On Continuity Properties of the Law of Integrals of Lévy Processes

Jean Bertoin* Alexander Lindner† Ross Maller‡

Abstract

Let (ξ, η) be a bivariate Lévy process such that the integral $\int_0^\infty e^{-\xi_{t-}}\, d\eta_t$ converges almost surely. We characterise, in terms of their Lévy measures, those Lévy processes for which (the distribution of) this integral has atoms. We then turn attention to almost surely convergent integrals of the form $I := \int_0^\infty g(\xi_t)\, dt$, where g is a deterministic function. We give sufficient conditions ensuring that I has no atoms, and under further conditions derive that I has a Lebesgue density. The results are also extended to certain integrals of the form $\int_0^\infty g(\xi_t)\, dY_t$, where Y is an almost surely strictly increasing stochastic process, independent of ξ.

1 Introduction

The aim of this paper is to study continuity properties of stationary distributions of generalised Ornstein-Uhlenbeck processes and of distributions of random variables of the form $\int_0^\infty g(\xi_t)\, dt$ for a Lévy process ξ and a general function $g : \mathbb{R} \to \mathbb{R}$.

For a bivariate Lévy process $(\xi, \zeta) = (\xi_t, \zeta_t)_{t\geq 0}$, the *generalised Ornstein-Uhlenbeck (O-U) process* $(V_t)_{t\geq 0}$ is defined as

$$V_t = e^{-\xi_t} \left(\int_0^t e^{\xi_{s-}}\, d\zeta_s + V_0 \right), \quad t \geq 0,$$

where V_0 is a finite random variable, independent of (ξ, ζ). This process appears as a natural continuous time generalisation of random recurrence equations, as shown by

*Laboratoire de Probabilités, Université Paris VI, 175 rue du Chevaleret, 75013 Paris, France, email: jbe@ccr.jussieu.fr

†Centre for Mathematical Sciences, Technical University of Munich, Boltzmannstraße 3, D-85747 Garching, Germany, email: lindner@ma.tum.de

‡Centre for Mathematical Analysis, and School of Finance & Applied Statistics, Australian National University, Canberra, ACT, email: Ross.Maller@anu.edu.au

de Haan and Karandikar [11], and has applications in many areas, such as risk theory (e.g. Paulsen [19]), perpetuities (e.g. Dufresne [6]), financial time series (e.g. Klüppelberg et al. [14]) or option pricing (e.g. Yor [23]), to name just a few. See also Carmona et al. [2, 3] for further properties of this process. Lindner and Maller [17] have shown that the existence of a stationary solution to the generalised O-U process is closely related to the almost sure convergence of the stochastic integral $\int_0^t e^{-\xi_{s-}}\, d\eta_s$ as $t \to \infty$, where (ξ, η) is a bivariate Lévy process, and η can be explicitly constructed in terms of (ξ, ζ). The stationary distribution is then given by $\int_0^\infty e^{-\xi_{s-}}\, d\eta_s$. Necessary and sufficient conditions for the convergence of $\int_0^\infty e^{-\xi_{s-}}\, d\eta_s$ were obtained by Erickson and Maller [7]. Distributional properties of the limit variable and hence of the stationary distribution of generalised O-U processes are of particular interest. Gjessing and Paulsen [9] determined the distribution in many cases when ξ and η are independent and the Lévy measure of (ξ, η) is finite. Carmona et al. [2] considered the case when $\eta_t = t$ and the jump part of ξ is of finite variation. Under some additional assumptions, they showed that $\int_0^\infty e^{-\xi_{s-}} ds$ is absolutely continuous, and its density satisfies a certain integro-differential equation. In Section 2 we shall be concerned with continuity properties of the limit variable $\int_0^\infty e^{-\xi_{s-}}\, d\eta_s$ without any restrictions on (ξ, η), assuming only convergence of the integral. We shall give a complete characterisation of when this integral has atoms, in terms of the characteristic triplet of (ξ, η). This characterisation relies on a similar result of Grincevičius [10] for "perpetuities" which are a kind of discrete time analogue of Lévy integrals.

Then, in Section 3, we turn our attention to continuity properties of the distribution of the integral $\int_0^\infty g(\xi_t)\, dt$, where $\xi = (\xi_t)_{t\geq 0}$ is a one-dimensional Lévy process with non-zero Lévy measure and g is a general deterministic Borel function. Such integrals appear in a variety of situations, for example concerning shattering phenomena in fragmentation processes, see, e.g., Haas [12].

Fourier analysis and Malliavin calculus are classical tools for establishing the absolute continuity of distributions of functionals of stochastic processes. In a different direction, the book of Davydov et al. [5] treats three different methods for proving absolute continuity of such functionals: the "stratification method", the "superstructure method" and the "method of differential operators". Chapter 4 in [5] pays particular attention to Poisson functionals, which includes integrals of Lévy processes. While it may be hard to check the conditions and apply these methods in general (in particular to find admissible semigroups for the stratification method), it has been carried out in some cases. For example, Davydov [4] gives sufficient conditions for absolute continuity of integrals of the form $\int_0^1 g(X_t)\, dt$ for *strictly stationary* processes $(X_t)_{t\geq 0}$ and quite general g. Concerning integrals of Lévy processes, Lifshits [16], p. 757, has shown that $\int_0^1 g(\xi_t)\, dt$ is absolutely continuous if ξ is a Lévy process with infinite and absolutely continuous Lévy measure, and g is locally Lipschitz-continuous and such that on a set of full Lebesgue measure in $[0,1]$ the derivative g' of g exists and is continuous and non-vanishing; see also Problem 15.1 in [5]. For our study of atoms of the distributions of integrals such as $\int_0^\infty g(\xi_t)\,dt$, we will impose less restrictive assumptions on g in Section 3. Note also that [5] and the references given there are usually concerned with the absolute continuity of functionals such as $\int_0^1 g(\xi_t)\,dt$ on the *compact* interval $[0,1]$, while we are concerned with integrals over $(0,\infty)$. That absolute continuity of the distribution of integrals over compact sets and over $(0,\infty)$ can be rather different topics is straightforward by considering the special case of compound Poisson processes. See also part (iii) of Theorem 2.2 below for situations where the integral over every finite time horizon may be absolutely continuous, while the limit variable can degenerate to a constant.

Section 3 is organised as follows: we start with some motivating examples, in some of which $\int_0^\infty g(\xi_t)\,dt$ has atoms while in others it does not. Then, in Section 3.2 we present some general criteria which ensure the continuity of the distribution of $\int_0^\infty g(\xi_t)\,dt$. The proofs there are based on the sample path behaviour and on excursion theory for Lévy processes. Then, in Section 3.3 we use a simple form of the stratification method to obtain absolute continuity of $\int_0^\infty g(\xi_t)\,dt$ for certain cases of g and ξ (which assume however no differentiability properties of g); the results are also extended to more general integrals of the form $\int_0^\infty g(\xi_t)\,dY_t$, where $Y = (Y_t)_{t\geq 0}$ is a strictly increasing stochastic process, independent of the Lévy process ξ.

Observe that our focus will be on continuity properties of the distribution of the integral $\int_0^\infty g(\xi_t)\,dt$ (or similar integrals), under the assumption that it is finite a.s. A highly relevant question is to ask under which conditions the integral does converge. It is important of course that any conditions we impose to ensure continuity of the integral, or its absence, be compatible with convergence. We only occasionally address this issue, when it is possible to give some simple sufficient (or, sometimes, necessary) conditions for convergence. Our approach is essentially to assume convergence and study the properties of the resulting integral. For a much fuller discussion of conditions for convergence *per se* we refer to Erickson and Maller [7, 8], who give an overview of known results as well as new results on the finiteness of Lévy integrals.

