

ON CONTACT STRUCTURES OF TANGENT SPHERE BUNDLES

YOSHIHIRO TASHIRO

(Received June 26, 1968; Revised October 4, 1968)

Introduction. The present author [6] has shown that an orientable hypersurface in an almost complex manifold can be given an induced almost contact structure, and studied conditions for the induced structure to be normal.

On the other hand, K.Yano and S.Kobayashi [8] and K.Yano and S.Ishihara [9] have introduced the notions of vertical, complete and horizontal lifts of tensor fields and connections to tangent bundles. Above all, if the base manifold is almost complex, then the complete or horizontal lift of the structure defines an almost complex structure in the tangent bundle. It is also known, cf. [5], that the tangent bundle of a Riemannian manifold is given an almost Kählerian structure.¹⁾

Basing on these two kinds of results, we shall be able to induce various kinds of almost contact structures into tangent sphere bundles. In this paper, we shall consider a class of hypersurfaces with some property in tangent bundles; the class contains the tangent sphere bundle of a manifold.

In §§1 to 3, we shall introduce a tensor field for the study of hypersurfaces in almost complex manifolds and state some theorems on normality of induced almost contact structures in the language of the tensor field. In §§4 to 8, various kinds of almost contact structures will be induced to hypersurfaces in tangent bundles and the normality of the structures will be discussed. In §8, we shall show that, given the almost Kählerien structure in the tangent bundle of a Riemannian manifold, the induced almost contact structure of the tangent unit-sphere bundle is *K*-contact if and only if the base manifold is of positive constant curvature.

1. Almost complex and almost contact structures.²⁾ Let M be an almost complex manifold of even dimension $2n$. Denote by $(x^A)^3)$ a local coordinate system and by $F=(F_B{}^A)$ the tensor field of the almost complex structure, which satisfies the equation

$$(1.1) \quad F_C{}^B F_B{}^A = -\delta_C^A,$$

1) As to differential geometry of tangent bundles, we refer K. Yano [10].

2) Refer [7] as to almost complex structures and [1, 2, 3, 4] as to almost contact structures.

3) In §§1 to 3, indices A, B, C, D, E run from 1 to $2n$, and $\alpha, \beta, \gamma, \delta, \varepsilon$ from 1 to $2n-1$.

$E=(\delta_B^A)$ being the unit tensor field in M . The $(1, 2)$ -tensor field $N=(N_{CB}^A)$ defined by

$$(1.2) \quad N_{CB}^A = F_C^E(\partial_E F_B^A - \partial_B F_E^A) - F_B^E(\partial_E F_C^A - \partial_C F_E^A)$$

is called the *Nijenhuis tensor* of the almost complex structure F . As is well known, the structure F or the manifold M is *complex* if and only if N vanishes.

If $\Gamma=(\Gamma_{CB}^A)$ is a symmetric affine connection of M and we denote by ∇ covariant differentiation with respect to the connection, then the Nijenhuis tensor N is written as

$$(1.3) \quad N_{CB}^A = F_C^E(\nabla_E F_B^A - \nabla_B F_E^A) - F_B^E(\nabla_E F_C^A - \nabla_C F_E^A).$$

A necessary and sufficient condition for F to be complex is that there exists a symmetric affine connection in M such as

$$(1.4) \quad \nabla_C F_B^A = 0.$$

In an almost complex manifold M , there is a Riemannian metric $g=(g_{BA})$ satisfying the relation

$$(1.5) \quad g_{BA} = F_B^D F_A^C g_{DC}.$$

Such a structure (F, g) or an almost complex manifold M with metric g is said to be *almost Hermitian*. The covariant tensor field (F_{BA}) given by

$$(1.6) \quad F_{BA} = F_B^C g_{CA}$$

is skew symmetric. We put

$$(1.7) \quad \Theta = (1/2) F_{BA} dx^B \wedge dx^A$$

and call it the *fundamental form* of M . If the form Θ is closed, that is,

$$(1.8) \quad \partial_C F_{BA} + \partial_B F_{AC} + \partial_A F_{CB} = 0,$$

then the metric structure (F, g) or the manifold M is said to be *almost Kählerian*. The left hand side of (1.8) is put and rewritten as

$$\begin{aligned} F_{CBA} &= \partial_C F_{BA} + \partial_B F_{AC} + \partial_A F_{CB} \\ &= \nabla_C F_{BA} + \nabla_B F_{AC} + \nabla_A F_{CB}, \end{aligned}$$

where ∇ indicates covariant differentiation with respect to the Riemannian connection of g . If F is complex, then an almost Hermitian or almost Kählerian manifold turns to the so-called *Hermitian* or *Kählerian* manifold, respectively. A necessary and sufficient condition for an almost Hermitian manifold to be Kählerian is that the Riemannian connection satisfies $\nabla_C F_B^A = 0$.

Next let \bar{M} be an almost contact manifold of odd dimension $2n-1$. Denote by (u^α) a local coordinate system and by (f, ξ, η) the almost contact structure, where $f = (f_\alpha^\beta)$ is a $(1,1)$ -tensor field of rank $2n-2$, $\xi = (\xi^\alpha)$ a contravariant vector field and $\eta = (\eta_\beta)$ a covariant vector field, and they satisfy the relations

$$(1.9) \quad \begin{cases} f_\beta^\varepsilon f_\varepsilon^\alpha - \eta_\beta \xi^\alpha = -\delta_\beta^\alpha, \\ \xi^\varepsilon f_\varepsilon^\alpha = 0, f_\beta^\varepsilon \eta_\varepsilon = 0, \xi^\alpha \eta_\alpha = 1. \end{cases}$$

There are defined the following tensor fields in \bar{M} :

$$(1.10) \quad \begin{cases} \bar{N}_{\gamma\beta}^\alpha = f_\gamma^\varepsilon (\partial_\varepsilon f_\beta^\alpha - \partial_\varepsilon f_\beta^\alpha) - f_\beta^\varepsilon (\partial_\varepsilon f_\gamma^\alpha - \partial_\gamma f_\varepsilon^\alpha) \\ \quad + \eta_\gamma \partial_\beta \xi^\alpha - \eta_\beta \partial_\gamma \xi^\alpha, \\ \bar{N}_{\gamma\beta} = -f_\gamma^\varepsilon (\partial_\varepsilon \eta_\beta - \partial_\beta \eta_\varepsilon) + f_\beta^\varepsilon (\partial_\varepsilon \eta_\gamma - \partial_\gamma \eta_\varepsilon), \\ \bar{N}_\beta^\alpha = \xi^\varepsilon (\partial_\varepsilon f_\beta^\alpha - \partial_\beta f_\varepsilon^\alpha) - f_\beta^\varepsilon \partial_\varepsilon \xi^\alpha, \\ \bar{N}_\beta = \xi^\varepsilon (\partial_\varepsilon \eta_\beta - \partial_\beta \eta_\varepsilon). \end{cases}$$

If $\bar{\Gamma} = (\bar{\Gamma}_{\gamma\beta}^\alpha)$ is a symmetric affine connection in \bar{M} and we denote by $\bar{\nabla}$ covariant differentiation with respect to the connection, the above tensor fields are written as

$$(1.11) \quad \begin{cases} \bar{N}_{\gamma\beta}^\alpha = f_\gamma^\varepsilon (\bar{\nabla}_\varepsilon f_\beta^\alpha - \bar{\nabla}_\beta f_\varepsilon^\alpha) - f_\beta^\varepsilon (\bar{\nabla}_\varepsilon f_\gamma^\alpha - \bar{\nabla}_\gamma f_\varepsilon^\alpha) \\ \quad + \eta_\gamma \bar{\nabla}_\beta \xi^\alpha - \eta_\beta \bar{\nabla}_\gamma \xi^\alpha, \\ \bar{N}_{\gamma\beta} = -f_\gamma^\varepsilon (\bar{\nabla}_\varepsilon \eta_\beta - \bar{\nabla}_\beta \eta_\varepsilon) + f_\beta^\varepsilon (\bar{\nabla}_\varepsilon \eta_\gamma - \bar{\nabla}_\gamma \eta_\varepsilon), \\ \bar{N}_\beta^\alpha = \xi^\varepsilon (\bar{\nabla}_\varepsilon f_\beta^\alpha - \bar{\nabla}_\beta f_\varepsilon^\alpha) - f_\beta^\varepsilon \bar{\nabla}_\varepsilon \xi^\alpha, \\ \bar{N}_\beta = \xi^\varepsilon (\bar{\nabla}_\varepsilon \eta_\beta - \bar{\nabla}_\beta \eta_\varepsilon). \end{cases}$$

If we denote by $\bar{\mathcal{L}}(\xi)$ Lie differentiation with respect to ξ in \bar{M} and take account of (1.9), then the third and fourth tensor fields of (1.10) can be written as

$$(1.12) \quad \begin{cases} \bar{N}_\beta^\alpha = \xi^\varepsilon \partial_\varepsilon f_\beta^\alpha + (\partial_\beta \xi^\varepsilon) f_\varepsilon^\alpha - (\partial_\varepsilon \xi^\alpha) f_\beta^\varepsilon = \bar{\mathcal{L}}(\xi) f_\beta^\alpha, \\ \bar{N}_\beta = \xi^\varepsilon \partial_\varepsilon \eta_\beta + (\partial_\beta \xi^\varepsilon) \eta_\varepsilon = \bar{\mathcal{L}}(\xi) \eta_\beta. \end{cases}$$

The first tensor field $\bar{N}=(\bar{N}_{\gamma\beta}^\alpha)$ is called the *Nijenhuis tensor* of the almost contact structure. If \bar{N} does vanish, then so do the other three and the almost contact structure is said to be *normal*.

