PACIFIC JOURNAF OF MATHEMATICS
Vol. 65, No. 1, 1976

ON CONTINUOUS IMAGE AVERAGING OF PROBABILITY MEASURES

SUN MAN CHANG

Let M be a compact space, and X a complete sparable metric space. Let $P(X)$ denote the probability measures on X. Let λ be a probability measure on M. Define a function φ_λ from $C(M, P(X))$ to $P(X)$ by $\varphi_\lambda(T)(f) = \int T(t)(f)d\lambda(t)$ for every $T \in C(M, P(X))$, $f \in C(X)$. We show that φ_λ is an open mapping.

1. Introduction. By a measure on a space X, we mean a regular Borel measure on X. A nonnegative measure is called a probability measure if its total mass is 1.

Let M be a compact space, and let X be a complete separable metric space. Let $P(X)$ denote the collection of all probability measures on X. Let $C(X)$ denote the set of all bounded continuous real-valued functions on X. Give $P(X)$ the weak topology as functionals on $C(X)$. Let $C(M, P(X))$ denote the set of all continuous functions from M into $P(X)$. Give $C(M, P(X))$ the topology of uniform convergence. Let λ be a fixed probability measure on M. For each $T \in C(M, P(X))$, define a functional $\varphi_\lambda(T)$ on $C(X)$ by

$$\varphi_\lambda(T)(f) = \int T(t)(f)d\lambda(t) .$$

By [3, p. 35 and p. 47], $\varphi_\lambda(T)$ may be considered as a measure in $P(X)$. Write $\varphi_\lambda(T) = \int T(t)d\lambda(t)$. Denote the mapping $T \to \varphi_\lambda(T)$ by φ_λ. Then φ_λ is a continuous function from $C(M, P(X))$ into $P(X)$. This paper is to show that φ_λ is an open mapping. This result contains a result due to Eifler [2, Theorem 2.4] as a special case when M consists of two points.

For a metric space X, we write $x_n \to x$ if $(x_n)_{n=1}^{\infty}$ converges to x in X.

ACKNOWLEDGMENT. The author wishes to thank Professor Robert M. Blumenthal for his suggestion of this problem and for his invaluable suggestion for the idea of the proof of (B) in Theorem 3.1.

2. Basic lemmas. We will use the following notation in Lemma 2.1: Let X and Y be complete separable metric spaces, and $\pi: Y \to X$ a continuous function. Then π induces a mapping also denoted by π, from $P(Y)$ to $P(X)$ and defined by $\pi\mu(E) = \mu(\pi^{-1}(E))$.

LEMMA 2.1. *Let X be a complete separable metric space. Then there exist a totally disconnected complete separable metric space G, a continuous function $\varphi\colon G \to X$, and a continuous function $\tilde{\varphi}\colon P(X) \to P(G)$ such that $\varphi\tilde{\varphi}(\mu) = \mu$ for all $\mu \in P(X)$. Moreover, $\tilde{\varphi}$ is affine:*

$$\tilde{\varphi}(a\mu + (1 - a)\nu) = a\tilde{\varphi}(\mu) + (1 - a)\tilde{\varphi}(\nu)$$

for every $0 < a < 1$, and measures $\mu, \nu \in P(X)$.

Proof. Such a space G is constructed by using a sequence $(F_n)_{n=1}^{\infty}$ of partitions of unity on X having the property that each F_n is subordinate to a cover of diameter less than $1/n$. The details of its construction can be found in [1].

Let X be a totally disconnected complete separable metric space. Consider sets of the form

$$(*) \qquad M_{\mu,\varepsilon}(G_1, \cdots, G_n) = \{\nu \in P(X) \colon |\nu(G_i) - \mu(G_i)| < \varepsilon \text{ for } i = 1, \cdots, n\}$$

where $\varepsilon > 0$, $\mu \in P(X)$, and $G_1, G_2, \cdots, G_n$ are mutually disjoint, both open and closed subsets of X such that $\bigcup_{i=1}^{n} G_i = X$.