We end this section by setting some notation. Recall that a *Lévy process* $X = (X_t)_{t\geq 0}$ in $\mathbb{R}^d$ ($d \in \mathbb{N}$) is a stochastically continuous process having independent and stationary increments, which has almost surely càdlàg paths and satisfies $X_0 = 0$. For each Lévy process, there exists a unique constant $\gamma = \gamma_X = (\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^d$, a symmetric positive semidefinite matrix $\Sigma = \Sigma_X$, and a Lévy measure $\Pi = \Pi_X$ on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d} \min\{1, |x|^2\}\,\Pi_X(dx) < \infty$, such that for all $t > 0$ and $\theta \in \mathbb{R}^d$ we have

$$(1/t)\log E\exp(i\langle\theta, X_t\rangle) = i\langle\gamma,\theta\rangle - \frac{1}{2}\langle\theta,\Sigma\theta\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,\theta\rangle} - 1 - i\langle z,\theta\rangle\mathbf{1}_{|z|\leq 1})\,\Pi_X(dz).$$

Here, $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the inner product and Euclidian norm in $\mathbb{R}^d$, and $\mathbf{1}_A$ is the indicator function of a set A. Together, (γ,Σ,Π) form the *characteristic triplet* of X. The Brownian motion part of X is described by the covariance matrix Σ_X. If $d=1$, then we will also write σ_X^2 for Σ_X, and if $d=2$ and $X=(\xi,\eta)$, the upper and lower diagonal elements of Σ_X are given by σ_ξ^2 and σ_η^2. We refer to Bertoin [1] and Sato [21] for further definitions and basic properties of Lévy processes. Integrals of the form $\int_a^b e^{-\xi_{t-}}\,d\eta_t$ for a bivariate Lévy process (ξ,η) are interpreted as the usual stochastic integral with respect to its completed natural filtration as in Protter [20], where $\int_a^b$ denotes integrals over the set $[a,b]$, and $\int_{a+}^b$ denotes integrals over the set $(a,b]$. If η (or a more general stochastic process $Y=(Y_t)_{t\geq 0}$ as an integrator) is of bounded variation on compacts, then the stochastic integral is equal to the pathwise computed Lebesgue-Stieltjes integral, and will also be interpreted in this sense. Integrals such as $\int_0^\infty$ are to be interpreted as limits of integrals of the form $\int_0^t$ as $t\to\infty$, where the convergence will typically be almost sure. The *jump* of a càdlàg process $(Z_t)_{t\geq 0}$ at time t will be denoted by $\Delta Z_t := Z_t - Z_{t-} = Z_t - \lim_{u\uparrow t} Z_u$, with the convention $Z_{0-}:=0$. The symbol "$\stackrel{D}{=}$" will be used to denote equality in distribution of two random variables, and "$\stackrel{P}{\to}$" will denote convergence in probability. Almost surely holding statements will be abbreviated by "a.s.", and properties which hold almost everywhere by "a.e.". The Lebesgue measure on $\mathbb{R}$ will be denoted by λ. Throughout the paper, in order to avoid trivialities, we will assume that ξ and η are different from the zero process $t\mapsto 0$.

2 Atoms of exponential Lévy integrals

Let $(\xi,\eta)=(\xi_t,\eta_t)_{t\geq 0}$ be a bivariate Lévy process. Erickson and Maller [7] characterised when the exponential integral $I_t := \int_0^t e^{-\xi_{s-}}\,d\eta_s$, $t>0$, converges almost surely to a finite random variable I as $t\to\infty$. They showed that this happens if and only if

$$\lim_{t\to\infty}\xi_t = +\infty \text{ a.s.,}\quad \text{and}\quad \int_{\mathbb{R}\setminus[-e,e]}\left(\frac{\log|y|}{A_\xi(\log|y|)}\right)\Pi_\eta(dy)<\infty. \tag{2.1}$$

Here, the function A_ξ is defined by

$$A_\xi(y) := 1+\int_1^y \Pi_\xi((z,\infty))\,dz,\quad y\geq 1.$$

As a byproduct of the proof, they obtained that I_t converges almost surely to a finite random variable I if and only if it converges in distribution to I, as $t\to\infty$. Observe that the convergence condition (2.1) depends on the marginal distributions of ξ and η only, but not on the bivariate dependence structure of ξ and η.

In this section we shall be interested in the question of whether the limit random variable I can have a distribution with atoms. A complete characterisation of this will be given in Theorem 2.2. A similar result for the characterisation of the existence of atoms for discrete time perpetuities was obtained by Grincevičius [10], Theorem 1. We will adapt his proof to show that $\int_0^\infty e^{-\xi_{t-}}\, d\eta_t$ has atoms if and only if it is constant. This will be a consequence of the following lemma, which is formulated for certain families of random fixed point equations.

Lemma 2.1. *For every $t \geq 0$, let Q_t, M_t and ψ_t be random variables such that $M_t \neq 0$ a.s., and ψ_t is independent of (Q_t, M_t). Suppose ψ is a random variable satisfying*

$$\psi = Q_t + M_t \psi_t \quad \text{for all } t \geq 0,$$

and such that

$$\psi \stackrel{D}{=} \psi_t \quad \text{for all } t \geq 0,$$

and suppose further that

$$Q_t \stackrel{P}{\to} \psi \quad \textit{as} \quad t \to \infty.$$

Then ψ has an atom if and only if it is a constant random variable.

Proof. We adapt the proof of Theorem 1 of [10]. Suppose that ψ has an atom at $a \in \mathbb{R}$, so that

$$P(\psi = a) =: \beta > 0.$$

Then for all $\varepsilon \in (0, \beta)$ there exists some $\delta > 0$ such that

$$P(|\psi - a| < 2\delta) < \beta + \varepsilon. \tag{2.2}$$

Since $Q_t \stackrel{P}{\to} \psi$ as $t \to \infty$, there exists $t' = t'(\varepsilon)$ such that

$$P(|\psi - Q_t| \geq \delta) = P(|M_t \psi_t| \geq \delta) < \varepsilon \quad \text{for all } t \geq t'. \tag{2.3}$$

Then (2.2) and (2.3) imply that, for all $t \geq t'$,

$$P(|Q_t - a| < \delta) \leq P(|Q_t - \psi| \geq \delta) + P(|\psi - a| < 2\delta) \leq \beta + 2\varepsilon. \tag{2.4}$$

Now observe that, for all $t \geq 0$,

$$\begin{aligned} \beta &= P(\psi = a) \leq P(\psi = a, |\psi - Q_t| < \delta) + P(|\psi - Q_t| \geq \delta) \\ &= \int_{\mathbb{R}} P(Q_t + M_t s = a, |M_t s| < \delta)\, dP(\psi_t \leq s) + P(|\psi - Q_t| \geq \delta) \\ &= \sum_{s \in D_t} P(Q_t + M_t s = a, |M_t s| < \delta)\, P(\psi_t = s) \ + P(|\psi - Q_t| \geq \delta). \end{aligned}$$

Here, the last equation follows from the fact that $P(Q_t + M_t s = a)$ can be positive for only a countable number of s, $s \in D_t$, say, since the number of atoms of any random variable is countable.

Since $\sum_{s\in D_t} P(\psi_t = s) \leq 1$ for all s, and since $P(|\psi - Q_t| \geq \delta) < \varepsilon$ for $t > t'$, by (2.3), it follows that for such t there is some $s_t \in \mathbb{R}$ such that

$$\beta_t := P(Q_t + M_t s_t = a, |M_t s_t| < \delta) \geq \beta - \varepsilon. \tag{2.5}$$

Observing that, for all $t \geq 0$

$$\{\psi = a\} \cup \{Q_t + M_t s_t = a, |M_t s_t| < \delta\} \subset \{|\psi - Q_t| \geq \delta\} \cup \{|Q_t - a| < \delta\},$$

we obtain for $t \geq t'$ that

$$\begin{aligned} &P(|\psi - Q_t| \geq \delta) + P(|Q_t - a| < \delta) \\ &\quad\geq\ P(\psi = a) + P(Q_t + M_t s_t = a, |M_t s_t| < \delta) - P(Q_t + M_t s_t = a, |M_t s_t| < \delta, \psi = a) \\ &\quad=\ \beta + \beta_t - \beta_t\, P(\psi_t = s_t). \end{aligned}$$

We used here that $P(M_t = 0) = 0$. From (2.3) and (2.4) it now follows that

$$\beta_t\, P(\psi_t = s_t) \geq \beta + \beta_t - \varepsilon - (\beta + 2\varepsilon) = \beta_t - 3\varepsilon.$$

Using (2.5) and the fact that $\psi \stackrel{D}{=} \psi_t$, we obtain

$$P(\psi = s_t) = P(\psi_t = s_t) \geq 1 - \frac{3\varepsilon}{\beta_t} > 1 - \frac{3\varepsilon}{\beta - \varepsilon}.$$

Letting $\varepsilon \to 0$ and observing that $P(\psi = a) > 0$, it follows that $P(\psi = a) = 1$. $\square$

As a consequence, we obtain:

Theorem 2.2. *Let (ξ, η) be a bivariate Lévy process such that ξ_t converges almost surely to ∞ as $t \to \infty$, and let $I_t := \int_0^t e^{-\xi_{s-}}\, d\eta_s$. Denote the characteristic triplet of (ξ, η) by $(\gamma, \Sigma, \Pi_{\xi,\eta})$, where $\gamma = (\gamma_1, \gamma_2)$, and denote the upper diagonal element of Σ by σ_ξ^2. Then the following assertions are equivalent:*

(i) *I_t converges a.s. to a finite random variable I as $t \to \infty$, where I has an atom.*

