

On Contra-Precontinuous Functions

¹SAEID JAFARI AND ²TAKASHI NOIRI

¹Department of Mathematics and Physics, Roskilde University, Postbox 260, 4000 Roskilde, Denmark

²Department of Mathematics, Yatsushiro College of Technology, Yatsushiro, Kumamoto, 866-8501 Japan

¹e-mail: sjafari@ruc.dk and ²e-mail: noiri@as.yatsushiro-nct.ac.jp

Abstract. The notion of contra-continuity was introduced and investigated by Dontchev [3]. In this paper, we introduce and investigate a new generalization of contra-continuity called contra-precontinuity.

1. Introduction

Dontchev [3] introduced the notions of contra-continuity and strong S -closedness in topological spaces. He defined a function $f: X \rightarrow Y$ to be contra-continuous if the preimage of every open set of Y is closed in X . In [3], he obtained very interesting and important results concerning contra-continuity, compactness, S -closedness and strong S -closedness. Recently a new weaker form of this class of functions called contra-semicontinuous functions is introduced and investigated by Dontchev and Noiri [5]. They also introduced the notion of RC -continuity [5] between topological spaces which is weaker than contra-continuity and stronger than B -continuity [35]. Quite recently, the present authors [12] introduced and investigated a new class of functions called contra-super-continuous functions which lies between classes of RC -continuous functions and contra-continuous functions.

The aim of this paper is to introduce and investigate a new class of functions called contra-precontinuous functions which is weaker than contra-continuous functions. In Section 3, we obtain several basic properties of contra-precontinuous functions. In Section 4, we introduce contra-preclosed graphs and investigate relations between contra-precontinuity and contra-preclosed graphs. In Section 5, we obtain some properties of strongly S -closed spaces and compact spaces. Decompositions of RC -continuity and perfect continuity are also obtained. In the last section, we deal with strong forms of connectedness.

2. Preliminaries

In what follows, spaces X and Y are always topological spaces. $C1(A)$ and $\text{Int}(A)$ designate the closure and interior of A which is a subset of X . A subset A is said to be *regular open* (resp. *regular closed*) if $A = \text{Int}(C1(A))$ (resp. $A = C1(\text{Int}(A))$).

Definition 2.1. A subset A of a space X is called

- (i) preopen [16] if $A \subset \text{Int}(C1(A))$,
- (ii) semi-open [15] if $A \subset C1(\text{Int}(A))$,
- (iii) α -open [22] if $A \subset \text{Int}(C1(\text{Int}(A)))$,
- (iv) β -open [1] if $A \subset C1(\text{Int}(C1(A)))$.

The complement of a preopen (resp. semi-open, α -open, β -open) set is said to be *preclosed* (resp. *semi-closed*, *α -closed*, *β -closed*). The collection of all closed (resp. preopen, semi-open, α -open and β -open) subsets of X will be denoted by $C(X)$ (resp. $PO(X)$, $SO(X)$, $\alpha(X)$ and $\beta(X)$). We set $C(X, x) = \{V \in C(X) \mid x \in V\}$ for $x \in X$. We define similarly $PO(X, x)$, $SO(X, x)$, $\alpha(X, x)$ and $\beta(X, x)$.

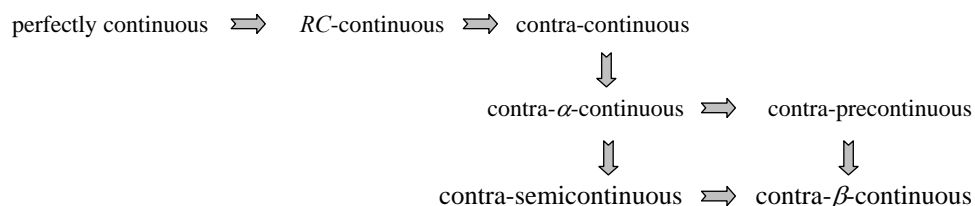
Definition 2.2. A function $f: X \rightarrow Y$ is called *perfectly continuous* [23] (resp. *RC-continuous* [5]) if for each open set V of Y , $f^{-1}(V)$ is clopen (resp. regular-closed) in X .