LEMMA 2.2. *The collection of sets of the form (*) is a base for the topology on $P(X)$.*

Proof. For any open subset U of X, let

$$N_{\mu,\varepsilon}(U) = \{\nu \in P(X) \colon \nu(U) + \varepsilon > \mu(U)\} .$$

Since sets of the form $N_{\mu,\epsilon}(U)$ is a sub-base for the topology on $P(X)$, it suffices to show that

$$N_{\mu,\varepsilon}(U) \cap M_{\mu,\varepsilon}(G_1, \cdots, G_n)$$

contains a set of the form (*). Let $V \subseteq U$ be a both open and closed subset of X such that $\mu(V) + \varepsilon/2 > \mu(U)$. Then $N_{\mu,\varepsilon/2}(V) \subseteq N_{\mu,\varepsilon}(U)$, and it is easy to check that

$$M_{\mu,\varepsilon/2n}(G_1 \cap V, \cdots, G_n \cap V, G_1 \backslash V, \cdots, G_n \backslash V) \subseteq N_{\mu,\varepsilon/2}(V) \cap M_{\mu,\varepsilon}(G_1, \cdots, G_n) .$$

This completes the proof.

3. Main result.

THEOREM 3.1. *Let M be a compact space, and let X be a complete separable metric space. Let λ be a probability measure on M. Then the function $\varphi_\lambda\colon C(M, P(X)) \to P(X)$ defined by*

$$\varphi_\lambda(T) = \int T(t)d\lambda(t)$$

is open.

Proof. The proof will be accomplished in two steps: (A) We establish the result when X is totally disconnected. (B) We use (A) to complete the proof.

(A) Let X be a totally disconnected complete separable metric space. Let $T \in C(M, P(X))$, and let $\mathscr{U}_T$ be a neighborhood of T. It suffices to show that $\varphi_\lambda(\mathscr{U}_T)$ is a neighborhood of $\varphi_\lambda(T)$. By Lemma 2.2, we may take $\mathscr{U}_T$ to be a set of the form:

$$\mathscr{U}_T = \{S \in C(M, P(X)) \colon S(M_i) \subseteqq \mathscr{V}_i\ , \quad \text{for} \quad i = 1, \cdots, m\}$$

where for each i, M_i is a compact subset of M, and $\mathscr{V}_i$ is a basic open subset of $P(X)$ of the form:

$$\mathscr{V}_i = \{\theta \in P(X) \colon |\theta(G_{ij}) - \theta_i(G_{ij})| < \varepsilon\ , \quad \text{for} \quad j = 1, \cdots, n_i\}$$

where $\theta_i \in P(X)$ and $\{G_{ij} \colon j = 1, \cdots, n_i\}$ is an open cover for X consisting of mutually disjoint open subsets of X.

Let $\mathscr{C}$ be the collection of all nonempty subsets U of X such that $U = G_{1j_1} \cap G_{2j_2} \cap \cdots \cap G_{mj_m}$. Write $\mathscr{C} = \{U_1, \cdots, U_n\}$. Then $\mathscr{C}$ is an open cover for X and $U_i \cap U_j = \varnothing$ if $i \neq j$.

Since each G_{ij} is both open and closed, we have

$$\delta = \operatorname*{Max}_{ij} \operatorname*{Max}_{t \in M_i} |T(t)(G_{ij}) - \theta_i(G_{ij})| < \varepsilon\ .$$

Let $\varepsilon_0 = \varepsilon - \delta > 0$. One sees immediately that if $S \in C(M, P(X))$ is such that $\operatorname{Max}_{t \in M} |S(t)(G_{ij}) - T(t)(G_{ij})| < \varepsilon_0$ for all i, j, then $S \in \mathscr{U}_T$.