(ii) *I_t converges a.s. to a constant random variable as $t \to \infty$.*

(iii) *$\exists\, k \in \mathbb{R} \setminus \{0\}$ such that* $P\left(\int_0^t e^{-\xi_{s-}}\, d\eta_s = k(1 - e^{-\xi_t}) \text{ for all } t > 0\right) = 1.$

(iv) *$\exists\, k \in \mathbb{R} \setminus \{0\}$ such that $e^{-\xi} = \mathcal{E}(-\eta/k)$, i.e. $e^{-\xi}$ is the stochastic exponential of $-\eta/k$.*

(v) $\exists\, k \in \mathbb{R} \setminus \{0\}$ *such that*

$$\Sigma_{\xi,\eta} = \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix} \sigma_\xi^2,$$

the Lévy measure $\Pi_{\xi,\eta}$ *of* (ξ, η) *is concentrated on* $\{(x, k(1 - e^{-x})) : x \in \mathbb{R}\}$, *and*

$$\gamma_1 - k^{-1}\gamma_2 = \sigma_\xi^2/2 + \int_{x^2+k^2(1-e^{-x})^2 \le 1} (e^{-x} - 1 + x)\, \Pi_\xi(dx). \tag{2.6}$$

Proof. To show the equivalence of (i) and (ii), suppose that I exists a.s. as a finite random variable and define

$$\psi := I = \int_0^\infty e^{-\xi_{s-}}\, d\eta_s, \quad Q_t := I_t = \int_0^t e^{-\xi_{s-}}\, d\eta_s \quad \text{and} \quad M_t := e^{-\xi_t}, \quad t \ge 0.$$

Then

$$\psi \stackrel{D}{=} \int_{t+}^\infty e^{-(\xi_{s-} - \xi_t)}\, d(\eta_{\cdot} - \eta_t)_s =: \psi_t.$$

So we have the setup of Lemma 2.1:

$$\psi = Q_t + M_t \psi_t,\ t \ge 0, \tag{2.7}$$

Q_t converges in probability (in fact, a.s.) to ψ as $t \to \infty$, and ψ_t is independent of (Q_t, M_t) for all $t \ge 0$. We conclude from Lemma 2.1 that $I = \psi$ is finite a.s. and has an atom if and only if it is constant, equivalently, if (ii) holds.

Now suppose that (ii) holds and that the constant value of the limit variable is k. Then it follows from (2.7) that, a.s.,

$$k = \int_0^t e^{-\xi_{s-}}\, d\eta_s + e^{-\xi_t} k, \ \text{for each } t > 0,$$

hence

$$\int_0^t e^{-\xi_{s-}}\, d\eta_s = k(1 - e^{-\xi_t}) \quad \text{for all } t > 0. \tag{2.8}$$

Observe that $k = 0$ is impossible by uniqueness of the solution to the stochastic differential equation $d\int_0^t X_{s-}\, d\eta_s = 0$ (which implies $e^{-\xi_s} = X_s = 0$, impossible). Since Q_t and $e^{-\xi_t}$ are càdlàg functions, (2.8) holds on an event of probability 1. This shows that (ii) implies (iii). The converse is clear, since $\lim_{t\to\infty} \xi_t = \infty$ a.s. by assumption.

Dividing (2.8) by $-k$, we obtain $e^{-\xi_t} = 1 + \int_0^t e^{-\xi_{s-}} d(-\eta_s/k)$, which is just the defining equation for $e^{-\xi} = \mathcal{E}(-\eta/k)$, see Protter [20], p. 84, giving the equivalence of (iii) and (iv).

The equivalence of (iv) and (v) follows by straightforward but messy calculations using the Doléans-Dade formula and the Lévy-Itô decomposition (for the calculation of γ), and is relegated to the appendix. $\square$

Remarks. (i) Under stronger assumptions, Theorem 2.2 may be strengthened to conclude that I has a density or is constant. Suppose (ξ, η) is a bivariate Lévy process such that ξ has no positive jumps and drifts to ∞, i.e. $\lim_{t\to\infty} \xi_t = \infty$ a.s. Assume further that $\int_{\mathbb{R}\setminus[-e,e]} (\log|y|)\, \Pi_\eta(dy) < \infty$. Then the condition (2.1) is fulfilled, and thus $I := \lim_{t\to\infty} \int_0^t e^{-\xi_{s-}} d\eta_s$ exists and is finite a.s. Applying the strong Markov property at the first passage time $T_x := \inf\{t \geq 0 : \xi_t > x\} = \inf\{t \geq 0 : \xi_t = x\}$ (since ξ has no positive jumps) yields the identity

$$I = \int_0^{T_x} e^{-\xi_{s-}} d\eta_s + e^{-x} I'$$

where I' has the same distribution as I and is independent of $\int_0^{T_x} e^{-\xi_{s-}} d\eta_s$. Thus I is a self-decomposable random variable, and as a consequence its law is infinitely divisible and unimodal and hence has a density, if it is not constant; see Theorem 53.1, p. 404, in Sato [21]. Thus I is continuous. A generalisation of this result to the case of multivariate η was recently obtained by Kondo et al. [15].

(ii) As another important special case, suppose ξ is a Brownian motion with a positive drift, and in addition that $\int_{\mathbb{R}\setminus[-e,e]} (\log|y|)\, \Pi_\eta(dy) < \infty$. Then I is finite a.s. From Condition (iii) of Theorem 2.2 we then see that $\Delta\eta_t = 0$, so the condition can hold only if η_t is also a Brownian motion. By Ito's lemma, Condition (iii) implies $d\eta_t = k(d\xi_t - \sigma_\xi^2 dt/2)$, or, equivalently, $\eta_t = k(\xi_t - \sigma_\xi^2 t/2)$. Similarly, if η is a Brownian motion, (iii) of Theorem 2.2 can only hold if ξ is a Brownian motion and the same relation is satisfied. Thus we can conclude that, apart from this degenerate case, $\int_0^\infty e^{-B_s} d\eta_s$ and $\int_0^\infty e^{-\xi_s} dB_s$, when convergent a.s., have continuous distributions, for a Brownian motion B_t.

3 Integrals with general g

We now turn our attention to the question of whether the integral $\int_0^\infty g(\xi_t)\, dt$ can have atoms, where g is a more general deterministic function, and $\xi = (\xi_t)_{t\geq 0}$ is a non-zero Lévy process. To start with, we shall discuss some natural motivating examples. Then we shall present a few criteria that ensure the absence of atoms. Finally, we shall obtain by a different technique, which is a variant of the stratification method, a sufficient condition for the absolute continuity of the integral.

3.1 Some examples

Example 3.1. *Let $(\xi_t)_{t\geq 0}$ be a compound Poisson process (with no drift) and $g : \mathbb{R} \to \mathbb{R}$ a deterministic function such that $g(0) \neq 0$ and such that $\int_0^\infty g(\xi_t)\, dt$ is finite almost surely. Then $\int_0^\infty g(\xi_t)\, dt$ has a Lebesgue density.*

Proof. Denote the time of the first jump of ξ by T_1. Recall that ξ is always assumed nondegenerate, so T_1 is a nondegenerate exponential random variable. We can write

$$\int_0^\infty g(\xi_t)\,dt = g(0)T_1 + \int_0^\infty g(\xi_{T_1+t})\,dt$$

(from which it is evident that the integral on the righthand side converges a.s.). Recall that the jump times in a compound Poisson process are independent of the jump sizes. By the strong Markov property of Lévy processes (see [1], Prop. 6, p. 20), the process $(\xi_{T_1+t})_{t\geq 0}$, and *a fortiori* the random variable $\int_0^\infty g(\xi_{T_1+t})\,dt$, are independent of T_1. From this follows the claim, since $g(0)T_1$ has a Lebesgue density and hence its sum with any independent random variable has also. □

The following example shows that this property does not carry over to compound Poisson processes with drift, at least not if the support of g is compact.

Example 3.2. *Let $\xi = (\xi_t)_{t\geq 0} = (at + Q_t)_{t\geq 0}$ be a compound Poisson process together with a deterministic drift $a \neq 0$, such that $\lim_{t\to\infty} \xi_t = \operatorname{sgn}(a)\infty$ a.s. Suppose that g is a deterministic integrable Borel function with compact support. Then $\int_0^\infty g(\xi_t)\,dt$ is finite almost surely and its distribution has atoms.*

Proof. Since ξ drifts to $\pm\infty$ a.s., there is a random time τ after which $\xi_t \notin \operatorname{supp} g$ for all t; that is, if ξ enters $\operatorname{supp} g$ at all; if it doesn't, then $g(\xi_t) = 0$ for all $t \geq 0$. In either case, $\int_\tau^\infty g(\xi_t)\,dt = 0$, and since g is integrable and the number of jumps of Q until time τ is almost surely finite, it follows that $\int_0^\infty g(\xi_t)\,dt < \infty$ a.s.