Definition 2.3. A function $f: X \rightarrow Y$ is called *precontinuous* [16] (resp. *semi-continuous* [15], *β -continuous* [1]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in PO(X, x)$ (resp. $U \in SO(X, x)$, $U \in \beta(X, x)$) such that $f(U) \subset V$.

Definition 2.4. A function $f: X \rightarrow Y$ is called *almost precontinuous* [20] if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists $U \in PO(X, x)$ such that $f(U) \subset \text{Int}(C1(V))$.

Definition 2.5. A function $f: X \rightarrow Y$ is called *contra-precontinuous* (resp. *contra-continuous* [3], *contra-semicontinuous* [5], *contra- α -continuous* [11], *contra- β -continuous* [3]) if $f^{-1}(V)$ is preclosed (resp. closed, semi-closed, α -closed, β -closed) in X for each open set V of Y .

For the functions defined above, we have the following diagram:



Remark 2.1. It should be noticed that contra-precontinuity and precontinuity are independent notions as shown by the following examples due to Dontchev [2].

Example 2.1. A continuous function need not be contra-precontinuous. The identity function on the real line with the usual topology is an example of a continuous function which is not contra-precontinuous.

Example 2.2. A contra-precontinuous function need not be precontinuous. Let $X = \{a, b\}$ be the Sierpinski space by setting $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, X\}$. The identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is contra-precontinuous. But it is neither precontinuous nor semi-continuous.

3. Some properties

Definition 3.1. Let A be a subset of a space (X, τ) .

- (1) The set $\bigcap \{U \in \tau \mid A \subset U\}$ is called the kernel of A [19] and is denoted by $\ker(A)$,
- (2) The set $\bigcap \{F \in X \mid A \subset F, F : \text{preclosed}\}$ is called the preclosure of A [7] and is denoted by $pCl(A)$.

Lemma 3.1. The following properties hold for subsets A, B of a space X :

- (1) $x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X, x)$.
- (2) $A \subset \ker(A)$ and $A = \ker(A)$ if A is open in X .
- (3) If $A \subset B$, then $\ker(A) \subset \ker(B)$.

Theorem 3.1. *The following are equivalent for a function $f : X \rightarrow Y$:*

- (1) f is contra-precontinuous;
- (2) for every closed subset F of Y , $f^{-1}(F) \in PO(X)$;
- (3) for each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in PO(X, x)$ such that $f(U) \subset F$;
- (4) $f(pCl(A)) \subset \ker(f(A))$ for every subset A of X ;
- (5) $pCl(f^{-1}(B)) \subset f^{-1}(\ker(B))$ for every subset of B of Y .

Proof. The implications (1) \Leftrightarrow (2) and (2) \Rightarrow (3) are obvious.

(3) \Rightarrow (2): Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in PO(X, x)$ such that $f(U_x) \subset F$. Therefore, we obtain $f^{-1}(F) = \bigcup \{U_x \mid x \in f^{-1}(F)\} \in PO(X)$.

(2) \Rightarrow (4): Let A be any subset of X . Suppose that $y \notin \ker(f(A))$. Then by Lemma 3.1 there exists $F \in C(X, Y)$ such that $f(A) \cap F = \emptyset$. Thus, we have $A \cap f^{-1}(F) = \emptyset$ and $pCl(A) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(pCl(A)) \cap F = \emptyset$ and $y \notin f(pCl(A))$. This implies that $f(pCl(A)) \subset \ker(f(A))$.

(4) \Rightarrow (5): Let B be any subset of Y . By (4) and Lemma 3.1, we have $f(pCl(f^{-1}(B))) \subset \ker(f(f^{-1}(B))) \subset \ker(B)$ and $pCl(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

(5) \Rightarrow (1): Let V be any open set of Y . Then, by Lemma 3.1 we have $pCl(f^{-1}(V)) \subset f^{-1}(\ker(V)) = f^{-1}(V)$ and $pCl(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is preclosed in X .

Theorem 3.2. *The following are equivalent for a function $f : X \rightarrow Y$:*

- (i) f is contra- α -continuous;
- (ii) f is contra-precontinuous and contra-semicontinuous.

Proof. This follows from the fact that $A \in \alpha(X)$ if and only if $A \in PO(X) \cap SO(X)$ [28, Lemma 1].