Let $\mu = \int T(t)d\lambda(t)$, and $a_i = \mu(U_i)$, $1 \leqq i \leqq n$. Then $\sum a_i = 1$ and we may assume that $a_n > 0$. Let N be an integer such that $N \cdot a_i > n^2$ whenever $a_i > 0$, $1 \leqq i \leqq n$. Define

$$\mathscr{V} = \{\nu \in P(X) \colon |\nu(U_i) - a_i| < \varepsilon_0/2N \quad \text{for} \quad i = 1, \cdots, n\}\ .$$

It suffices to show that $\varphi_\lambda(\mathscr{U}_T) \supseteqq \mathscr{V}$.

Let $\nu \in \mathscr{V}$. Then $\nu = \nu_1 + \cdots + \nu_n$, where ν_i is a measure on X defined as $\nu_i(A) = \nu(A \cap U_i)$. Let $b_i = \nu(U_i)$. Then $|a_i - b_i| < \varepsilon_0/2N$, and $b_i > 0$ whenever $a_i > 0$.

Now, go back to the function T. Let $f_i(t) = T(t)(U_i)$. Then all f_i, $i = 1, \cdots, n$, are continuous functions on M, and $\int f_i(t)d\lambda(t) = a_i$. We will construct continuous functions $g_1, \cdots, g_n$ on M such that

(1) $\int g_i(t)d\lambda(t) = b_i$,

(2) $\mathrm{Max}_{t\in M}|g_i(t)-f_i(t)|<\varepsilon_0/n$, and

(3) $0\leqq g_i(t)\leqq 1$ and $\sum_{i=1}^n g_i(t)=1$ for all t.

Given $i=1,\cdots,n-1$, define g_i as follows:

(a) If $b_i=a_i$, let $g_i(t)=f_i(t)$ for all t.

(b) If $b_i>a_i$, set $\delta_i=b_i-a_i<\varepsilon_0/2N$. Let $g_i(t)=f_i(t)+(\delta_i/a_n)f_n(t)$. Then,

$$f_i(t)\leqq g_i(t)\leqq f_i(t)+(\varepsilon_0/2N\cdot a_n)f_n(t)$$
$$\leqq f_i(t)+(\varepsilon_0/2n^2)f_n(t)\ .$$

(c) If $b_i<a_i$, set $\delta_i=a_i-b_i<\varepsilon_0/2N$. Since $a_i>0$, so that $b_i>0$. Define $h_i(t)=0$, if $f_i(t)\leqq\delta_i$; $h_i(t)=f_i(t)-\delta_i$, otherwise. Then $b_i\leqq\int h_i(t)d\lambda(t)\leqq a_i$. Let $b_i'=\int h_i(t)d\lambda(t)$ and $g_i(t)=(b_i/b_i')h_i(t)$. Then $g_i(t)\leqq f_i(t)$ and

$$f_i(t)-g_i(t)\leqq\delta_i+h_i(t)(1-b_i/b_i')$$
$$\leqq\delta_i+\varepsilon_0/2N\cdot a_i<\varepsilon_0/n^2\ .$$

Thus for $i=1,\cdots,n-1$, $0\leqq g_i\leqq 1$, $\int g_i(t)d\lambda(t)=b_i$, and

$$\underset{t\in M}{\mathrm{Max}}|g_i(t)-f_i(t)|<\varepsilon_0/n^2\ .$$

Moreover, $g_i(t)\leqq f_i(t)+(\varepsilon_0/2n^2)f_n(t)$. Hence, $g_1(t)+\cdots+g_{n-1}(t)\leqq 1$ for all t. Let $g_n(t)=1-g_1(t)-\cdots-g_{n-1}(t)$. Then the functions $g_1,\cdots,g_n$ are as required. This completes the construction.