Suppose now that $a > 0$, so that ξ drifts to $+\infty$ a.s., and let $r = \sup(\operatorname{supp} g)$. If $r \leq 0$ there is a positive probability that ξ does not enter $\operatorname{supp} g$, except, possibly, when $r = t = 0$, and then $g(\xi_t) = 0$; in either case, $\int_0^\infty g(\xi_t)\,dt = 0$ with positive probability, giving an atom at 0. If $r > 0$, let $T = r/a$. The event A that the first jump of ξ occurs at or after time T has positive probability. On A, $\xi_t = at$ for all $0 \leq t \leq T$. Also, since ξ drifts to $+\infty$ a.s., on a subset of A with positive probability ξ does not re-enter $\operatorname{supp} g$ after time T. On this subset, we have $\int_0^\infty g(\xi_t)\,dt = \int_0^T g(at)\,dt$, which is constant. Similarly if $a < 0$. □

Our third example relies on the following classical criterion for the continuity of infinitely divisible distributions (cf. Theorem 27.4, p. 175, in Sato [21]), that we shall further use in the sequel.

Lemma 3.3. *Let μ be an infinitely divisible distribution on $\mathbb{R}$ with an infinite Lévy measure, or with a non-zero Gaussian component. Then μ is continuous.*

If ξ has infinite Lévy measure, or no drift, Example 3.2 may fail, as shown next:

Example 3.4. *Suppose that ξ is a subordinator with infinite Lévy measure, or is a non-zero subordinator with no drift. Then $\int_0^\infty 1_{[0,1]}(\xi_t)\,dt$ is finite a.s. and has no atoms.*

Proof. Since ξ_t drifts to ∞ a.s. it is clear that $\int_0^\infty 1_{[0,1]}(\xi_t)\,dt$ is finite almost surely. For $x > 0$ define

$$L_x := \inf\{t > 0 : \xi_t > x\}.$$

Then $\int_0^\infty 1_{[0,1]}(\xi_t)\,dt = L_1$, and for $a > 0$ we have

$$\begin{aligned} \{L_1 = a\} &= \{\inf\{u : \xi_u > 1\} = a\} \\ &= \{\xi_{a-\varepsilon} \le 1 \text{ for all } \varepsilon > 0, \quad \xi_{a+\varepsilon} > 1 \text{ for all } \varepsilon > 0\} \\ &\subseteq \{\xi_a = 1\} \cup \{\Delta\xi_a > 0\}. \end{aligned}$$

A Lévy process is stochastically continuous so $P(\Delta\xi_a > 0) = 0$. If ξ is a subordinator with infinite Lévy measure, then $P(\xi_a = 1) = 0$ by Lemma 3.3. Thus we get $P(L_1 = a) = 0$. If ξ is a subordinator with no drift, then $\Delta\xi_{L_1} > 0$ a.s. ([1], p. 77) (and this includes the case of a compound Poisson), so again

$$P(L_1 = a) = P(L_1 = a, \Delta\xi_{L_1} > 0) \le P(\Delta\xi_a > 0) = 0. \qquad \square$$

3.2 Some criteria for continuity

We shall now present some fairly general criteria which ensure the continuity of the distribution of the integral $\int_0^\infty g(\xi_t)dt$ whenever the latter is finite a.s. and the Lévy process ξ is transient (see Bertoin [1], Section I.4 or Sato [21], Section 35 for definitions and properties of transient and recurrent Lévy processes).

Remarks. (i) One might expect that the existence of $\int_0^\infty g(\xi_t)\,dt$ already implies the transience of ξ. That this is not true in general was shown by Erickson and Maller [8], Remark (ii) after Theorem 6. As a counterexample, we may take ξ to be a compound Poisson process with Lévy measure $\Pi(dx) = \sqrt{2}\delta_1 + \delta_{-\sqrt{2}}$. Note that $\int x\Pi(dx) = 0$, so ξ is recurrent. Nonetheless ξ never returns to 0 after its first exit-time and thus $0 < \int_0^\infty \mathbf{1}_{\{\xi_t=0\}}\,dt < \infty$ a.s.

(ii) Sufficient conditions under which the existence of $\int_0^\infty g(\xi_t)\,dt$ implies the transience of ξ are mentioned in Remark (iii) after Theorem 6 of [8]. One such sufficient condition is that there is some non-empty open interval $J \subset \mathbb{R}$ such that $\inf\{g(x) : x \in J\} > 0$.

We shall now turn to the question of atoms of $\int_0^\infty g(\xi_t)\,dt$. For the next theorem, denote by E° the set of inner points of a set E, by $\overline{E}$ its topological closure and by ∂E its boundary.

Theorem 3.5. *Let $g : \mathbb{R} \to [0, \infty)$ be a deterministic Borel function. Assume that its support,* $\operatorname{supp} g$*, is compact, that $g > 0$ on $(\operatorname{supp} g)^\circ$, and that $0 \in (\operatorname{supp} g)^\circ$. Write* $\partial\operatorname{supp} g := \operatorname{supp} g \setminus (\operatorname{supp} g)^\circ$ *for the boundary of* $\operatorname{supp} g$*. Let ξ be a transient Lévy process, and assume that $I := \int_0^\infty g(\xi_t)\,dt$ is almost surely finite. If either*
(i) *ξ is of unbounded variation and $\partial\operatorname{supp} g$ is finite,*
or
(ii) *ξ is of bounded variation with zero drift and $\partial\operatorname{supp} g$ is at most countable,*
then the distribution of I has no atoms.

Proof. If ξ is a compound Poisson process without drift, the result follows from Example 3.1, so we will assume that ξ has unbounded variation, or is of bounded variation with zero drift such that its Lévy measure is infinite, and that g has the properties specified in the statement of the theorem.

Write

$$I(x) := \int_0^x g(\xi_t)\,dt, \quad x \in (0, \infty].$$

Then $x \mapsto I(x)$ is increasing and $I = I(\infty)$ is finite a.s. by assumption, so $I(x) < \infty$ a.s. for all $x \geq 0$. Plainly $I(x)$ is a.s. continuous at each $x > 0$. Assume by way of contradiction that there is some $a \geq 0$ such that $P(I = a) > 0$, and proceed as follows.

Define

$$T_s := \inf\{u \geq 0 : I(u) = s\}, \quad s \geq 0.$$

Since ξ_t is adapted to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of $(\xi_t)_{t\geq 0}$, so is $g(\xi_\cdot)$ (g is Borel), thus $\{T_s > u\} = \{\int_0^u g(\xi_t)dt < s\} \in \mathcal{F}_u$, because $I(\cdot)$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. Thus T_s is a stopping time for each $s \geq 0$. Further, $T_s > 0$ for all $s > 0$. Since $0 \in (\operatorname{supp} g)^\circ$, it is clear that $a \neq 0$. By assumption, ξ is transient, so there is a finite random time σ such that $\xi_t \notin \operatorname{supp} g$ for all $t \geq \sigma$. Then $I(\infty) = I(\sigma)$, and it follows that $P\{T_a < \infty\} > 0$.

Define the stopping times $\tau_n := T_{a-1/n} \wedge n$. Then $(\tau_n)_{n\in\mathbb{N}}$ is strictly increasing to T_a, showing that T_a is announceable; it follows that $t \mapsto \xi_t$ is continuous at $t = T_a$ on $\{T_a < \infty\}$, see e.g. Bertoin [1], p. 21 or p. 39.

Let $B = \{T_a < \infty,\ I(\infty) = a\}$. We restrict attention to $\omega \in B$ from now on. Since T_a is the first time $I(\cdot)$ reaches a, for every $\varepsilon > 0$ there must be a subset $J_\varepsilon \subset (T_a - \varepsilon, T_a)$ of positive Lebesgue measure such that $g(\xi_t) > 0$ for all $t \in J_\varepsilon$. Thus $\xi_t \in \operatorname{supp} g$ for all $t \in J_\varepsilon$, and so $\xi_{T_a} \in \operatorname{supp} g$. Since we assume that $\partial\operatorname{supp} g := \operatorname{supp} g \setminus (\operatorname{supp} g)^\circ$ is countable, and that ξ has infinite Lévy measure or a non-zero Gaussian component, we have by Lemma 3.3 that $P(\xi_t \in \partial\operatorname{supp} g) = 0$ for all $t > 0$. Consequently

$$E(\lambda\{t \geq 0 : \xi_t \in \partial\operatorname{supp} g\}) = \int_0^\infty P(\xi_t \in \partial\operatorname{supp} g)dt = 0.$$

It follows that there are times $t < T_a$ arbitrarily close to T_a with ξ_t in $(\operatorname{supp} g)^\circ$. By the continuity of $t \mapsto \xi_t$ at $t = T_a$, we then have $\xi_{T_a} \in \overline{(\operatorname{supp} g)^\circ}$ for $\omega \in B' \subseteq B$, where $P(B') = P(B) > 0$. Since $g > 0$ on $(\operatorname{supp} g)^\circ$ it follows that $\xi_{T_a} \in \partial((\operatorname{supp} g)^\circ)$ on the event $B' \subseteq \{I(\infty) = a\}$; for, if not, this would imply, by an application of the Markov property, that $I(t) > a$ for $t > T_a$, which is impossible.

