature $S = 2n(2n + 1)$ is of constant sectional curvature.

In the next theorem we consider a contact metric manifold of constant ϕ-sectional curvature, that is, a manifold M such that at any point $m \in M$ the sectional curvature $K(X, \phi X)$ (denote it by H_m) is independent of the choice of the tangent vector $X \in M_m$, $0 \neq X \perp \xi$. By H we denote the ϕ-sectional curvature of M, i.e., $H\colon M \to \boldsymbol{R}$, $H(m) = H_m$.

THEOREM 4.3. *Let M be a contact metric manifold of constant ϕ-sectional curvature. Then the scalar curvature S and the ϕ-sectional curvature H satisfy the inequality $n(n+1)H + 3n^2 + n \geqq S$. Equality holds if and only if M is Sasakian.*

PROOF. By our assumption we have $g(R_{X\phi X}X, \phi X) + H_m\|X\|^4 = 0$ at any point $m \in M$ and for any $X \in M_m$, $X \perp \xi$. It is clear that this condition implies

$$g(R_{\phi X\phi^2 X}\phi X, \phi^2 X) + H_m\|\phi X\|^4 = 0 \tag{4.4}$$

at any point $m \in M$ and for any $X \in M_m$. Set

$$P(X, Y, Z, W) = g(R_{\phi X\phi^2 Y}\phi Z, \phi^2 W) + H_m g(\phi X, \phi Z)g(\phi Y, \phi W) .$$

The tensor P satisfies $P(X, Y, Z, W) = P(Z, W, X, Y)$. Therefore (4.4) is equivalent to

$$\begin{aligned} &P(X, Y, Z, W) + P(X, Y, W, Z) + P(Y, X, Z, W) + P(Y, X, W, Z) \\ &+ P(X, W, Y, Z) + P(X, W, Z, Y) + P(W, X, Y, Z) + P(W, X, Z, Y) \\ &+ P(X, Z, Y, W) + P(X, Z, W, Y) + P(Z, X, Y, W) \\ &+ P(Z, X, W, Y) = 0 . \end{aligned} \tag{4.5}$$

Choosing a ϕ-basis, taking $X = W = E_i$, $Y = Z = E_j$ into (4.5) and summing over i and j we obtain

$$\sum_{i,j=1}^{2n+1}\{P(E_i, E_j, E_j, E_i) + P(E_i, E_j, E_i, E_j) + P(E_i, E_i, E_j, E_j)\} = 0 ,$$

which by the definition of P, the first Bianchi identity and (2.3) gives

$$4n(n+1)H - 3S^* - S + 2(2n - \operatorname{tr} h^2) = 0 .$$

Comparing the last identity with (3.8) we obtain

$$n(n+1)H + 3n^2 + n - S = (5/4)\operatorname{tr} h^2 + (3/8)\{\|\nabla\phi\|^2 - 4n\} ,$$

which completes the proof.

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