In an almost contact manifold, there is a Riemannian metric $\bar{g}=(\bar{g}_{\beta\alpha})$ satisfying the relations

$$(1.13) \quad \begin{cases} \bar{g}_{\beta\alpha} - \eta_\beta\eta_\alpha = f_\beta^\delta f_\alpha^\gamma \bar{g}_{\delta\gamma}, \\ \eta_\beta = \bar{g}_{\beta\alpha} \xi^\alpha. \end{cases}$$

The structure (f, ξ, η, \bar{g}) of an almost contact structure associated with metric \bar{g} is called an *almost contact metric structure* of \bar{M} . The covariant tensor field $(f_{\beta\alpha})$ given by

$$(1.14) \quad f_{\beta\alpha} = f_\beta^\gamma \bar{g}_{\gamma\alpha}$$

is skew symmetric, and we put

$$(1.15) \quad \theta_1 = \eta_\alpha du^\alpha, \quad \theta_2 = (1/2)f_{\beta\alpha} du^\beta \wedge du^\alpha.$$

If $d\theta_1 = \theta_2$, or

$$(1.16) \quad f_{\beta\alpha} = \partial_\beta \eta_\alpha - \partial_\alpha \eta_\beta = \bar{\nabla}_\beta \eta_\alpha - \bar{\nabla}_\alpha \eta_\beta,$$

where $\bar{\nabla}$ indicates covariant differentiation with respect to the Riemannian connection of \bar{g} , then the structure (f, ξ, η, \bar{g}) or the manifold \bar{M} is said to be *contact metric*. If, in addition, the vector field ξ is a Killing vector field with respect to the metric \bar{g} , that is,

$$(1.17) \quad \bar{\mathcal{L}}(\xi)\bar{g}_{\beta\alpha} = \bar{\nabla}_\beta \eta_\alpha + \bar{\nabla}_\alpha \eta_\beta = 0,$$

then the structure or the manifold is said to be *K-contact metric*, and we have the equation

$$(1.18) \quad f_{\beta\alpha} = 2\bar{\nabla}_\beta \eta_\alpha.$$

If a contact metric structure is normal, then the structure (f, ξ, η, \bar{g}) is said to be *Sasakian* and it is characterized by the equations

$$(1.19) \quad \begin{cases} 2\bar{\nabla}_\beta \eta_\alpha = f_{\beta\alpha}, \\ 2\bar{\nabla}_\gamma f_{\beta\alpha} = \eta_\beta \bar{g}_{\gamma\alpha} - \eta_\alpha \bar{g}_{\gamma\beta}. \end{cases}$$

The first equation means that a Sasakian structure is a special one of K -contact structures.

2. Almost contact structure of a hypersurface in an almost complex manifold. Now we consider an almost complex manifold M with structure tensor F and an orientable hypersurface \bar{M} in M . Assume that \bar{M} is locally represented by parametric equations

$$(2.1) \quad x^A = x^A(u^\alpha).$$

We put

$$(2.2) \quad B_\alpha{}^A = \partial_\alpha x^A,$$

which span the tangent hyperplane of \bar{M} at each point. Further we choose a vector field $C=(C^A)$ complementary to the tangent hyperplane of \bar{M} at each point, and call it a *pseudo-normal* to \bar{M} . The matrix $\begin{pmatrix} B_\alpha{}^A \\ C^A \end{pmatrix}$ is regular and the inverse matrix will be denoted by $(B_\beta{}^A, C_B)$. Then we have the equations

$$(2.3) \quad \begin{cases} B_\beta{}^A B_\alpha{}^A = \delta_\beta^\alpha, & C^A B_\alpha{}^A = 0, \\ B_\beta{}^A C_A = 0, & C^A C_A = 1 \end{cases}$$

and

$$(2.4) \quad B_\alpha{}^A B_\beta{}^B + C^A C_B = \delta_\beta^A.$$

For an arbitrary $(1,1)$ -tensor field $F=(F_B{}^A)$ in M , we put

$$(2.5) \quad \begin{aligned} f_\beta{}^\alpha &= B_\beta{}^B F_B{}^A B_A{}^\alpha, & f_\infty{}^\alpha &= C^B F_B{}^A B_A{}^\alpha, \\ f_\beta{}^\infty &= B_\beta{}^B F_B{}^A C_A, & f_\infty{}^\infty &= C^B F_B{}^A C_A. \end{aligned}$$

If, in particular, F is an almost complex structure in M , then the pseudo-normal vector field $C=(C^A)$ can be chosen such as

$$(2.6) \quad f_\infty{}^\infty = C^B F_B{}^A C_A = 0.$$

Indeed, since there is an almost Hermitian metric g associated with F , we may take C as the unit normal vector field to the hypersurface \bar{M} with respect to g . If we once choose such a pseudo-normal vector C and put

$$(2.7) \quad \begin{cases} \xi^\alpha = f_\infty^\alpha = C^B F_B^A B_A^\alpha, \\ \eta_\beta = -f_\beta^\infty = -B_\beta^B F_B^A C_A, \end{cases}$$

then we see that (f, ξ, η) defines an almost contact structure in the hypersurface \bar{M} , which will be said to be *induced* in \bar{M} from an almost complex structure F in M , [6].

Given a symmetric affine connection $\Gamma = (\Gamma_{cB}^A)$ in M , the induced connection $\bar{\Gamma} = (\bar{\Gamma}_{\gamma\beta}^\alpha)$ in the hypersurface \bar{M} is defined by

$$(2.8) \quad \bar{\Gamma}_{\gamma\beta}^\alpha = (\partial_\gamma B_\beta^A + \Gamma_{cB}^A B_\gamma^c B_\beta^B) B_A^\alpha,$$

and the tensor fields $h = (h_{\gamma\beta})$, $l = (l_\gamma^\alpha)$ and $m = (m_\gamma)$ in \bar{M} by

$$(2.9) \quad \begin{cases} h_{\gamma\beta} = (\partial_\gamma B_\beta^A + \Gamma_{cB}^A B_\gamma^c B_\beta^B) C_A, \\ l_\gamma^\alpha = (\partial_\gamma C^A + \Gamma_{cB}^A B_\gamma^c C^B) B_A^\alpha, \\ m_\gamma = (\partial_\gamma C^A + \Gamma_{cB}^A B_\gamma^c C^B) C_A, \end{cases}$$

respectively. Then the so-called van der Waerden-Bortolloti covariant derivatives of B_β^A and C^A are expressed by

$$(2.10) \quad \begin{cases} \nabla_\gamma B_\beta^A = \partial_\gamma B_\beta^A + B_\gamma^c B_\beta^B \Gamma_{cB}^A - \bar{\Gamma}_{\gamma\beta}^\alpha B_\alpha^A = h_{\gamma\beta} C^A, \\ \nabla_\gamma C^A = \partial_\gamma C^A + B_\gamma^c C^B \Gamma_{cB}^A = l_\gamma^\alpha B_\alpha^A + m_\gamma C^A, \end{cases}$$

and those of B_A^α and C_B by

$$(2.11) \quad \begin{cases} \nabla_\gamma B_A^\alpha = -l_\gamma^\alpha C_B, \\ \nabla_\gamma C_B = -h_{\gamma\beta} B_\beta^\alpha - m_\gamma C_B. \end{cases}$$

For any $(1, 1)$ -tensor field F in M , the covariant derivatives of the four tensor fields defined by (2.5) with respect to the induced connection $\bar{\Gamma}$ are written in the forms

$$(2.12) \quad \begin{cases} \bar{\nabla}_\gamma f_\beta^\alpha = B_\gamma^c B_\beta^B B_A^\alpha \nabla_c F_B^A - l_\gamma^\alpha f_\beta^\infty + h_{\gamma\beta} f_\infty^\alpha, \\ \bar{\nabla}_\gamma f_\beta^\infty = B_\gamma^c B_\beta^B C_A \nabla_c F_B^A - h_{\gamma\alpha} f_\beta^\alpha - m_\gamma f_\beta^\infty + h_{\gamma\beta} f_\infty^\infty, \\ \bar{\nabla}_\gamma f_\infty^\alpha = B_\gamma^c C^B B_A^\alpha \nabla_c F_B^A + l_\gamma^\alpha f_\beta^\alpha + m_\gamma f_\infty^\alpha - l_\gamma^\alpha f_\infty^\infty, \\ \bar{\nabla}_\gamma f_\infty^\infty = B_\gamma^c C^B C_A \nabla_c F_B^A + l_\gamma^\beta f_\beta^\infty - h_{\gamma\alpha} f_\infty^\alpha. \end{cases}$$

If F is an almost complex structure in M , then the covariant derivatives of the induced almost contact structure (f, ξ, η) in \bar{M} are given by

$$(2.13) \quad \left\{ \begin{array}{l} \bar{\nabla}_\gamma f_\beta^\alpha = B_\gamma^c B_\beta^B B_A^\alpha \nabla_c F_B^A + l_\gamma^\alpha \eta_\beta + h_{\gamma\beta} \xi^\alpha, \\ \bar{\nabla}_\gamma \eta_\beta = -B_\gamma^c B_\beta^B C_A \nabla_c F_B^A + h_{\gamma\alpha} f_\beta^\alpha - m_\gamma \eta_\beta, \\ \bar{\nabla}_\gamma \xi^\alpha = B_\gamma^c C^B B_A^\alpha \nabla_c F_B^A + l_\gamma^\beta f_\beta^\alpha + m_\gamma \xi^\alpha, \end{array} \right.$$

and we have

$$(2.14) \quad B_\gamma^c C^B C_A \nabla_c F_B^A = l_\gamma^\beta \eta_\beta + h_{\gamma\alpha} \xi^\alpha.$$

Substituting these into (1.11), we obtain the equations

$$(2.15) \quad \left\{ \begin{array}{l} \bar{N}_{\gamma\beta}^\alpha = B_\gamma^c B_\beta^B B_A^\alpha N_{CB}^A + \eta_\gamma C^E B_\beta^B B_A^\alpha \nabla_E F_B^A - \eta_\beta C^E B_\gamma^c B_A^\alpha \nabla_E F_C^A \\ \quad + (f_\gamma^\epsilon l_\epsilon^\alpha - l_\gamma^\epsilon f_\epsilon^\alpha - m_\gamma \xi^\alpha) \eta_\beta - (f_\beta^\epsilon l_\epsilon^\alpha - l_\beta^\epsilon f_\epsilon^\alpha - m_\beta \xi^\alpha) \eta_\gamma, \\ \bar{N}_{\gamma\beta} = B_\gamma^c B_\beta^B C_A N_{CB}^A + \eta_\gamma C^E B_\beta^B C_A \nabla_E F_B^A - \eta_\beta C^E B_\gamma^c C_A \nabla_E F_C^A \\ \quad + (l_\gamma^\alpha \eta_\alpha + f_\gamma^\alpha m_\alpha) \eta_\beta - (l_\beta^\alpha \eta_\alpha + f_\beta^\alpha m_\alpha) \eta_\gamma, \\ \bar{N}_\beta^\alpha = C^c B_\beta^B B_A^\alpha N_{CB}^A - f_\beta^\epsilon C^c B_\epsilon^B B_A^\alpha \nabla_c F_B^A \\ \quad - (l_\beta^\alpha + f_\beta^\epsilon l_\epsilon^\delta f_\delta^\alpha + f_\beta^\epsilon m_\epsilon \xi^\alpha - \eta_\beta \xi^\epsilon l_\epsilon^\alpha), \\ \bar{N}_\beta = -C^c B_\beta^B C_A N_{CB}^A + f_\beta^\epsilon C^c B_\epsilon^B C_A \nabla_c F_B^A - (f_\beta^\epsilon l_\epsilon^\alpha \eta_\alpha + \eta_\beta \xi^\alpha m_\alpha - m_\beta). \end{array} \right.$$