Now suppose (i), so that ξ is of infinite variation. Then it follows from Shtatland's (1965) result ([22], see also Sato [21], Thm 47.1, p. 351) that 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$. Since ξ_{T_a} belongs to the finite set $\partial \operatorname{supp} g$, there is an open interval $U \subset (\operatorname{supp} g)^\circ$ which has ξ_{T_a} either as left or right end point. In either case, the regularity of 0 for $(0, \infty)$ and for $(-\infty, 0)$ implies that immediately after time T_a there must be times t such that ξ_t is strictly less than ξ_{T_a} and other times t such that ξ_t is strictly greater than ξ_{T_a}. By the continuity of ξ at T_a, it follows that there must be times after T_a such that $\xi_t \in U$. Consequently, there is some $\varepsilon = \varepsilon(\omega) > 0$ such that $\xi_{T_a+\varepsilon} \in (\operatorname{supp} g)^\circ$. By the right-continuity of ξ at $T_a + \varepsilon$ it follows further that $I(\infty) > I(T_a) = a$ on B', where $P(B') > 0$ and $B' \subseteq \{I(\infty) = a\}$, a contradiction.

Alternatively, suppose (ii), so that ξ has finite variation and zero drift (and infinite Lévy measure). Then it follows that ξ almost surely does not hit single points (by Kesten's theorem [13]; see [1], p. 67). Thus, since $\partial((\operatorname{supp} g)^\circ) \subseteq \operatorname{supp} g \setminus (\operatorname{supp} g)^\circ$ and the latter is at most countable, ξ almost surely does not hit $\partial((\operatorname{supp} g)^\circ)$. But on the set B', where $P(B') > 0$ and $B' \subseteq \{T_a < \infty,\ I(\infty) = a\}$, we have $\xi_{T_a} \in \partial((\operatorname{supp} g)^\circ)$, contradicting $P(I(\infty) = a) > 0$. □

Remarks. (i) The assumptions on the topological structure of $\{x : g(x) > 0\}$ in the previous theorem are easy to check. That they cannot be completely relaxed can be seen from the following example: let $g(x) = 1$ for all $x \in \mathbb{Q} \cap [-1, 1]$ and $g(x) = 0$ otherwise, then $\operatorname{supp} g = [-1, 1]$, $(\operatorname{supp} g)^\circ = (-1, 1)$, but $g > 0$ on $(-1, 1)$ does not hold. And in fact, it is easy to see that in that case we have for every Lévy process of unbounded variation or infinite Lévy measure that

$$E \int_0^\infty g(\xi_t)\, dt = E \int_0^\infty 1_{\mathbb{Q}\cap[-1,1]}(\xi_t)\, dt = \int_0^\infty P(\xi_t \in \mathbb{Q} \cap [-1, 1])\, dt = 0$$

by Lemma 3.3, so that $\int_0^\infty g(\xi_t)\, dt = 0$ a.s.

(ii) Suppose g is as in Theorem 3.5, and assume $\int_0^\infty g(x)dx < \infty$. Let ξ be a Brownian motion with non-zero drift. Then $\int_0^\infty g(\xi_t)dt < \infty$ a.s. by Theorem 6 of [8] and the integral has a continuous distribution by Theorem 3.5.

Theorem 3.5 allows a wide class of transient Lévy processes (we have to exclude ξ which are of bounded variation with nonzero drift, by Ex. 3.2), but restricts us, essentially, to nonnegative g which have compact support. Another approach which combines excursion theory and Lemma 3.3 allows a much wider class of g at the expense of placing restrictions on the local behaviour of ξ. Here is the first result in this vein. We refer e.g. to Chapters IV and V in [1] for background on local time and excursion theory for Lévy processes.

Theorem 3.6. *Let $g : \mathbb{R} \to [0,\infty)$ be a measurable function such that $g > 0$ on some neighbourhood of 0. Suppose that ξ is a transient Lévy process such that 0 is regular for itself, in the sense that $\inf\{t > 0 : \xi_t = 0\} = 0$ a.s., and that the integral $I := \int_0^\infty g(\xi_t)dt$ is finite a.s. Then the distribution of I has no atoms.*

Proof. Thanks to Example 3.1, we may assume without losing generality that ξ is not a compound Poisson. Then 0 is an instantaneous point, in the sense that $\inf\{t > 0 : \xi_t \neq 0\} = 0$ a.s. The assumption that ξ is transient implies that its last-passage time at 0, defined by

$$\ell := \sup\{t \geq 0 : \xi_t = 0\},$$

is finite a.s. Since the point 0 is regular for itself, there exists a continuous nondecreasing local time process at level 0 which we denote by $L = (L_t, t > 0)$; we also introduce its right-continuous inverse

$$L^{-1}(t) := \inf\{s \geq 0 : L_s > t\}, \qquad t \geq 0$$

with the convention that $\inf \emptyset = \infty$. The largest value of L, namely, L_∞, is finite a.s.; more precisely, L_∞ has an exponential distribution, and we have $L^{-1}(L_\infty-) = \ell$ and $L^{-1}(t) = \infty$ for every $t \geq L_\infty$ ([1], Prop. 7 and Thm 8, pp. 113–115). We denote the set of discontinuity times of the inverse local time before explosion by

$$\mathcal{D} := \{t < L_\infty : L^{-1}(t-) < L^{-1}(t)\}$$

and then, following Itô, we introduce for every $t \in \mathcal{D}$ the excursion $\varepsilon(t)$ with finite lifetime $\zeta_t := L^{-1}(t) - L^{-1}(t-)$ by

$$\varepsilon_s(t) := \xi_{L^{-1}(t-)+s}, \qquad 0 \leq s < \zeta_t.$$

Itô's excursion theory shows that conditionally on L_∞, the family of finite excursions $(\varepsilon(t), t \in \mathcal{D})$ is distributed as the family of the atoms of a Poisson point process with intensity $L_\infty \mathbf{1}_{\{\zeta<\infty\}} n$, where n denotes the Itô measure of the excursions of the Lévy process ξ away from 0, and ζ the lifetime of a generic excursion ([1], Thm 10, p. 118).

Since ξ is not a compound Poisson process, the set of times t at which $\xi_t = 0$ has zero Lebesgue measure a.s., and we can express the integral in the form $I = A + B$ with

$$A := \sum_{t\in\mathcal{D}} \int_{L^{-1}(t-)}^{L^{-1}(t)} g(\xi_s)ds = \sum_{t\in\mathcal{D}} \int_0^{\zeta_t} g(\varepsilon_s(t))ds \tag{3.9}$$

and

$$B := \int_{\ell}^{\infty} g(\xi_s) ds \, .$$

Excursion theory implies that A and B are independent, and hence we just need to check that A has no atom. Now, the conditional distribution of A given L_∞ is infinitely divisible, with Lévy measure Λ given by the image of $L_\infty \mathbf{1}_{\{\zeta<\infty\}} n$ under the map $\varepsilon \to \int_0^\zeta g(\varepsilon_s) ds$.

The fact that 0 is an instantaneous point implies that the measure $\mathbf{1}_{\{\zeta<\infty\}} n$ is infinite, and further that the excursions $\varepsilon(t)$ leave 0 continuously for all $t \in \mathcal{D}$ a.s. The assumption that $g > 0$ on some neighbourhood of 0 then entails that $\int_0^{\zeta_t} g(\varepsilon_s(t)) ds > 0$ for every $t \in \mathcal{D}$. Thus $\Lambda\{(0, \infty)\} = \infty$, and we conclude from Lemma 3.3 that the conditional distribution of A given L_∞ has no atoms. It follows that $P(A = a) = E(P(A = a | L_\infty)) = 0$ for every $a > 0$, completing the proof of our statement. □

Remark. See Bertoin [1], Ch. V and Sato [21], Section 43, for discussions relevant to Lévy processes for which 0 is regular for itself.

An easy modification of the argument in Theorem 3.6 yields the following criterion in the special case when the Lévy process has no positive jumps. This extends the result of Theorem 3.5 by allowing a drift, as long as there is no upward jump.