Theorem 3.3. *If a function $f : X \rightarrow Y$ is contra-precontinuous and Y is regular, then f is precontinuous.*

Proof. Let x be an arbitrary point of X and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $C1(W) \subset V$. Since f is contra-precontinuous, so by Theorem 3.1 there exists $U \in PO(X, x)$ such that $f(U) \subset C1(W)$. Then $f(U) \subset C1(W) \subset V$. Hence, f is precontinuous.

Remark 3.1. By Example 2.1, a precontinuous functions $f: X \rightarrow Y$ is not always contra-precontinuous even if Y is regular.

Recall that a function $f: X \rightarrow Y$ is called *M-preopen* [17] if the image of each preopen set is preopen.

Theorem 3.4. *If $f: X \rightarrow Y$ is an M-preopen contra-precontinuous function, then f is almost precontinuous.*

Proof. Let x be any arbitrary point of X and V be an open neighborhood $f(x)$. Since f is contra-precontinuous, then by Theorem 3.1 (3), there exists $U \in PO(X, x)$ such that $f(U) \subset C1(V)$. Since f is M-preopen, $f(U)$ is preopen in Y . Therefore $f(U) \subset \text{Int}(C1(f(U))) \subset \text{Int}(C1(V))$. This shows that f is almost precontinuous.

Definition 3.2. *A function $f: X \rightarrow Y$ is said to be almost weakly continuous [13] if $f^{-1}(V) \subset \text{Int}(C1(f^{-1}(C1(V))))$ for every open set V of Y .*

It is shown in [25, Theorem 3.1] that a function $f: X \rightarrow Y$ is almost weakly continuous if and only if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists $U \in PO(X, x)$ such that $f(U) \subset C1(V)$.

Remark 3.2. The following implications are obvious:

$$\text{precontinuity} \Rightarrow \text{almost precontinuity} \Rightarrow \text{almost weak continuity},$$

where the converses are false as shown in Examples 2.1 and 2.2 [10].

As shown in Example 2.2, a contra-precontinuous function need not be precontinuous. However, every contra-precontinuous function is necessarily almost weakly continuous.

Theorem 3.5. *If a function $f: X \rightarrow Y$ is contra-precontinuous, then f is almost weakly continuous.*

Proof. Let V be any open set of Y . Since $C1(V)$ is closed in Y , $f^{-1}(C1(V))$ is preopen in X and we have $f^{-1}(V) \subset f^{-1}(C1(V)) \subset \text{Int}(C1(f^{-1}(C1(V))))$. This shows that f is almost weakly continuous.

The *prefrontier* [26] $\text{pFr}(A)$ of A , where $A \subset X$, is defined by $\text{pFr}(A) = \text{pC1}(A) \cap \text{pC1}(X - A)$.

Theorem 3.6. *The set of all points of x of X at which $f: X \rightarrow Y$ is not contra-precontinuous is identical with the union of the prefrontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof. “Necessity”. Suppose that f is not contra-precontinuous at $x \in X$. There exists $F \in C(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \emptyset$ for every $U \in PO(X, x)$. This implies that $U \cap f^{-1}(Y - F) \neq \emptyset$. Therefore, we have $x \in \text{pC1}(f^{-1}(Y - F)) = \text{pC1}(X - f^{-1}(F))$. However, since $x \in f^{-1}(F)$, $x \in \text{pC1}(f^{-1}(F))$. Therefore, we obtain $x \in \text{pFr}(f^{-1}(F))$.

“Sufficiency”. Suppose that $x \in \text{pFr}(f^{-1}(F))$ for some $F \in C(Y, f(x))$. Now, we assume that f is contra-precontinuous at x . Then there exists $U \in PO(X, x)$ such that $f(U) \subset F$. Therefore, we have $x \in U \subset f^{-1}(F)$ and hence $x \in \text{pInt}(f^{-1}(F)) \subset X - \text{pFr}(f^{-1}(F))$. This is a contradiction. This means that f is not contra-precontinuous.

Recall that a family E of subsets of a space (X, τ) is called a *network* for a topology τ on (X, τ) if every set in τ is the union of some subfamily of E .