If we put

$$(2.16) \quad L_\beta^\alpha = l_\beta^\alpha - \eta_\beta \xi^\epsilon l_\epsilon^\alpha + f_\beta^\epsilon (l_\epsilon^\delta f_\delta^\alpha + m_\epsilon \xi^\alpha) + C^c B_\epsilon^B B_A^\alpha \nabla_c F_B^A,$$

then we see the tensor field $L = (L_\beta^\alpha)$ satisfies the equations

$$(2.17) \quad \left\{ \begin{array}{l} f_\gamma^\beta L_\beta^\alpha = f_\gamma^\beta l_\beta^\alpha - l_\gamma^\beta f_\beta^\alpha - m_\gamma \xi^\alpha - C^c B_\beta^B B_A^\alpha \nabla_c F_B^A \\ \quad + \eta_\gamma \xi^\beta (l_\beta^\delta f_\delta^\alpha + m_\beta \xi^\alpha) + C^c B_\beta^B B_A^\alpha \nabla_c F_B^A, \\ L_\beta^\alpha \eta_\alpha = l_\beta^\alpha \eta_\alpha + f_\beta^\alpha m_\alpha - C^c B_\beta^B C_A \nabla_c F_B^A \\ \quad - \eta_\beta (\xi^\epsilon l_\epsilon^\alpha \eta_\alpha - C^c C^D F_D^B C_A \nabla_c F_B^A), \\ \xi^\beta L_\beta^\alpha = 0, \\ f_\gamma^\beta L_\beta^\alpha \eta_\alpha = f_\gamma^\beta l_\beta^\alpha \eta_\alpha - m_\gamma + \eta_\gamma \xi^\alpha m_\alpha - f_\gamma^\beta C^c B_\beta^B C_A \nabla_c F_B^A, \end{array} \right.$$

and the expressions (2.15) are written as

$$(2.18) \quad \left\{ \begin{array}{l} \bar{N}_{\gamma\beta}^{\alpha} = B_{\gamma}^{\alpha}B_{\beta}^{\beta}B_{\alpha}^{\alpha}N_{CB}^A - \eta_{\gamma}f_{\beta}^{\epsilon}L_{\epsilon}^{\alpha} + \eta_{\beta}f_{\gamma}^{\epsilon}L_{\epsilon}^{\alpha}, \\ \bar{N}_{\gamma\beta} = B_{\gamma}^{\alpha}B_{\beta}^{\beta}C_AN_{CB}^A - \eta_{\gamma}L_{\beta}^{\alpha}\eta_{\alpha} + \eta_{\beta}L_{\gamma}^{\alpha}\eta_{\alpha}, \\ \bar{N}_{\beta}^{\alpha} = C^{\alpha}B_{\beta}^{\beta}B_{\alpha}^{\alpha}N_{CB}^A - L_{\beta}^{\alpha}, \\ \bar{N}_{\beta} = -C^{\alpha}B_{\beta}^{\beta}C_AN_{CB}^A - f_{\beta}^{\epsilon}L_{\epsilon}^{\alpha}\eta_{\alpha}. \end{array} \right.$$

Now we can obtain the following

THEOREM 1.⁴⁾ *Let M be a complex manifold and \bar{M} an orientable hypersurface in M . Then the conditions in each of the following triples are equivalent to one another and the first implies the second:*

(1) *The induced almost contact structure in \bar{M} is normal, i.e.,*

$$\bar{N}_{\gamma\beta}^{\alpha} = 0 \iff \bar{N}_{\beta}^{\alpha} = 0 \iff L_{\beta}^{\alpha} = 0.$$

$$(2) \quad \bar{N}_{\gamma\beta} = 0 \iff \bar{N}_{\beta} = 0 \iff L_{\beta}^{\alpha}\eta_{\alpha} = 0.$$

PROOF. It is known [2] that $\bar{N}_{\gamma\beta}^{\alpha}=0$ implies $\bar{N}_{\beta}^{\alpha}=0$, $\bar{N}_{\gamma\beta}=0$ and $\bar{N}_{\beta}=0$ and that $\bar{N}_{\beta}=0$ implies $\bar{N}_{\beta}=0$. By the assumption, the Nijenhuis tensor N of M vanishes. Hence $\bar{N}_{\beta}^{\alpha}=0$ implies $L_{\beta}^{\alpha}=0$, then $\bar{N}_{\gamma\beta}^{\alpha}=0$. On the other hand, $\bar{N}_{\beta}=0$ implies $f_{\beta}^{\epsilon}L_{\epsilon}^{\alpha}\eta_{\alpha}=0$, from which and the third equation of (2.17) follows $L_{\epsilon}^{\alpha}\eta_{\alpha}=0$, then $\bar{N}_{\beta}=0$. Q.E.D.

3. Almost contact metric structure of a hypersurface in an almost Hermitian manifold. Suppose now that M is an almost Hermitian manifold and \bar{M} an orientable hypersurface in M . We take $C=(C^A)$ as the unit normal vector field to \bar{M} . Then the induced metric $\bar{g}=(\bar{g}_{\beta\alpha})$ is defined by

$$(3.1) \quad \bar{g}_{\beta\alpha} = g_{BA}B_{\beta}^{\beta}B_{\alpha}^{\alpha},$$

and it is associated with the induced almost contact structure (f, ξ, η) in \bar{M} ,

4) Cf. Y. Tashiro [6, Theorem 2].

see [6]. If the contravariant components of the metric tensor \bar{g} are denoted by $\bar{g}^{\beta\alpha}$, then we have

$$(3.2) \quad B^\alpha_B = \bar{g}^{\alpha\beta} g_{BA} B_\beta^A, \quad C_B = g_{BA} C^A,$$

and the tensor fields h, l and m have the properties

$$(3.3) \quad l_\gamma^\alpha = -\bar{g}^{\beta\alpha} h_{\beta\gamma}, \quad m_\gamma = 0.$$

The covariant components of the tensor field L defined by (2.16) are equal to

$$(3.4) \quad L_{\beta\alpha} = -h_{\beta\alpha} + \eta_\beta \xi^\varepsilon h_{\varepsilon\alpha} + f_\beta^\varepsilon (h_{\varepsilon\delta} f_\alpha^\delta + C^\varepsilon B_\varepsilon^B B_\alpha^A \nabla_C F_{BA})$$

and satisfy the equation

$$(3.5) \quad f_\gamma^\beta L_{\beta\alpha} = -f_\gamma^\beta h_{\beta\alpha} - f_\alpha^\beta h_{\beta\gamma} - C^\varepsilon B_\gamma^B B_\alpha^A \nabla_C F_{BA} \\ + \eta_\gamma \xi^\varepsilon (f_\alpha^\delta h_{\varepsilon\delta} + C^\varepsilon B_\varepsilon^B B_\alpha^A \nabla_C F_{BA}).$$

In this case, the second equation of (2.13) is written in the form

$$(3.6) \quad \bar{\nabla}_\gamma \eta_\beta = -B_\gamma^C B_\beta^B C^A \nabla_C F_{BA} + h_{\gamma\alpha} f_\beta^\alpha.$$

Substituting (3.6) into the equation (1.16), which characterizes for \bar{M} to be contact metric, and using (3.5), we have the equation

$$(3.7) \quad f_{\gamma\beta} = -B_\gamma^C B_\beta^B C^A (\nabla_C F_{BA} - \nabla_B F_{CA}) + h_{\gamma\alpha} f_\beta^\alpha - h_{\beta\alpha} f_\gamma^\alpha \\ = -C^\varepsilon B_\gamma^B B_\beta^A (F_{CBA} - \nabla_C F_{BA}) + h_{\gamma\alpha} f_\beta^\alpha - h_{\beta\alpha} f_\gamma^\alpha \\ = -C^\varepsilon B_\gamma^B B_\beta^A F_{CBA} - f_\gamma^\alpha L_{\alpha\beta} - 2f_\gamma^\alpha h_{\alpha\beta} \\ + \eta_\gamma \xi^\varepsilon (f_\beta^\delta h_{\varepsilon\delta} + C^\varepsilon B_\varepsilon^B B_\beta^A \nabla_C F_{BA}).$$

If, in particular, the manifold M is almost Kählerian, $F_{CBA} = 0$, then we contract (3.7) with ξ^γ and see that

$$(3.8) \quad \xi^\varepsilon (f_\beta^\delta h_{\varepsilon\delta} + C^\varepsilon B_\varepsilon^B B_\beta^A \nabla_C F_{BA}) = 0$$

and consequently, from (3.5),

$$f_\gamma^\alpha (\bar{g}_{\alpha\beta} + L_{\alpha\beta} + 2h_{\alpha\beta}) = 0.$$

Hence we can put

$$\bar{g}_{\alpha\beta} + L_{\alpha\beta} + 2h_{\alpha\beta} = \eta_\alpha v_\beta$$

or

$$(3.9) \quad L_{\alpha\beta} = -2h_{\alpha\beta} - \bar{g}_{\alpha\beta} + \eta_\alpha v_\beta.$$

Contracting this expression with ξ^β , and using the third equation of (2.17), we obtain

$$v_\alpha = \eta_\alpha + 2\xi^\epsilon h_{\epsilon\alpha}.$$

Thus we have the following

THEOREM 2. *Let M be an almost Kählerian manifold and \bar{M} an orientable hypersurface in M . If the induced almost contact metric structure in \bar{M} is contact metric, then the tensor field $L_{\beta\alpha}$ has the form*

$$(3.10) \quad L_{\beta\alpha} = -2h_{\beta\alpha} - \bar{g}_{\beta\alpha} + \eta_\beta(\eta_\alpha + 2\xi^\epsilon h_{\epsilon\alpha}).$$

Now we can state the following

THEOREM 3. *Let M be a Kählerian manifold and \bar{M} an orientable hypersurface in M . If the induced almost contact metric structure in \bar{M} is K -contact, then the structure is normal, that is, Sasakian. A necessary and sufficient condition for the case is that the second fundamental tensor $h_{\beta\alpha}$ has the form*

$$(3.11) \quad 2h_{\beta\alpha} = -\bar{g}_{\beta\alpha} + \alpha\eta_\beta\eta_\alpha,$$

α being a scalar field in \bar{M} .⁵⁾

PROOF. Since we have $\nabla F=0$ in a Kählerian manifold M , the equation (3.6) reduces to

$$\bar{\nabla}_\gamma\eta_\beta = h_{\gamma\alpha}f_\beta^\alpha,$$

5) Cf. Y. Tashiro [6, Theorem 8]. The difference of the coefficients of (3.11) from those of the theorem is not essential.

hence it follows from the Killing equation (1.17) that

$$h_{\gamma\alpha}f_{\beta}^{\alpha} + h_{\beta\alpha}f_{\gamma}^{\alpha} = 0,$$

and from (3.5)

$$f_{\gamma}^{\beta}L_{\beta\alpha} = \eta_{\gamma}\xi^{\epsilon}f_{\alpha}^{\delta}h_{\epsilon\delta}$$

which is equal to zero as is seen by contraction with ξ^{γ} . By means of this equation and the third of (2.17), we see that the tensor field L vanishes. By virtue of Theorem 1, the induced structure is normal.