Proposition 3.7. *Let $g : \mathbb{R} \to [0, \infty)$ be a measurable function with $g > 0$ on some neighbourhood of 0. Suppose that $\xi_t = at - \sigma_t$, where $a > 0$ and σ is a subordinator with infinite Lévy measure and no drift, and such that the integral $I := \int_0^\infty g(\xi_t) dt$ is finite a.s. Assume further that $a \neq E\sigma_1$, so that ξ is transient. Then the distribution of I has no atoms.*

Remark. We point out that in the case when ξ is a Lévy process with no positive jumps and infinite variation, then 0 is regular for itself ([1], Cor. 5, p. 192), and thus Theorem 3.6 applies. Recall also Example 3.2 for the case of compound Poisson processes with drift. Therefore our analysis covers entirely the situation when the Lévy process has no positive jumps and is not the negative of a subordinator.

Proof. Introduce the supremum process $\bar{\xi}_t := \sup_{0 \le s \le t} \xi_s$. We shall use the fact that the reflected process $\bar{\xi} - \xi$ is Markovian and that $\bar{\xi}$ can be viewed as its local time at 0; see Theorem VII.1 in [1], p. 189. The first-passage process $T_x := \inf\{t \ge 0 : \xi_t \ge x\}$ $(x \ge 0)$ thus plays the role of the inverse local time. It is well-known that $T_\cdot$ is a subordinator (killed at some independent exponential time when ξ drifts to $-\infty$); more precisely, the hypothesis that $\xi_t = at - \sigma_t$ has bounded variation implies that the drift coefficient of $T_\cdot$ is $a^{-1} > 0$.

Let us consider first the case when ξ drifts to ∞, so the first-passage times T_x are finite a.s. We write $\mathcal{D}$ for the set of discontinuities of T. and for every $x \in \mathcal{D}$, we define the excursion of the reflected Lévy process away from 0 as

$$\varepsilon_s(x) \,=\, x - \xi_{T_{x-}+s}\,, \qquad 0 \le s < \zeta_x := T_x - T_{x-}\,.$$

According to excursion theory, the point measure

$$\sum_{x\in\mathcal{D}} \delta_{(x,\varepsilon(x))}$$

is then a Poisson random measure with intensity $dx \otimes \bar{n}$, where $\bar{n}$ denotes the Itô measure of the excursions of the reflected process $\bar{\xi} - \xi$ away from 0. Let $b > 0$ be such that $g > 0$ on $[-b, b]$. We can express

$$\int_0^\infty g(\xi_s)ds \,=\, A + B + C$$

where

$$\begin{aligned} A &= a^{-1}\int_0^\infty g(x)dx\,, \\ B &= \sum_{x\in\mathcal{D},x\le b}\int_{T_{x-}}^{T_x} g(\xi_s)ds \,=\, \sum_{x\in\mathcal{D},x\le b}\int_0^{\zeta_x} g(x-\varepsilon_s(x))ds\,, \\ C &= \sum_{x\in\mathcal{D},x> b}\int_{T_{x-}}^{T_x} g(\xi_s)ds \,=\, \sum_{x\in\mathcal{D},x> b}\int_0^{\zeta_x} g(x-\varepsilon_s(x))ds\,. \end{aligned}$$

The first term A is deterministic, and B and C are independent infinitely divisible random variables (by the superposition property of Poisson random measures). More precisely, the Lévy measure of B is the image of $\mathbf{1}_{\{0\le x\le b\}}dx \otimes \bar{n}$ by the map

$$(x,\varepsilon) \mapsto \int_0^\zeta g(x-\varepsilon_s)ds\,.$$

Observe that the value of this map evaluated at any $x \in [0, b]$ and excursion ε is strictly positive (because excursions return continuously to 0, as ξ has no positive jumps). On the other hand, the assumption that the Lévy measure of the subordinator $\sigma_t = at - \xi_t$ is infinite ensures that 0 is an instantaneous point for the reflected process $\bar{\xi} - \xi$, and hence the Itô measure $\bar{n}$ is infinite. It thus follows from Lemma 3.3 that the infinitely divisible variable B has no atom, which establishes our claim.

The argument in case ξ drifts to $-\infty$ is similar; the only difference is that the excursion process is now stopped when an excursion with infinite lifetime arises. This occurs at time (in the local-time scale $\bar{\xi}$) $\bar{\xi}_\infty = \sup_{t\ge0}\xi_t$, where this variable has an exponential distribution. □

3.3 A criterion for absolute continuity

Next we will investigate some different sufficient conditions, and some of them also ensure the existence of Lebesgue densities. We will work with more general integrals of the form $\int_0^\infty g(\xi_t)\,dY_t$ for a process $(Y_t)_{t\geq 0}$ of bounded variation, independent of the Lévy process ξ. The method will be a variant of the stratification method, by conditioning on almost every quantity apart from certain jump times. Such an approach was also used by Nourdin and Simon [18] for the study of absolute continuity of solutions to certain stochastic differential equations.

We need the following lemma, which concerns only deterministic functions. Part (a) is just a rewriting of Theorem 4.2 in Davydov et al. [5], and it is this part which will be invoked when studying $\int_0^\infty g(\xi_t)\,dY_t$ for $Y_t = t$.

Lemma 3.8. *Let $Y : [0,1] \to \mathbb{R}$ be a right-continuous deterministic function of bounded variation. Let $f : [0,1] \to \mathbb{R}$ be a deterministic Borel function such that*

$$f \neq 0 \quad a.e. \tag{3.10}$$

and such that the Lebesgue-Stieltjes integral $\int_0^1 f(t)\,dY_t$ exists and is finite. Let

$$H : (0,1] \to \mathbb{R}, \quad x \mapsto \int_{0+}^x f(t)\,dY_t,$$

and denote by $\mu := H(\lambda_{|(0,1]})$ the image measure of λ under H. Then the following are sufficient conditions for (absolute) continuity of μ:
(a) Suppose the absolute continuous part of the measure induced by Y on $[0,1]$ has a density which is different from zero a.e. Then μ is absolutely continuous.
(b) Suppose that Y is strictly increasing and that f is in almost every point $t \in [0,1]$ right- or left-continuous. Then μ is continuous.

Proof. (a) Denoting the density of the absolute continuous part of Y by ϕ, it follows that H is almost everywhere differentiable with derivative $f\phi \neq 0$ a.e., and the assertion follows from Theorem 4.2 in Davydov et al. [5].

(b) Suppose that Y is strictly increasing and denote

$$K := \{t \in (0,1) : f \text{ is right- or left-continuous in } t\}.$$

By assumption, K has Lebesgue measure 1. Using the right-/left-continuity, for every $t \in K$ such that $f(t) > 0$ there exists a unique maximal interval $J_+(t) \subset (0,1)$ of positive length such that $t \in J_+(t)$ and $f(y) > 0$ for all $y \in J_+(t)$. By the axiom of choice there exists a subfamily $K_+ \subset K$ such that $(J_+(t) : t \in K_+)$ are pairwise disjoint and their union covers $K \cap \{t \in (0,1) : f(t) > 0\}$. Since each of these intervals has positive length, there can only be countably many such intervals, so K_+ must be countable.

Similarly, we obtain a countable cover $(J_-(t) : t \in K_-)$ of $K \cap \{t \in (0,1) : f(t) < 0\}$ with disjoint intervals. Now let $a \in \operatorname{Range}(H)$. Then

$$\begin{aligned} H^{-1}(\{a\}) \quad \subset \quad & \left(\bigcup_{t \in K_+} (H^{-1}(\{a\}) \cap J_+(t)) \right) \cup \left(\bigcup_{t \in K_-} (H^{-1}(\{a\}) \cap J_-(t)) \right) \\ & \cup ([0,1] \setminus K) \cup \{t \in [0,1] : f(t) = 0\} \cup \{0,1\}. \end{aligned}$$

Observing that

$$\lambda \left(H^{-1}(\{a\}) \cap J_\pm(t) \right) = \lambda \left((H_{|J_\pm(t)})^{-1}(\{a\}) \right) = 0$$

since H is strictly increasing (decreasing) on $J_+(t)$ ($J_-(t)$) as a consequence of $f > 0$ on $J_+(t)$ ($f < 0$ on $J_-(t)$) and strict increase of Y, it follows that $\lambda(H^{-1}(\{a\})) = 0$, showing continuity of μ. □

We now come to the main result of this subsection. Note that the case $Y_t = t$ falls under the case (i) considered in the following theorem, giving particularly simple conditions for absolute continuity of $\int_0^\infty g(\xi_t)\, dt$. In particular, part (b) shows that if ξ has infinite Lévy measure and g is strictly monotone on a neighbourhood of 0, then $\int_0^\infty g(\xi_t)\, dt$ is absolutely continuous.