Now let I, J be subsets of $\{1,2,\cdots,n\}$ such that $I=\{i\colon b_i>0\}$, $J=\{j\colon b_j=0\}$. For each $j\in J$, pick a measure $\alpha_j\in P(U_j)$. Define a continuous function $S\colon M\to P(X)$ by $S(t)=\sum_{i\in I}(g_i(t)/b_i)\nu_i+\sum_{j\in J}g_j(t)\alpha_j$. Clearly,

$$\varphi_\lambda(S)=\sum_{i\in I}\Big(\int\frac{g_i(t)}{b_i}d\lambda(t)\Big)\nu_i+\sum_{j\in J}\Big(\int g_j(t)d\lambda(t)\Big)\alpha_j$$
$$=\sum_{i\in I}\nu_i=\nu,\quad\text{and}\quad\underset{t\in M}{\mathrm{Max}}|S(t)(U_i)-T(t)(U_i)|<\varepsilon_0/n$$

for all i. Since each G_{ij} is a disjoint union of U_k, it follows that $\mathrm{Max}_{t\in M}|S(t)(G_{ij})-T(t)(G_{ij})|<\varepsilon_0$. Therefore, $S\in\mathscr{U}_T$. This completes the proof of (A).

(B) Let X be a complete separable metric space. To show that the mapping φ_λ is open, it is equivalent to show the following: Let $T\in C(M,P(X))$, and $\mu=\varphi_\lambda(T)$. Let μ_n be a sequence converging to μ in $P(X)$. Then there is a sequence $T_n\to T$ in $C(M,P(X))$ such that $\varphi_\lambda(T_n)=\mu_n$.

For this purpose, we use Lemma 2.1 to pick a totally disconnected space G, continuous functions $\varphi\colon G\to X$ and $\tilde{\varphi}\colon P(X)\to P(G)$, such that $\varphi\tilde{\varphi}(\mu) = \mu$, and that $\tilde{\varphi}$ is affine. Let $\tilde{\mu}_n = \tilde{\varphi}\mu_n$, $\tilde{\mu} = \tilde{\varphi}\mu$. Then $\tilde{\mu}_n \to \tilde{\mu}$ in $P(G)$. Let $\tilde{T}(t) = \tilde{\varphi}T(t)$ for each t. Then $\tilde{T} \in C(M, P(G))$. It is easy to check that $\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T)$. In fact, this is obvious if there is a finite subset $\{t_1, \cdots, t_n\} \subseteqq M$ with $\lambda\{t_1, \cdots, t_n\} = 1$. In general, we may pick a net $\lambda_\alpha \to \lambda$ in $P(M)$ such that for each α, $\lambda_\alpha(F_\alpha) = 1$ for some finite subset F_α of M. Thus, $\varphi_{\lambda_\alpha}(\tilde{T}) = \tilde{\varphi}\varphi_{\lambda_\alpha}(T)$. Let $\alpha \to \infty$, then we obtain

$$\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T) .$$

Hence $\varphi_\lambda(\tilde{T}) = \tilde{\mu}$. Since by (A), the function

$$\varphi_\lambda : C(M, P(G)) \longrightarrow P(G)$$

is open, hence, we may pick $\tilde{T}_n \to \tilde{T}$ in $C(M, P(G))$ such that $\varphi_\lambda(\tilde{T}_n) = \tilde{\mu}_n$. Let $T_n(t) = \varphi\tilde{T}_n(t)$. Then $T_n \to \varphi\tilde{T} = T$ in $C(M, P(X))$, and the same argument in proving $\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T)$ will give $\varphi_\lambda(T_n) = \varphi\varphi_\lambda(\tilde{T}_n)$. Therefore,

$$\varphi_\lambda(T_n) = \varphi\tilde{\mu}_n = \mu_n .$$

This proves (B), and so completes the proof of this theorem.

As a special case of Theorem 3.1, we let $M = \{1, 2\}$ with the discrete topology. We obtain Eifler's result [2]:

COROLLARY 3.2. *Let X be a complete separable metric space, and let $0 < \lambda < 1$. Then the function*

$$\lambda : P(X) \times P(X) \longrightarrow P(X)$$

defined by $(\mu, \nu) \to \lambda\mu + (1 - \lambda)\nu$ is open.

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Received October 21, 1975.

UNIVERSITY OF WASHINGTON