Then it follows from (3.10) that

$$2h_{\beta\alpha} = -\bar{g}_{\beta\alpha} + \eta_{\beta}(\eta_{\alpha} + 2\xi^{\epsilon}h_{\epsilon\alpha}).$$

Since both $h_{\beta\alpha}$ and $g_{\beta\alpha}$ are symmetric, we may put

$$\eta_{\alpha} + 2\xi^{\epsilon}h_{\epsilon\alpha} = \alpha\eta_{\alpha},$$

where α is a scalar field in \bar{M} given by

$$\alpha = 2h_{\beta\alpha}\bar{g}^{\beta\alpha} + n - 1.$$

Conversely, if $h_{\beta\alpha}$ has the form (3.11), it follows from (3.4) that the tensor field L vanishes.

Q.E.D.

4. Hypersurfaces in a tangent bundle. Let M be an n -dimensional manifold and $T_x(M)$ the tangent space of M at a point x and $T(M)$ the tangent bundle of M . If (x^h) is a local coordinate system in M and (y^h) the cartesian coordinate in the tangent space $T_x(M)$ at each point x with respect to the natural base $\partial_h = \partial/\partial x^h$, then (x^h, y^h) form a local coordinate system in the tangent bundle $T(M)$, called the *induced coordinate system* from (x^h) . We write often (\tilde{x}) for (y^h) and (x^A) for (x^h, y^h) .⁶⁾

6) We refer [10] as to differential geometry of tangent bundles. From now on, various kinds of indices run respectively on the following ranges:

$A, B, C, \dots = 1, 2, \dots, n, n+1, \dots, 2n$;

$h, i, j, \dots = 1, 2, \dots, n$;

$\bar{h}, \bar{i}, \bar{j}, \dots = n+1, \dots, 2n$;

$\alpha, \beta, \gamma, \dots = 1, 2, \dots, n, n+1, \dots, 2n-1$;

$\kappa, \lambda, \mu, \dots = n+1, \dots, 2n-1$.

A regular hypersurface in the tangent space $T_x(M)$ at a point x is represented by parametric equations $y^h = y^h(u^\lambda)$ with matrix $(\partial_\lambda y^h)$ of rank $n-1$. Differentiable functions $y^h = y^h(x^i, u^\lambda)$ of local coordinates (x^i) and $n-1$ parameters (u^λ) give locally a hypersurface in the tangent space $T_x(M)$ at each point x with coordinates (x^i) , and consequently a field of hypersurfaces in the tangent bundle $T(M)$. Such a field will be denoted by $S(M)$. If the local coordinates (x^h) themselves are regarded as parameters, the field $S(M)$ is a hypersurface represented by

$$(4.1) \quad x^A = x^A(x^i, u^\lambda) \quad \text{or} \quad \begin{cases} x^h = x^h \\ y^h = y^h(x^i, u^\lambda) \end{cases}$$

in the tangent bundle $T(M)$. We write (u^α) for parameters (x^i, u^λ) .

Now we consider $S(M)$ as a hypersurface \bar{M} treated in the preceding paragraphs. The tangent vectors $B_\alpha{}^A = \partial x^A / \partial u^\alpha$ of $S(M)$ in $T(M)$ are given by

$$(4.2) \quad B_\alpha{}^A \begin{cases} B_i{}^h = \delta_i^h, & B_\lambda{}^h = 0, \\ B_i{}^{\bar{h}} = \partial_i y^h, & B_\lambda{}^{\bar{h}} = \partial_\lambda y^h. \end{cases}$$

We suppose that the square matrix $(B_\lambda{}^{\bar{h}}, y^h)$ is regular at each point of $S(M)$. This means that $S(M)$ is a regular hypersurface in $T(M)$ and the tangent hyperplane to $S(M)$ at each point does not pass through the origin of $T_x(M)$. Then we can take the vector field C with components

$$(4.3) \quad (C^A) = \begin{pmatrix} 0 \\ y^h \end{pmatrix}$$

as a pseudo-normal vector field of $S(M)$ in $T(M)$. The vectors $B^\alpha{}_B$ and C_B have the components

$$(4.4) \quad B^\alpha{}_B \begin{cases} B^h{}_i = \delta_i^h, & B^h{}_{\bar{i}} = 0, \\ B^{\bar{h}}{}_i = -B_i{}^{\bar{h}} B^h{}_{\bar{h}}, & B^{\bar{h}}{}_{\bar{i}}, \end{cases}$$

and

$$(4.5) \quad C_A : C_i = -B_i{}^{\bar{h}} C_{\bar{h}}, \quad C_{\bar{h}},$$

respectively, among which $B^{\bar{h}}{}_{\bar{i}}$ and $C_{\bar{h}}$ are determined by the non-trivial relations

$$(4.6) \quad \begin{cases} B_{\lambda}{}^{\bar{h}} B_{\bar{h}}{}^{\kappa} = \delta_{\lambda}^{\kappa}, & y^h B_{\bar{h}}{}^{\kappa} = 0, \\ B_{\lambda}{}^{\bar{h}} C_{\bar{h}} = 0, & y^h C_{\bar{h}} = 1, \end{cases}$$

and satisfy the relation

$$(4.7) \quad B_{\lambda}{}^{\bar{h}} B_{\bar{i}}{}^{\lambda} + y^h C_{\bar{i}} = \delta_i^h$$

following from (2.3) and (2.4).

In the next paragraphs, we shall induce almost contact structures in $S(M)$ from lifted almost complex structures in $T(M)$, and research their properties.

5. The case of complete lifts. Let $\Gamma = (\Gamma_{ji}^h)$ be a symmetric affine connection in M . The complete lift of Γ to the tangent bundle $T(M)$ is denoted by Γ^σ . Its components $(\tilde{\Gamma}_{CB}^A)$ with respect to an induced coordinate system are given by

$$(5.1) \quad \begin{cases} \tilde{\Gamma}_{ji}^h = \Gamma_{ji}^h, & \tilde{\Gamma}_{j\bar{i}}^h = \tilde{\Gamma}_{\bar{j}i}^h = \tilde{\Gamma}_{\bar{j}\bar{i}}^h = 0, \\ \tilde{\Gamma}_{ji}^h = \partial \Gamma_{ji}^h, & \tilde{\Gamma}_{j\bar{i}}^h = \tilde{\Gamma}_{\bar{j}i}^h = \Gamma_{ji}^h, \quad \tilde{\Gamma}_{\bar{j}\bar{i}}^h = 0, \end{cases}$$

where we have put

$$\partial = y^k \partial_k.$$

By computations using (2.8), (5.1) and the results in §4, the induced connection $\tilde{\Gamma}^\sigma = (\tilde{\Gamma}_{\gamma\beta}^\alpha)$ in $S(M)$ from Γ^σ has components

$$(5.2) \quad \begin{cases} \bar{\Gamma}_{ji}^h = \Gamma_{ji}^h, \quad \bar{\Gamma}_{j\bar{\lambda}}^h = \bar{\Gamma}_{\mu i}^h = \bar{\Gamma}_{\mu\lambda}^h = 0, \\ \bar{\Gamma}_{ji}^h = B_{\lambda}{}^{\bar{h}} (\partial_j B_{i}{}^{\bar{\lambda}} + \partial \Gamma_{ji}^h + \Gamma_{ji}^h B_{i}{}^{\bar{l}} + \Gamma_{ii}^h B_{j}{}^{\bar{l}} - B_{i}{}^{\bar{h}} \Gamma_{ji}^l) \\ \quad = B_{\lambda}{}^{\bar{h}} \mathcal{L}(y) \Gamma_{ji}^h, \\ \bar{\Gamma}_{\mu i}^h = B_{\lambda}{}^{\bar{h}} (\partial_\mu \partial_i y^h + \Gamma_{ji}^h B_{\mu}{}^j) = B_{\lambda}{}^{\bar{h}} \partial_\mu \nabla_i y^h, \\ \bar{\Gamma}_{\mu\lambda}^h = B_{\lambda}{}^{\bar{h}} \partial_\mu \partial_\lambda y^h, \end{cases}$$

where the operators $\mathcal{L}(y)$ and ∇ indicate formal Lie and covariant derivatives:

$$\begin{aligned} \mathcal{L}(y) \Gamma_{ji}^h &= \partial_j \partial_i y^h + y^k \partial_k \Gamma_{ji}^h + \Gamma_{ji}^h \partial_i y^l + \Gamma_{ii}^h \partial_j y^l - \Gamma_{ji}^l \partial_i y^h, \\ \nabla_i y^h &= \partial_i y^h + \Gamma_{ij}^h y^j. \end{aligned}$$

The tensor fields h , l , and m in $S(M)$ defined by (2.9) have components

$$(5.3) \quad h_{\gamma\beta} \left\{ \begin{array}{l} h_{ji} = (\partial_j B_i^{\bar{h}} + \partial \Gamma_{ji}^h + \Gamma_{j\bar{i}}^h B_i^{\bar{i}} + \Gamma_{i\bar{i}}^h B_j^{\bar{i}}) C_{\bar{h}} \\ \quad = (\mathcal{L}(y) \Gamma_{ji}^h) C_{\bar{h}}, \\ h_{\mu i} = (\partial_\mu \nabla_i y^h) C_{\bar{h}}, \\ h_{\mu\lambda} = (\partial_\mu \partial_\lambda y^h) C_{\bar{h}}, \end{array} \right.$$

$$(5.4) \quad l_{\gamma}^a \left\{ \begin{array}{l} l_j^h = 0, \quad l_\lambda^h = 0, \\ l_j^k = (\nabla_i y^h) B_{\bar{h}}^k, \quad l_\lambda^k = \delta_\lambda^k, \end{array} \right.$$

$$(5.5) \quad m_\gamma: m_j = (\nabla_j y^h) C_{\bar{h}}, \quad m_\mu = 0,$$

respectively.