Theorem 3.9. *Let $\xi = (\xi_t)_{t \geq 0}$ be a transient Lévy process with non-zero Lévy measure Π_ξ. Let $Y = (Y_t)_{t \geq 0}$ be a stochastic process of bounded variation on compacts which has càdlàg paths and which is independent of ξ. Denote the density of the absolutely continuous part of the measure induced by the paths $t \mapsto Y_t(\omega)$ by ϕ_ω. Let $g : \mathbb{R} \to \mathbb{R}$ be a deterministic Borel function and suppose that the integral*

$$I := \int_{(0,\infty)} g(\xi_t)\, dY_t$$

exists almost surely and is finite.
(a) [general Lévy process] *Suppose that there are a compact interval $J \subset \mathbb{R} \setminus \{0\}$ with $\Pi_\xi(J) > 0$ and some constant $t_0 > 0$ such that*

$$\lambda(\{|t| \geq t_0 : g(t) = g(t+z)\}) = 0 \quad \text{for all } z \in J. \tag{3.11}$$

Case (i): *If $\lambda(\{t \in [t_0, \infty) : \phi(t) = 0\}) = 0$ a.s., then I is absolutely continuous.*
Case (ii): *If Y is strictly increasing on $[t_0, \infty)$ and g has only countably many discontinuities, then I does not have atoms.*

(*b*) [infinite activity Lévy process] *Suppose the Lévy measure* Π_ξ *is infinite. Suppose further that there is* $\varepsilon > 0$ *such that*

$$\lambda(\{t \in (-\varepsilon, \varepsilon) : g(t) = g(t+z)\}) = 0 \quad \text{for all } z \in [-\varepsilon, \varepsilon]. \tag{3.12}$$

Case (i): *If* $\lambda(\{t \in (0,\varepsilon) : \phi(t) = 0\}) = 0$ *a.s., then* I *is absolutely continuous.*
Case (ii): *If* Y *is strictly increasing on* $(0,\varepsilon)$ *and* g *has only countably many discontinuities, then* I *does not have atoms.*

Proof. (*a*) Let J be an interval such that (3.11) is satisfied, and define

$$R_t := \sum_{0<s\le t, \Delta\xi_s \in J} \Delta\xi_s, \quad M_t := \xi_t - R_t, \quad t \ge 0.$$

Then $R = (R_t)_{t\ge 0}$ is a compound Poisson process, independent of $M = (M_t)_{t\ge 0}$. For $i \in \mathbb{N}$ denote by T_i and Z_i the time and size of the i^{th} jump of R, respectively, and let $T_0 := 0$. Further, denote

$$\begin{aligned} I_i &:= \int_{(T_{2i-2},T_{2i}]} g(\xi_t)\, dY_t \\ &= \int_{(T_{2i-2},T_{2i-1}]} \left(g\left(M_t + \sum_{j=1}^{2i-2} Z_j\right) - g\left(M_t + \sum_{j=1}^{2i-1} Z_j\right)\right) dY_t \\ &\quad + \int_{(T_{2i-2},T_{2i}]} g\left(M_t + \sum_{j=1}^{2i-1} Z_j\right) dY_t \\ &\quad + \left[g(\xi_{T_{2i-1}}) - g(\xi_{T_{2i-1}} - Z_{2i-1})\right] \Delta Y_{T_{2i-1}} \\ &\quad + \left[g(\xi_{T_{2i}}) - g(\xi_{T_{2i}} - Z_{2i}\right] \Delta Y_{T_{2i}}. \end{aligned} \tag{3.13}$$

We now condition on all random quantities present except the odd numbered T_i. Thus, for every Borel set $B \subset \mathbb{R}$, we write

$$P(I \in B) = E\, P\left(\sum_{i=1}^\infty I_i \in B \big| Y, M, (T_{2j})_{j\in\mathbb{N}}, (Z_j)_{j\in\mathbb{N}}\right).$$

To show that I has no atoms, it is hence sufficient to show that

$$P\left(\sum_{i=1}^\infty I_i \in B \big| Y, M, (T_{2j})_{j\in\mathbb{N}}, (Z_j)_{j\in\mathbb{N}}\right) = 0 \quad \text{a.s.} \tag{3.14}$$

for every Borel set B of the form $B = \{a\}$ with $a \in \mathbb{R}$. Similarly, for showing that I is absolutely continuous it is sufficient to show that (3.14) holds for every Borel set B of Lebesgue measure 0. Observe that the $(I_i)_{i\in\mathbb{N}}$ are conditionally independent given

$$V := (Y, M, (T_{2j})_{j\in\mathbb{N}}, (Z_j)_{j\in\mathbb{N}}).$$

Thus the conditional probability that $I = \sum_{i=1}^{\infty} I_i \in B$ is the convolution of the conditional probabilities that $I_i \in B$, $i \in \mathbb{N}$. Hence it suffices to show that there is some random integer $i_0 \in \mathbb{N}$ such that almost surely, the conditional distribution of I_{i_0} given V is absolutely continuous (case (i)) or has no atoms (case (ii)), respectively.

Define the integer i_0 as the first index i such that

$$\min\left\{ \inf_{t\in(T_{2i-2},T_{2i}]} \{|M_t + \sum_{j=1}^{2i-2} Z_j|\}, T_{2i-2} \right\} \geq t_0, \tag{3.15}$$

with t_0 as in (3.11). Since ξ is transient i_0 is almost surely finite. As a function of V, i_0 is constant under the conditioning by V. The right hand side of (3.13) is comprised of four summands. The second and fourth summands are constant given V. The third summand is still random, after conditioning, since T_{2i-1} enters in ΔY; but here R and Y are independent, so that the third summand equals 0 a.s. Thus it is sufficient to show that, given V, the first summand, evaluated at i_0, namely

$$\widetilde{I}_{i_0} := \int_{(T_{2i_0-2},T_{2i_0-1}]} \left(g\left(M_t + \sum_{j=1}^{2i_0-2} Z_j \right) - g\left(M_t + \sum_{j=1}^{2i_0-1} Z_j \right)\right) dY_t,$$

is almost surely absolutely continuous (case (i)) or has no atoms (case (ii)). Define the functions $f = f_V : [T_{2i_0-2}, T_{2i_0}] \to \mathbb{R}$ and $H = H_V : (T_{2_{i_0}-2}, T_{2i_0}] \to \mathbb{R}$ by

$$\begin{aligned} f(t) &= g\left(M_t + \sum_{j=1}^{2i_0-2} Z_j \right) - g\left(M_t + \sum_{j=1}^{2i_0-1} Z_j \right), \\ H(x) &:= \int_{(T_{2i_0-2},x]} f(t)\, dY_t. \end{aligned}$$

Observing that T_{2i_0-1} is uniformly distributed on (T_{2i_0-2}, T_{2i_0}) given V, it follows from Fubini's theorem that for any Borel set $B \subset \mathbb{R}$

$$\begin{aligned} P(\widetilde{I}_{i_0} \in B|V) &= E(\mathbf{1}_{\{H(T_{2i_0-1})\in B\}}|V) = \int_{(T_{2i_0-2},T_{2i_0})} \mathbf{1}_{\{H(x)\in B\}} \frac{dx}{T_{2i_0} - T_{2i_0-2}} \\ &= \frac{\lambda(H^{-1}(B))}{T_{2i_0} - T_{2i_0-2}}. \end{aligned}$$

We shall apply Lemma 3.8 to show that $\widetilde{I}_{i_0}$ given V is absolutely continuous or has no atoms, respectively. For this, observe that (3.10) is satisfied because of (3.11) and (3.15), and note that $z := Z_{2i_0-1} \in J$, since all the jumps of R are in the interval J. In case (i) this then gives absolute continuity of $\widetilde{I}_{i_0}$ conditional on V by Lemma 3.8 (a) and hence of the distribution of I. Now concentrate on case (ii), when Y is strictly increasing on $[t_0, \infty)$ and g has only countably many discontinuities. Denote this set of discontinuities of g by F. By assumption, F is countable. This then implies that almost surely, the function f is almost everywhere right-continuous. For by the a.s. right-continuity of the paths of Lévy processes, f can happen to be non-right-continuous at a point t only if $\xi_t^{(1)} := M_t + \sum_{j=1}^{2i_0-1} Z_j \in F$ or $\xi_t^{(2)} := M_t + \sum_{j=1}^{2i_0-2} Z_j \in F$. But

$$E(\lambda\{t \geq 0 : \xi_t^{(1)} \in F \text{ or } \xi_t^{(2)} \in F\}) = \int_0^\infty P(\xi_t^{(1)} \in F \text{ or } \xi_t^{(2)} \in F)\, dt,$$

and by Lemma 3.3 the last integral is zero if ξ has infinite Lévy measure, so that almost surely, f is almost everywhere right-continuous if Π_ξ is infinite. If ξ has finite Lévy measure, then f is trivially almost everywhere right-continuous. So we see that in case (ii) our assumptions imply the conditions of Lemma 3.8 (b), which then gives the claim.