Given a $(1,s)$ -tensor field $P = (P_{i_1 \dots i_s}^h)$ in M , the complete lift $P^c = (\tilde{P}_{B_1 \dots B_s}^A)$ of P to the tangent bundle $T(M)$ has components

$$(5.6) \quad \left\{ \begin{array}{l} \tilde{P}_{i_1 \dots i_s}^h = P_{i_1 \dots i_s}^h, \\ \tilde{P}_{i_1 \dots i_t}^{\bar{h}} = \partial P_{i_1 \dots i_t}^h, \\ \tilde{P}_{i_1 \dots i_t \dots i_s}^{\bar{h}} = P_{i_1 \dots i_t \dots i_s}^h \quad (t=1, \dots, s), \end{array} \right.$$

all the others being zero, with respect to the induced coordinate system. The complete lift Γ^c of an affine connection Γ has the property

$$(5.7) \quad \nabla^c P^c = (\nabla P)^c$$

for any tensor field P in M . In particular, the complete lift F^c to $T(M)$ of a $(1,1)$ -tensor field $F = (F_i^h)$ in M has components

$$(5.8) \quad \tilde{F}_B^A \left\{ \begin{array}{l} \tilde{F}_i^h = F_i^h, \quad \tilde{F}_{\bar{i}}^h = 0, \\ \tilde{F}_i^{\bar{h}} = \partial F_i^h, \quad \tilde{F}_{\bar{i}}^{\bar{h}} = F_i^h, \end{array} \right.$$

and the covariant derivative $\nabla^c F^c$ of the lift F^c with respect to Γ^c has components

$$(5.9) \quad \left\{ \begin{array}{l} \tilde{\nabla}_j \tilde{F}_i^h = \nabla_j F_i^h, \quad \tilde{\nabla}_j \tilde{F}_{\bar{i}}^h = \tilde{\nabla}_{\bar{j}} \tilde{F}_i^h = \tilde{\nabla}_{\bar{j}} \tilde{F}_{\bar{i}}^h = 0, \\ \tilde{\nabla}_j \tilde{F}_i^{\bar{h}} = \partial \nabla_j F_i^h, \quad \tilde{\nabla}_j \tilde{F}_{\bar{i}}^{\bar{h}} = \tilde{\nabla}_{\bar{j}} \tilde{F}_i^{\bar{h}} = \nabla_j F_i^h, \quad \tilde{\nabla}_{\bar{j}} \tilde{F}_{\bar{i}}^{\bar{h}} = 0. \end{array} \right.$$

Suppose now that M is an almost complex manifold and F is the structure. Then it is known that the complete lift F^c of F defines an almost complex structure in $T(M)$ and the Nijenhuis tensor \tilde{N} of F^c coincides with the complete lift N^c to $T(M)$ of the Nijenhuis tensor N of F . Therefore the complete lift F^c is complex if and only if F is complex.

For the pseudo-normal vector field $C=(0, y^h)$, the equation (2.6) becomes

$$(5.10) \quad C^B \tilde{F}_B{}^A C_A = y^i F_i{}^h C_{\bar{h}} = 0$$

with respect to the complete lift F^c . If an almost Hermitian metric $g=(g_{ij})$ in M is associated with the almost complex structure F and we consider the unit spheres defined by

$$g_{ih} y^i y^h = 1$$

in the tangent spaces $T_x(M)$, then the vectors (y^h) and $(C_i)=(g_{ih} y^h)$ satisfy (5.10). Hence the tangent unit-sphere bundle possesses the property. We shall confine ourselves with fields of hypersurfaces in $T(M)$ such that the vector field $C=(0, y^h)$ satisfies the equation (5.10), and call them *S-hypersurfaces*.

The tensor fields $f=(f_\beta{}^\alpha)$, $\xi=(\xi^\alpha)$ and $\eta=(\eta_\beta)$ of the induced almost contact structure in an *S*-hypersurface $S(M)$ from the complete lift F^c to $T(M)$ have components

$$(5.11) \quad f_\beta{}^\alpha \left\{ \begin{array}{l} f_i{}^h = F_i{}^h, \\ f_\lambda{}^h = 0, \\ f_i{}^\kappa = [\partial F_i{}^h + B_i{}^{\bar{j}} F_j{}^h - B_j{}^{\bar{k}} F_i{}^j] B_{\bar{k}}{}^\kappa = [\mathcal{L}(y) F_i{}^h] B_{\bar{k}}{}^\kappa, \\ f_\lambda{}^\kappa = B_\lambda{}^{\bar{i}} F_i{}^h B_{\bar{k}}{}^\kappa, \end{array} \right.$$

$$(5.12) \quad \xi^\alpha \left\{ \begin{array}{l} \xi^h = 0, \\ \xi^\kappa = y^i F_i{}^h B_{\bar{k}}{}^\kappa, \end{array} \right.$$

and

$$(5.13) \quad \eta_\beta \left\{ \begin{array}{l} \eta_i = -[\partial F_i{}^h + B_i{}^{\bar{j}} F_j{}^h - B_j{}^{\bar{k}} F_i{}^j] C_{\bar{k}} = -[\mathcal{L}(y) F_i{}^h] C_{\bar{k}}, \\ \eta_\lambda = -B_\lambda{}^{\bar{i}} F_i{}^h C_{\bar{k}}, \end{array} \right.$$

respectively. Substituting these components and (5.3), (5.4), (5.5) and (5.9) into the four types of the components, L_i^h , L_λ^h , L_i^κ , L_λ^κ of the tensor field L defined by (2.16), we see that all the components vanish.

THEOREM 4. *For the induced almost contact structure of an S-hypersurface $S(M)$ from the complete lift F^c in $T(M)$ of an almost complex structure F of M , the tensor field L defined by (2.16) vanishes identically.*

Hence the Nijenhuis tensor \bar{N} and the others given by (2.17) of the induced almost contact structure in $S(M)$ have the components

$$\begin{aligned} \bar{N}_{\gamma\beta}^\alpha & \left\{ \begin{array}{l} \bar{N}_{ji}^h = N_{ji}^h, \quad \bar{N}_{\mu i}^h = \bar{N}_{j\lambda}^h = \bar{N}_{\mu\lambda}^h = 0, \\ \bar{N}_{ji}^\kappa = [-N_{ji}^l B_l^h + \partial N_{ji}^h + B_j^l N_{il}^h + B_i^l N_{jl}^h] B_\lambda^h \\ \quad = [\mathcal{L}(y) N_{ji}^h] B_\lambda^h, \\ \bar{N}_{\mu i}^\kappa = B_\mu^j N_{ji}^h B_\lambda^h, \quad \bar{N}_{\mu\lambda}^\kappa = 0, \end{array} \right. \\ \bar{N}_{\gamma\beta} & \left\{ \begin{array}{l} \bar{N}_{ji} = (\mathcal{L}(y) N_{ji}^h) C_h, \\ \bar{N}_{\mu i} = B_\mu^j N_{ji}^h C_h, \quad \bar{N}_{\mu\lambda} = 0, \end{array} \right. \\ \bar{N}_\beta & \left\{ \begin{array}{ll} \bar{N}_i^h = 0, & \bar{N}_\lambda^h = 0, \\ \bar{N}_i^\kappa = y^j N_{ji}^h B_\lambda^h, & \bar{N}_\lambda^\kappa = 0, \end{array} \right. \\ \bar{N}_\beta & : \quad \bar{N}_i = -y^j N_{ji}^h C_h, \quad \bar{N}_\lambda = 0. \end{aligned}$$

Therefore we have the following

THEOREM 5. *Let F be an almost complex structure in a manifold M , F^c the complete lift of F to the tangent bundle $T(M)$ and $\bar{F}^c = (f, \xi, \eta)^c$ the induced almost contact structure in an S-hypersurface $S(M)$. The structure \bar{F}^c is normal if and only if F is complex.*

6. The case of horizontal lifts. Let $\Gamma = (\Gamma_{ji}^h)$ be a symmetric affine connection in M . The horizontal lift of Γ to the tangent bundle $T(M)$ will be denoted by Γ^H . The components $(\tilde{\Gamma}_{\sigma\beta}^A)$ of Γ^H are given by

$$(6.1) \quad \left\{ \begin{array}{l} \tilde{\Gamma}_{ji}^h = \Gamma_{ji}^h, \quad \tilde{\Gamma}_{ji}^h = \tilde{\Gamma}_{ji}^h = \tilde{\Gamma}_{ji}^h = 0, \\ \tilde{\Gamma}_{ji}^h = \partial \Gamma_{ji}^h - K_{ji}^h, \quad \tilde{\Gamma}_{ji}^h = \tilde{\Gamma}_{ji}^h = \Gamma_{ji}^h, \quad \tilde{\Gamma}_{ji}^h = 0 \end{array} \right.$$

with respect to an induced local coordinate system in $T(M)$, where K_{kji}^h are the components of the curvature tensor field K of the affine connection Γ and we put $K_{.ji}^h = y^k K_{kji}^h$.

The induced connection, denoted by $\bar{\Gamma}^H = (\bar{\Gamma}_{\gamma\beta}^\alpha)$, in $S(M)$ from the horizontal lift Γ^H has the components

$$(6.2) \quad \begin{cases} \bar{\Gamma}_{ji}^h = \Gamma_{ji}^h, & \bar{\Gamma}_{j\lambda}^h = \bar{\Gamma}_{\mu i}^h = \bar{\Gamma}_{\mu\lambda}^h = 0, \\ \bar{\Gamma}_{jt}^k = [\mathcal{L}(y)\Gamma_{jt}^h - K_{.ji}^h]B_{\bar{k}}^h = (\nabla_j \nabla_i y^h)B_{\bar{k}}^h, \\ \bar{\Gamma}_{\mu i}^k = (\partial_\mu \nabla_i y^h)B_{\bar{k}}^h, \\ \bar{\Gamma}_{\mu\lambda}^k = (\partial_\mu \partial_\lambda y^h)B_{\bar{k}}^h. \end{cases}$$

The components of the tensor fields h, l and m in $S(M)$ defined by (2.9) are given by

$$(6.3) \quad h_{\gamma\beta} \quad \begin{cases} h_{ji} = (\nabla_j \nabla_i y^h)C_{\bar{k}}, \\ h_{\mu i} = (\partial_\mu \nabla_i y^h)C_{\bar{k}}, \\ h_{\mu\lambda} = (\partial_\mu \partial_\lambda y^h)C_{\bar{k}}, \end{cases}$$

$$(6.4) \quad l_{\gamma}^\alpha \quad \begin{cases} l_j^h = 0, & l_\mu^h = 0, \\ l_j^k = (\nabla_j y^h)B_{\bar{k}}^h, & l_\mu^k = \delta_\mu^k, \end{cases}$$

$$(6.5) \quad m_\gamma : \quad m_j = (\nabla_j y^h)C_{\bar{k}}, \quad m_\mu = 0,$$

respectively.

On the other hand, the components $(\tilde{P}_{B_1 \dots B_s}{}^A)$ of the horizontal lift P^H to $T(M)$ of a $(1,s)$ -tensor field $P = (P_{i_1 \dots i_s}{}^h)$ in M are given by

$$(6.6) \quad \begin{cases} \tilde{P}_{i_1 \dots i_s}{}^h = P_{i_1 \dots i_s}{}^h, \\ \tilde{P}_{i_1 \dots i_t}{}^{\bar{k}} = -\Gamma_t^h P_{i_1 \dots i_t}{}^l + \sum_l \Gamma_{i_t}^l P_{i_1 \dots i_{t-1} \dots i_s}{}^h, \\ \tilde{P}_{i_1 \dots i_t \dots i_s}{}^{\bar{k}} = P_{i_1 \dots i_t \dots i_s}{}^h \quad (t = 1, 2, \dots, s), \end{cases}$$

the others being zero, with respect to the induced coordinate system in $T(M)$, where we have put $\Gamma_i^h = \Gamma_{ij}^h y^j$. The components of the horizontal lift F^H of a $(1,1)$ -tensor field $F = (F_i{}^h)$ in M are given by

$$(6.7) \quad \tilde{F}_B{}^A \quad \begin{cases} \tilde{F}_i{}^h = F_i{}^h, & \tilde{F}_{\bar{i}}{}^h = 0, \\ \tilde{F}_i{}^{\bar{k}} = -\Gamma_t^h F_i{}^l + \Gamma_i^l F_t{}^h, & \tilde{F}_{\bar{i}}{}^{\bar{k}} = F_i{}^h, \end{cases}$$

and those of the covariant derivative $\nabla^H F^H$ of F^H with respect to Γ^H by

$$(6.8) \quad \left\{ \begin{array}{l} \widetilde{\nabla}_j \widetilde{F}_i^h = \nabla_j F_i^h, \quad \widetilde{\nabla}_j \widetilde{F}_{\bar{i}}^h = \widetilde{\nabla}_{\bar{i}} \widetilde{F}_i^h = \widetilde{\nabla}_{\bar{j}} \widetilde{F}_{\bar{i}}^h = 0, \\ \widetilde{\nabla}_j \widetilde{F}_i^{\bar{h}} = -\Gamma_j^h \nabla_j F_i^l + \Gamma_i^l \nabla_l F_i^h, \\ \widetilde{\nabla}_j \widetilde{F}_{\bar{i}}^{\bar{h}} = \nabla_j F_i^h, \quad \widetilde{\nabla}_{\bar{j}} \widetilde{F}_i^h = 0, \quad \widetilde{\nabla}_{\bar{j}} \widetilde{F}_{\bar{i}}^h = 0. \end{array} \right.$$

Suppose now that M is an almost complex manifold and F is the structure. Then it is seen [9] that the horizontal lift F^H defines an almost complex structure in $T(M)$, and the components of the Nijenhuis tensor \widetilde{N} of F^H are given by

$$(6.9) \quad \left\{ \begin{array}{l} \widetilde{N}_{ji}^h = N_{ji}^h, \quad \widetilde{N}_{\bar{j}\bar{i}}^h = \widetilde{N}_{\bar{j}i}^h = \widetilde{N}_{j\bar{i}}^h = 0, \\ \widetilde{N}_{ji}^{\bar{h}} = -\Gamma_i^h N_{ji}^l + \Gamma_i^l (F_j^m \nabla_m F_i^h + F_i^m \nabla_j F_m^h) \\ \quad - \Gamma_i^l (F_i^m \nabla_m F_l^h + F_l^m \nabla_i F_m^h), \\ \widetilde{N}_{j\bar{i}}^{\bar{h}} = F_j^l \nabla_l F_i^h + F_i^l \nabla_j F_l^h, \\ \widetilde{N}_{\bar{j}\bar{i}}^{\bar{h}} = 0. \end{array} \right.$$

Similarly to §5, we consider an S -hypersurface $S(M)$ and denote by $\bar{F}^H = (f, \xi, \eta)^H$ the induced almost contact structure in $S(M)$ from the horizontal lift F^H to $T(M)$. The components of the tensor fields f , ξ and η of the structure \bar{F}^H are given by

$$(6.10) \quad f_\beta^\alpha \left\{ \begin{array}{l} f_i^h = F_i^h, \quad F_\lambda^h = 0, \\ f_i^{\bar{h}} = (-\Gamma_i^h F_i^l + \Gamma_i^l F_i^h + B_i^l \bar{F}_i^h - F_i^l B_i^{\bar{h}}) B_{\bar{k}}^{\lambda}, \\ \quad = (-F_i^l \nabla_l y^h + F_i^h \nabla_l y^l) B_{\bar{k}}^{\lambda}, \\ f_\lambda^k = B_\lambda^l F_i^h B_{\bar{k}}^{\bar{h}}, \end{array} \right.$$

$$(6.11) \quad \xi^\alpha \left\{ \begin{array}{l} \xi^h = 0, \\ \xi^{\bar{h}} = y^i F_i^h B_{\bar{k}}^{\bar{h}}, \end{array} \right.$$

$$(6.12) \quad \eta_\beta \left\{ \begin{array}{l} \eta_i = -(-F_i^l \nabla_l y^h + F_i^h \nabla_l y^l) C_{\bar{k}}, \\ \eta_\lambda = -B_\lambda^l F_i^h C_{\bar{k}}, \end{array} \right.$$

respectively.

Substituting (6.4), (6.5), (6.8), (6.10), (6.11) and (6.12) into the four kinds of the components, $L_i^h, L_\lambda^h, L_i^\kappa, L_\lambda^\kappa$, of the tensor field L defined by (2.16), we see that

THEOREM 6. *For the induced almost contact structure \bar{F}^H of an S -hypersurface from the horizontal lift F^H to $T(M)$ of an almost complex structure F in M , the tensor field L defined by (2.16) vanishes identically.*

Hence the Nijenhuis tensor \bar{N} and the others given by (2.17) of the induced almost contact structure \bar{F}^H in $S(M)$ have components

$$\begin{aligned} \bar{N}_{\gamma\beta}^\alpha &= \left\{ \begin{array}{l} \bar{N}_{ji}^h = N_{ji}^h, \quad \bar{N}_{\mu i}^h = \bar{N}_{j\lambda}^h = \bar{N}_{\mu\lambda}^h = 0, \\ \bar{N}_{ji}^\kappa = [-N_{ji}^i \nabla_i y^h + (\nabla_i y^i)(F_j^m \nabla_m F_i^h + F_i^m \nabla_j F_m^h) \\ \quad - (\nabla_j y^i)(F_i^m \nabla_m F_i^h + F_i^m \nabla_i F_m^h)]B_h^\kappa, \\ \bar{N}_{j\lambda}^\kappa = B_\lambda^i(F_j^i \nabla_i F_i^h + F_i^i \nabla_j F_i^h)B_h^\kappa, \\ \bar{N}_{\mu\lambda}^\kappa = 0, \end{array} \right. \\ \bar{N}_{\beta\gamma} &= \left\{ \begin{array}{l} \bar{N}_{ji} = [-N_{ji}^i \nabla_i y^h + (\nabla_i y^i)(F_j^m \nabla_m F_i^h + F_i^m \nabla_j F_m^h) \\ \quad - (\nabla_j y^i)(F_i^m \nabla_m F_i^h + F_i^m \nabla_i F_m^h)]C_h, \\ \bar{N}_{j\lambda} = B_\lambda^i(F_j^i \nabla_i F_i^h + F_i^i \nabla_j F_i^h)C_h, \\ \bar{N}_{\mu\lambda} = 0, \end{array} \right. \\ \bar{N}_\beta^\alpha &= \left\{ \begin{array}{l} \bar{N}_i^h = \bar{N}_\lambda^h = 0, \\ \bar{N}_i^\kappa = -y^i(F_i^i \nabla_i F_j^h + F_j^i \nabla_i F_i^h)B_h^\kappa, \\ \bar{N}_\lambda^\kappa = 0, \end{array} \right. \\ \bar{N}_\beta &= \left\{ \begin{array}{l} \bar{N}_i = y^i(F_i^i \nabla_i F_j^h + F_j^i \nabla_i F_i^h)C_h, \\ \bar{N}_\lambda = 0. \end{array} \right. \end{aligned}$$

Now we have the following

THEOREM 7. *Let M be an almost complex manifold with structure F and symmetric affine connection Γ . In order that the induced almost contact structure $\bar{F}^H = (f, \xi, \eta)^H$ in an S -hypersurface $S(M)$ from the horizontal lift F^H to $T(M)$ of F is normal, it is necessary and sufficient that the*

structure F is complex and the affine connection Γ satisfies the equation

$$(6.13) \quad F_j^i \nabla_i F_i^h + F_i^i \nabla_j F_i^h = 0.$$

PROOF. It is known that $\bar{N}_{\gamma\beta}^\alpha = 0$ implies $\bar{N}_{\gamma\beta} = \bar{N}_\beta^\alpha = \bar{N}_\beta = 0$. It follows from the first components $\bar{N}_{ji}^h = 0$ of \bar{N} that the Nijenhuis tensor N vanishes and F is complex. Moreover, it follows from $\bar{N}_{j\lambda}^\kappa = 0$ and $\bar{N}_{j\lambda} = 0$ that

$$B_\lambda^i (F_j^i \nabla_i F_i^h + F_i^i \nabla_j F_i^h) = 0$$

and from $\bar{N}_i^\kappa = 0$ and $\bar{N}_i = 0$ that

$$y^i (F_j^i \nabla_i F_i^h + F_i^i \nabla_j F_i^h) = 0.$$

By use of these equations, we obtain the equation (6.13).