(b) The proof is similar to the proof of (a): for $0 < \delta < \varepsilon/2$, let

$$R_t^{(\delta)} := \sum_{|\Delta\xi_s| \in [\delta, \varepsilon/2]} \Delta\xi_s, \quad M_t^{(\delta)} := \xi_t - R_t^{(\delta)}, \quad t \geq 0,$$

and denote the time and size of the i^{th} jump of $R^{(\delta)} = (R_t^{(\delta)})_{t\geq 0}$ by $T_i^{(\delta)}$ and $Z_i^{(\delta)}$, respectively. Further, define the set Ω_δ by

$$\Omega_\delta := \{T_2^{(\delta)} \leq \varepsilon, \sup_{0 \leq t < T_2^{(\delta)}} |M_t^{(\delta)}| \leq \varepsilon/2\}.$$

Let $P_\delta(\cdot) := P(\cdot|\Omega_\delta)$, and denote expectation with respect to P_δ by E_δ. Since $P(\Omega_\delta) \to 1$ as $\delta \downarrow 0$ because the Lévy measure of ξ is infinite, it is sufficient to show that, given $\delta > 0$, we have $P_\delta(B) = 0$ for all Borel sets B such that $\lambda(B) = 0$ (case (i)), or such that $B = \{a\}$, $a \in \mathbb{R}$ (case (ii)), respectively. Let

$$V_\delta := (Y, M^{(\delta)}, (T_j^{(\delta)})_{j\geq 2}, (Z_j^{(\delta)})_{j\in\mathbb{N}}.$$

Then we can write

$$P_\delta(I \in B) = E_\delta P_\delta(I \in B|V_\delta),$$

and it suffices to show that $P_\delta(I \in B|V_\delta) = 0$ a.s. for the sets B under consideration. But, conditional on V_δ, I almost surely differs from

$$\widetilde{I}_1 := \int_{(0,T_2^{(\delta)}]} \left(g\left(M_t^{(\delta)}\right) - g\left(M_t^{(\delta)} + Z_1^{(\delta)}\right)\right) dY_t$$

only by a constant. It then follows in complete analogy to the proof of (a) that under P_δ, $\widetilde{I}_1$ given V_δ has no atoms or is absolutely continuous, respectively, which then transfers to I under P_δ and hence to I under P. □

Remarks. (i) The preceding proof has shown that the independence assumption on ξ and Y can be weakened. Indeed, we need only assume that the processes $(R_t)_{t\geq 0}$ and Y are independent.
(ii) In addition to the assumptions of Theorem 3.9, assume that g is continuous. Then almost surely, $g(\xi_{t-}) = g(\xi_t)_-$ exist for all $t > 0$, and the assertions of Theorem 3.9 remain true for integrals of the form

$$\int_{(0,\infty)} g(\xi_t)_- \, dY_t.$$

This follows in complete analogy to the proof of Theorem 3.9.
(iii) Similar statements as in Theorem 3.9 can be made for integrals of the form $\int_0^\infty (g(\xi_t + \psi(t))\,dt$, where ψ is some deterministic function behaving nicely. We omit the details.

Appendix

Proof of the equivalence of (iv) and (v) in Theorem 2.2. Assume (iv), and observe that by the Doléans-Dade formula (e.g. [20], p. 84), $e^{-\xi} = \mathcal{E}(-\eta/k)$, where $k \neq 0$, if and only if $\Pi_\eta(\{y \in \mathbb{R} : k^{-1}y \geq 1\}) = 0$ and $\xi_t = X_t$, where

$$X_t := k^{-1}\eta_t + k^{-2}\sigma_\eta^2 t/2 - \sum_{0\leq s\leq t} \left(\log(1 - k^{-1}\Delta\eta_s) + k^{-1}\Delta\eta_s\right), \quad t \geq 0. \tag{3.16}$$

Now (X, η) is a bivariate Lévy process, whose Gaussian covariance matrix is given by $\Sigma_{X,\eta} = \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix} \sigma_X^2$. Further, (3.16) implies $\Delta X_t = -\log(1 - k^{-1}\Delta\eta_t)$, showing that the Lévy measure $\Pi_{X,\eta}$ of (X, η) is concentrated on $\{(x, k(1 - e^{-x})) : x \in \mathbb{R}\}$.

Conversely, if (Y, η) is a bivariate Lévy process with Gaussian covariance matrix given by $\Sigma_{Y,\eta} = \Sigma_{X,\eta}$, whose Lévy measure is concentrated on $\{(x, k(1 - e^{-x})) : x \in \mathbb{R}\}$, then $\Delta Y_t = -\log(1 - k^{-1}\Delta\eta_t)$, and it follows that there is some $c \in \mathbb{R}$ such that $Y_t = X_t + ct$, so that $e^{-Y_t + ct} = (\mathcal{E}(-\eta/k))_t$. Hence we have established the equivalence of (iv) and (v) in Theorem 2.2, subject to relating γ_1 and γ_2 as in (2.6).

To do this, let X_t as in (3.16) and use the Lévy–Itô decomposition. Define

$$\begin{pmatrix} X_t^{(1)} \\ \eta_t^{(1)} \end{pmatrix} := \lim_{\varepsilon\downarrow 0} \left(\sum_{\substack{0<s\leq t \\ (\Delta X_s)^2 + (\Delta\eta_s)^2 > \varepsilon^2}} \begin{pmatrix} \Delta X_s \\ \Delta\eta_s \end{pmatrix} - t \int\int_{x_1^2 + x_2^2 \in (\varepsilon^2, 1]} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Pi_{X,\eta}(d(x_1, x_2)) \right)$$

and $(X_t^{(2)}, \eta_t^{(2)})' := (X_t, \eta_t)' - (X_t^{(1)}, \eta_t^{(1)})'$ where the limit is a.s. as $\varepsilon \downarrow 0$. (Note that the expression in big brackets on the right is not precisely the compensated sum of jumps.) Then $(X_t^{(2)}, \eta_t^{(2)})'_{t\geq 0}$ is a Lévy process with characteristic triplet $(\gamma, \Sigma, 0)$, so has the form

$(X_t^{(2)}, \eta_t^{(2)})' = (\gamma_1 t, \gamma_2 t)' + \vec{B}_t$, $t \geq 0$, where $(\vec{B}_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^2$. From this follows that

$$X_t^{(2)} - k^{-1}\eta_t^{(2)} = (\gamma_1 - k^{-1}\gamma_2)t + \widetilde{B}_t, \quad t \geq 0, \tag{3.17}$$

for some Brownian motion $(\widetilde{B}_t)_{t\geq 0}$ in $\mathbb{R}^1$. We wish to determine $\gamma_1 - k^{-1}\gamma_2$. To do this, observe that from (3.16) and $\sigma_X^2 = k^{-2}\sigma_\eta^2$, we have

$$\begin{aligned}
&(X_t - X_t^{(1)}) - k^{-1}(\eta_t - \eta_t^{(1)}) \\
&= \sigma_X^2 t/2 + \sum_{0\leq s\leq t} (\Delta X_s - k^{-1}\Delta\eta_s) \\
&- \lim_{\varepsilon\downarrow 0}\left(\sum_{\substack{0<s\leq t \\ (\Delta X_s)^2+(\Delta\eta_s)^2>\varepsilon^2}} (\Delta X_s - k^{-1}\Delta\eta_s) - t\int\int_{x_1^2+x_2^2\in(\varepsilon^2,1]} (x_1 - k^{-1}x_2)\,\Pi_{X,\eta}(d(x_1,x_2))\right).
\end{aligned}$$

Noting that $k^{-1}\Delta\eta_s = 1 - e^{-\Delta X_s}$ and that $\sum_{0<s\leq t}(\Delta X_s - 1 + e^{-\Delta X_s})$ converges absolutely, we obtain, letting $\varepsilon \downarrow 0$, that

$$\begin{aligned}
X_t^{(2)} - k^{-1}\eta_t^{(2)} &= \sigma_X^2 t/2 + t\int\int_{x_1^2+x_2^2\leq 1} (x_1 - k^{-1}x_2)\Pi_{X,\eta}(d(x_1,x_2)) \\
&= \sigma_X^2 t/2 + t\int_{x^2+k^2(1-e^{-x})^2\leq 1} (x - 1 + e^{-x})\,\Pi_X(dx).
\end{aligned}$$

Comparing this with (3.17) gives (2.6). □

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