Q.E.D.

If Γ is the symmetric affine connection which leaves F invariant, $\nabla F = 0$ and which exists in a complex manifold, then the equation (6.13) is satisfied.

7. Tangent sphere bundle of a Riemannian manifold. Let M be a Riemannian manifold with metric tensor $g = (g_{ji})$ and $\Gamma = (\Gamma_{ji}^h)$ the Riemannian connection of \tilde{g} . Putting

$$(7.1) \quad dy^h = dy^h + \Gamma_j^h dx^j, \quad \Gamma_j^h = \Gamma_{ji}^h y^i,$$

many authors consider the Riemannian metric $\tilde{g} = (\tilde{g}_{CB})$ in the tangent bundle $T(M)$ defined by

$$(7.2) \quad \tilde{g}_{CB} dx^c dx^B = g_{ji} dx^j dx^i + g_{ji} \delta y^j \delta y^i$$

with respect to an induced coordinate system. The components of the metric tensor \tilde{g} are

$$(7.3) \quad \tilde{g}_{CB} \quad \begin{cases} \tilde{g}_{ji} = g_{ji} + g_{ts} \Gamma_j^t \Gamma_i^s, & \tilde{g}_{i\bar{j}} = \Gamma_{i\bar{j}}, \\ \tilde{g}_{\bar{i}j} = \Gamma_{\bar{i}j}, & \tilde{g}_{\bar{j}\bar{i}} = g_{ji}, \end{cases}$$

and its contravariant components are

$$(7.4) \quad \tilde{g}^{BA} \quad \begin{cases} \tilde{g}^{th} = g^{th}, & \tilde{g}^{i\bar{h}} = -\Gamma_i^h g^{th}, \\ \tilde{g}^{i\bar{h}} = -\Gamma_i^h g^{th}, & \tilde{g}^{\bar{i}\bar{h}} = g^{th} + g^{ts} \Gamma_t^i \Gamma_s^h, \end{cases}$$

where we have put

$$\Gamma_{ji} = \Gamma_j^h g_{hi}.$$

The components of the Riemannian connection, denoted by $\Gamma^M = (\widetilde{\Gamma}_{CB}^A)$, of the metric \widetilde{g} in $T(M)$ are given by

$$(7.5) \quad \left\{ \begin{array}{l} \widetilde{\Gamma}_{ji}^h = \Gamma_{ji}^h + \frac{1}{2}(K_{.sj}^h \Gamma_i^s + K_{.si}^h \Gamma_j^s), \\ \widetilde{\Gamma}_{ji}^h = \frac{1}{2} K_{.ji}^h, \quad \widetilde{\Gamma}_{ji}^h = \frac{1}{2} K_{.ij}^h, \quad \widetilde{\Gamma}_{ji}^h = 0, \\ \widetilde{\Gamma}_{ji}^h = \partial \Gamma_{ji}^h - \frac{1}{2} [K_{.ji}^h + K_{.ij}^h + (K_{.sj}^i \Gamma_i^s + K_{.si}^i \Gamma_j^s) \Gamma_i^h], \\ \widetilde{\Gamma}_{ji}^h = \Gamma_{ji}^h - \frac{1}{2} K_{.ji}^i \Gamma_i^h, \\ \widetilde{\Gamma}_{ji}^h = 0. \end{array} \right.$$

In this and the next paragraphs, we consider the tangent sphere bundle $S(M)$, which consists of the unit spheres defined by

$$(7.6) \quad g_{ih}(x)y^i y^h = 1$$

with respect to the metric $g(x)$ in the tangent space $T_x(M)$ at each point x . The sphere bundle $S(M)$ is represented by parametric equations

$$(7.7) \quad x^h = x^h, \quad x^{\bar{i}} = y^h = y^h(x^i, u^\lambda)$$

satisfying the equation (7.6). Differentiating (7.6) covariantly in x^j and partially in u^λ , we have

$$(7.8) \quad g_{ih}(\nabla_j y^i) y^h = g_{ih}(B_j^{\bar{i}} + \Gamma_j^i) y^h = g_{ih} B_j^{\bar{i}} y^h + \Gamma_{jh} y^h = 0$$

and

$$(7.9) \quad g_{ih} B_\lambda^{\bar{i}} y^h = 0,$$

respectively.

By means of (7.8) and (7.9), the vector field C , having the components

$\begin{pmatrix} C^h \\ C^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ y^h \end{pmatrix}$, on $S(M)$ satisfies the equations

$$(7.10) \quad \tilde{g}_{BA} B_j^B C^A = (\tilde{g}_{ih} B_j^i + \tilde{g}_{ih} B_j^{\bar{i}}) y^h = (\Gamma_{jh} + g_{ih} B_j^{\bar{i}}) y^h = 0,$$

$$\tilde{g}_{BA} B_{\lambda}^B C^A = g_{ih} B_{\lambda}^i y^h = g_{ih} B_{\lambda}^{\bar{i}} y^h = 0$$

and

$$(7.11) \quad \tilde{g}_{BA} C^B C^A = \tilde{g}_{ih} y^i y^h = g_{ih} y^i y^h = 1.$$

Therefore the vector field $C = \begin{pmatrix} 0 \\ y^h \end{pmatrix}$ is the unit normal to the tangent sphere bundle $S(M)$ with respect to the metric \tilde{g} . The covariant components of C are equal to

$$(7.12) \quad C_B : C_i = \Gamma_{ih} y^h, \quad C_{\bar{i}} = g_{ih} y^h = y_i.$$

The induced metric $\bar{g} = (\bar{g}_{\gamma\beta})$ in the tangent sphere bundle $S(M)$ from the metric \tilde{g} is given by

$$\bar{g}_{\gamma\beta} = \tilde{g}_{CB} B_{\gamma}^C B_{\beta}^B$$

and have covariant components

$$(7.13) \quad \left\{ \begin{array}{l} \bar{g}_{ji} = g_{ji} + g_{is} \Gamma_j^i \Gamma_s^i + \Gamma_{jh} B_i^{\bar{h}} + \Gamma_{ih} B_j^{\bar{h}} + g_{ih} B_j^{\bar{i}} B_i^{\bar{h}} \\ \quad = g_{ji} + g_{is} (\nabla_j y^i) (\nabla_i y^s), \\ \bar{g}_{\mu i} = \Gamma_{\mu i} B_{\mu}^j + g_{js} B_{\mu}^j B_{\mu}^{\bar{s}} = B_{\mu}^j \nabla_i y_j = g_{is} (\partial_{\mu} y^i) (\nabla_i y^s), \\ \bar{g}_{\mu\lambda} = g_{ji} B_{\mu}^j B_{\lambda}^{\bar{i}} = g_{ji} (\partial_{\mu} y^j) (\partial_{\lambda} y^i), \end{array} \right.$$

and contravariant components

$$(7.14) \quad \left\{ \begin{array}{l} \bar{g}^{ih} = g^{ih}, \\ \bar{g}^{\lambda h} = -g^{hi} (\nabla_i y^h) B_{\lambda}^i, \\ \bar{g}^{\lambda\kappa} = [g^{ih} + g^{ts} (\nabla_i y^h) (\nabla_s y^t)] B_{\lambda}^i B_{\kappa}^{\bar{i}}. \end{array} \right.$$

Since the Riemannian connection $\bar{\Gamma}^M = (\bar{\Gamma}_{\gamma\beta}^\alpha)$ of the induced metric \bar{g} are related to the Riemannian connection Γ^M in $T(M)$ by the equation (2.8), we obtain by straightforward computations

$$(7.15) \quad \left\{ \begin{array}{l} \bar{\Gamma}_{ji}^h = \Gamma_{ji}^h + \frac{1}{2}(K_{s_j}{}^h \nabla_i y^s + K_{s_i}{}^h \nabla_j y^s), \\ \bar{\Gamma}_{\mu i}^h = \frac{1}{2} K_{s_i}{}^h B_\mu{}^s, \\ \bar{\Gamma}_{\mu\lambda}^h = 0, \\ \bar{\Gamma}_{ji}^k = [\nabla_j \nabla_i y^h + \frac{1}{2} K_{j_i}{}^h - \frac{1}{2} K_{i_j}{}^h \\ \quad - \frac{1}{2}(K_{s_j}{}^l \nabla_i y^s + K_{s_i}{}^l \nabla_j y^s) \nabla_l y^h] B_k{}^h, \\ \bar{\Gamma}_{\mu i}^k = (\partial_\mu \nabla_i y^h - \frac{1}{2} K_{s_i}{}^l B_\mu{}^s \nabla_l y^h) B_k{}^h, \\ \bar{\Gamma}_{\mu\lambda}^k = (\partial_\mu \partial_\lambda y^h) B_k{}^h. \end{array} \right.$$

The second fundamental tensor h in $S(M)$ has components

$$(7.16) \quad h_{\gamma\beta} = \begin{cases} h_{ji} = y_h \nabla_j \nabla_i y^h = -(\nabla_j y_h)(\nabla_i y^h), \\ h_{\mu i} = y_h \partial_\mu \nabla_i y^h = -(\partial_\mu y_h)(\nabla_i y^h), \\ h_{\mu\lambda} = y_h \partial_\mu \partial_\lambda y^h = -(\partial_\mu y_h)(\partial_\lambda y^h). \end{cases}$$

8. Contact structure in a tangent sphere bundle. In the tangent bundle $T(M)$ of a Riemannian manifold M , there exists the 1-form

$$(8.1) \quad \theta = y_i dx^i = g_{ji} y^j dx^i$$

and the derived form $d\theta$ is equal to

$$(8.2) \quad \begin{aligned} d\theta &= [(\partial_j g_{ki}) y^k dx^i + g_{ji} dy^i] \wedge dx^i \\ &= [(\Gamma_{jk}^i g_{ii} + \Gamma_{ji}^i g_{ik}) y^k dx^i + g_{ji} dy^i] \wedge dx^i \\ &= \frac{1}{2} [(\Gamma_{ji} - \Gamma_{ij}) dx^i \wedge dx^i + g_{ji} dy^i \wedge dx^i - g_{ji} dx^i \wedge dy^i]. \end{aligned}$$

Putting

$$(8.3) \quad d\theta = \frac{1}{2} \tilde{F}_{CB} dx^C \wedge dx^B,$$

we have the skew-symmetric tensor field

$$(8.4) \quad \widetilde{F}_{GB} \begin{cases} \widetilde{F}_{ji} = \Gamma_{ji} - \Gamma_{ij}, & \widetilde{F}_{ji} = g_{ji}, \\ \widetilde{F}_{ji} = -g_{ji}, & \widetilde{F}_{ji} = 0, \end{cases}$$

and the $(1, 1)$ -tensor field $F^M = (\widetilde{F}_B^A)$ defined by

$$\widetilde{F}_B^A = \widetilde{F}_{BG} g^{CA},$$

whose components are

$$(8.5) \quad \widetilde{F}_B^A \begin{cases} \widetilde{F}_i^h = \Gamma_i^h, & \widetilde{F}_i^h = \delta_i^h, \\ \widetilde{F}_i^{\bar{h}} = -\Gamma_i^h \Gamma_i^{\bar{h}} - \delta_i^h, & \widetilde{F}_i^{\bar{h}} = -\Gamma_i^{\bar{h}}. \end{cases}$$

The tensor field F^M satisfies the equation

$$(8.6) \quad \widetilde{F}_G^B \widetilde{F}_B^A = -\delta_G^A,$$

that is, it defines an almost Hermitian structure in $T(M)$ with the metric \widetilde{g} . Since the fundamental form $d\theta$ is closed, the structure (F^M, \widetilde{g}) is almost Kählerian.

Furthermore, since (\widetilde{F}_{BA}) is skew symmetric and we have

$$C^B \widetilde{F}_B^A C_A = \widetilde{F}_{BA} C^B C^A = 0,$$

the tangent sphere bundle $S(M)$ is an S -hypersurface in $T(M)$ with respect to the almost Kählerian structure (F^M, \widetilde{g}) . Hence, to the tangent sphere bundle $S(M)$, the contact metric structure, denoted by $(\bar{F}, \bar{g})^M = (f, \xi, \eta, \bar{g})^M$ is induced, from the almost Kählerian structure in $T(M)$, and the tensor fields f, ξ and η have components

$$(8.7) \quad f_\beta^\alpha \begin{cases} f_i^h = \nabla_i y^h, & f_\lambda^h = B_\lambda^{\bar{h}}, \\ f_i^{\bar{h}} = -[(\nabla_i y^l)(\nabla_l y^h) + \delta_i^h] B_h^{\bar{h}}, & \\ f_\lambda^{\bar{h}} = -B_\lambda^i (\nabla_i y^h) B_h^{\bar{h}}, & \end{cases}$$

$$(8.8) \quad \xi^\alpha \begin{cases} \xi^h = y^h, \\ \xi^{\bar{h}} = -y^i (\nabla_i y^h) B_h^{\bar{h}}, \end{cases}$$

$$(8.9) \quad \eta_\beta \begin{cases} \eta_i = y_i, \\ \eta_\lambda = 0. \end{cases}$$

The covariant components of $f = (f_{\beta}^{\alpha})$ are

$$(8.10) \quad f_{\gamma\beta} \begin{cases} f_{ji} = \nabla_j y_i - \nabla_i y_j, \\ f_{\mu i} = -f_{i\mu} = \partial_{\mu} y_i, \\ f_{\mu\lambda} = 0, \end{cases}$$

and we have

$$(8.11) \quad f_{\gamma\beta} = \partial_{\gamma}\eta_{\beta} - \partial_{\beta}\eta_{\gamma}.$$

Therefore the induced structure $(\bar{F}, \bar{g})^M$ in the tangent sphere bundle $S(M)$ is a contact metric structure.

Now we prove the following

THEOREM 8. *Let M be a Riemannian manifold with metric g and $(\bar{F}, \bar{g})^M = (f, \xi, \eta, \bar{g})^M$ the induced contact metric structure defined above in the tangent sphere bundle $S(M)$. In order that the structure $(\bar{F}, \bar{g})^M$ is K -contact, that is, the vector field ξ in $S(M)$ is an infinitesimal isometry with respect to the induced metric \bar{g} , it is necessary and sufficient that the manifold M is of positive constant curvature. Then the induced structure $(\bar{F}, \bar{g})^M$ is Sasakian.*

PROOF. The Killing equation of ξ in $S(M)$ is now written as

$$(8.12) \quad \bar{\mathcal{L}}(\xi)\bar{g}_{\gamma\beta} = \bar{\nabla}_{\gamma}\eta_{\beta} + \bar{\nabla}_{\beta}\eta_{\gamma} = \partial_{\gamma}\eta_{\beta} + \partial_{\beta}\eta_{\gamma} - 2\bar{\Gamma}_{\gamma\beta}^{\alpha}\eta_{\alpha} = 0,$$

where $\bar{\nabla}$ indicates covariant differentiation with respect to the metric \bar{g} . Substituting the expressions (7.15) and (8.9) of the components η_{β} and $\bar{\Gamma}_{\gamma\beta}^{\alpha}$ into (8.12), we have the equations

$$(8.13) \quad \begin{cases} \bar{\mathcal{L}}(\xi)\bar{g}_{ji} = \nabla_j y_i + \nabla_i y_j - (K_{sj} \cdot \nabla_i y^s + K_{si} \cdot \nabla_j y^s) = 0, \\ \bar{\mathcal{L}}(\xi)\bar{g}_{\mu i} = \partial_{\mu} y_i - K_{si} \cdot B_{\mu}^s = 0, \\ \bar{\mathcal{L}}(\xi)\bar{g}_{\mu\lambda} = 0. \end{cases}$$

It follows from the second equation that

$$B_\mu^J(g_{ji} - K_{.ji.}) = 0$$

and, taking account of the symmetry of $g_{ji} - K_{.ji.}$ in i and j , we may put

$$g_{ji} - K_{.ji.} = \alpha y_j y_i.$$

Contracting this expression with y^i , we see that $\alpha=1$ and hence we have the equation

$$(8.14) \quad g_{ji} - K_{.ji.} = y_j y_i.$$

By means of (7.11), this equation is written as

$$K_{kjih} y^k y^h = g_{ji} g_{kh} y^k y^h - g_{jk} g_{ih} y^k y^h.$$

Since this equation is valid for an arbitrary tangent vector y , we have the equation

$$K_{kjih} + K_{hjik} = 2g_{ji} g_{kh} - g_{jk} g_{ih} - g_{jh} g_{ik}.$$

By interchanging the indices k and j , taking the difference and using Bianchi's identity, we obtain the equation

$$(8.15) \quad K_{kjih} = g_{kh} g_{ji} - g_{jh} g_{ki},$$

which means for M to be of positive constant curvature. We see that the first equation of (8.13) is satisfied by (8.15). Thus the first half of the theorem has been established.

When the curvature tensor of M has the form (8.15), by substituting (7.15) and (8.10) into the components of the covariant derivative

$$\bar{\nabla}_\gamma f_{\beta\alpha} = \partial_\gamma f_{\beta\alpha} - \bar{\Gamma}_{\gamma\beta}^\epsilon f_{\epsilon\alpha} - \bar{\Gamma}_{\gamma\alpha}^\epsilon f_{\beta\epsilon},$$

and comparing the results with (7.13) and (8.9), we obtain the equations

$$\begin{aligned} \bar{\nabla}_j f_{ih} &= \frac{1}{2} \left[y_i (g_{jh} + (\nabla_j y^l)(\nabla_h y_l)) - y_h (g_{ji} + (\nabla_j y^l)(\nabla_i y_l)) \right] \\ &= \frac{1}{2} (\eta_i \bar{g}_{jh} - \eta_h \bar{g}_{ji}), \\ \bar{\nabla}_j f_{i\kappa} &= \frac{1}{2} y_i (\nabla_j y^l)(\partial_\kappa y_l) \end{aligned}$$

$$= \frac{1}{2} \eta_i \bar{g}_{jk},$$

$$\bar{\nabla}_j f_{ik} = 0,$$

$$\bar{\nabla}_\mu f_{ih} = \frac{1}{2} \left[y_i (\partial_\mu y^l) (\nabla_h y_l) - y_h (\partial_\mu y^l) (\nabla_i y_l) \right] = \frac{1}{2} (\eta_i \bar{g}_{\mu h} - \eta_h \bar{g}_{\mu i}),$$

$$\bar{\nabla}_\mu f_{ik} = \frac{1}{2} y_i (\partial_\mu y^l) (\partial_k y_l) = \frac{1}{2} \eta_i \bar{g}_{\mu k},$$

$$\bar{\nabla}_\mu f_{ik} = 0.$$

These equations are combined up to the tensor equation

$$2\bar{\nabla}_\gamma f_{\beta\alpha} = \eta_\beta \bar{g}_{\gamma\alpha} - \eta_\alpha \bar{g}_{\gamma\beta},$$

which means that the induced structure $(\bar{F}, \bar{g})^M$ is Sasakian. Q.E.D.

BIBLIOGRAPHY

- [1] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure, I, Tôhoku Math. J., 12(1960), 459-476.
- [2] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure, II, Tôhoku Math. J., 13(1961), 281-294.
- [3] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures, J. Math. Soc. Japan, 14(1962), 249-271.
- [4] S. SASAKI, Almost Contact Manifolds, I, II, Lecture Note at Tôhoku Univ., 1965, 1967.
- [5] S. TACHIBANA AND M. OKUMURA, On the almost complex structure of tangent bundles of Riemannian spaces, Tôhoku Math. J., 14(1962), 156-161.
- [6] Y. TASHIRO, On contact structure of hypersurfaces in complex manifolds, I, II, Tôhoku Math. J., 15(1963), 62-73, 167-175.
- [7] K. YANO, Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press, Oxford, 1965.
- [8] K. YANO AND S. KOBAYASHI, Prolongations of tensor fields and connections to tangent bundles, I, J. Math. Soc. Japan, 18(1966), 194-210.
- [9] K. YANO AND S. ISHIHARA, Horizontal lifts of tensor fields to tangent bundles, J. Math. Mech., 16(1967), 1015-1029.
- [10] K. YANO, Differential Geometry of Tangent Bundles, to appear.

THE COLLEGE OF GENERAL EDUCATION
OKAYAMA UNIVERSITY
OKAYAMA, JAPAN