Definition 3.3. *A space (X, τ) is said to be*

- (1) *extremally disconnected* [33] *if the closure of every open set of X is open in X ,*
- (2) *locally indiscrete* [21] *if every open set of X is closed in X .*
- (3) *submaximal* [29] *if every dense set of X is open in X , equivalently if every preopen set is open,*
- (4) *strongly irresolvable* [8] *if no nonempty open set is resolvable, equivalently if every preopen subset is α -open,*
- (5) *mildly Hausdorff* [6] *if the δ -closed sets form a network for its topology τ , where a δ -closed set is the intersection of regular closed sets,*
- (6) *strongly S -closed* [3] *if every closed cover of X has a finite subcover,*
- (7) *door space* [4] *if every subset of X is either open or closed.*

Remark 3.3. It should be noted that every door space X is submaximal [4, Theorem 2.7] and every mildly Hausdorff strongly S -closed space is locally indiscrete [6].

The following results follow immediately from Definition 3.3 and Remark 3.3:

Theorem 3.7. *If a function $f: X \rightarrow Y$ is continuous and X is locally indiscrete, then f is contra-continuous.*

Corollary 3.1. *If a function $f: X \rightarrow Y$ is continuous and X is mildly Hausdorff strongly S -closed, then f is contra-continuous.*

Theorem 3.8. *Let $f: X \rightarrow Y$ be a contra-precontinuous function.*

- (1) *If X is submaximal, then f is contra-continuous,*
- (2) *If X is strongly irresolvable, then f is contra- α -continuous.*

Corollary 3.2. *If a function $f: X \rightarrow Y$ is contra-precontinuous and X is a door space, then f is contra-continuous.*

Lemma 3.2. *For a subset A of a space X , the following are equivalent:*

- (1) *A is regular closed;*
- (2) *A is preclosed and semi-open;*
- (3) *A is α -closed and β -open.*

Proof. (1) \Rightarrow (2): Let A be regular closed. Then $A = C1(\text{Int}(A))$ and A is preclosed and semi-open.

(2) \Rightarrow (3): Let A be preclosed and semi-open. Then $C1(\text{Int}(A)) \subset A$ and $A \subset C1(\text{Int}(A))$. Therefore, we have $C1(\text{Int}(A)) = C1(A)$ and hence $C1(\text{Int}(C1(A))) = C1(\text{Int}(C1(\text{Int}(A)))) = C1(\text{Int}(A)) \subset A$. This shows that A is α -closed. Since $SO(X) \subset \beta O(X)$, it is obvious that A is β -open.

(3) \Rightarrow (1): Let A be α -closed and β -open. Then $A = C1(\text{Int}(C1(A)))$ and hence $C1(\text{Int}(A)) = C1(\text{Int}[C1(\text{Int}(C1(A)))] = C1(\text{Int}(C1(A))) = A$. Therefore, A is regular closed.

As a consequence of the above lemma, we have the following result:

Theorem 3.9. *The following statements are equivalent for a function $f: X \rightarrow Y$:*

- (1) *f is RC-continuous;*
- (2) *f is contra-precontinuous and semi-continuous;*
- (3) *f is contra- α -continuous and β -continuous.*

4. Contra-preclosed graphs

We begin with the following notion:

Definition 4.1. The graph $G(f)$ of a function $f: X \rightarrow Y$ is said to be contra-preclosed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in PO(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 4.1. The graph $G(f)$ of $f: X \rightarrow Y$ is contra-preclosed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in PO(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.

Theorem 4.1. If $f: X \rightarrow Y$ is contra-precontinuous and Y is Urysohn, then $G(f)$ is contra-preclosed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exist open sets V, W such that $f(x) \in V, y \in W$ and $C1(V) \cap C1(W) = \emptyset$. Since f is contra-precontinuous, there exists $U \in PO(X, x)$ such that $f(U) \subset C1(V)$. Therefore, we obtain $f(U) \cap C1(W) = \emptyset$. This shows that $G(f)$ is contra-preclosed.

Definition 4.2. A space X is said to be

- (1) strongly compact [17] if every preopen cover of X has a finite subcover,
- (2) S -closed [34] if every semi-open cover $\{V_\alpha \mid \alpha \in \nabla\}$ of X , there exists a finite subset ∇_\circ of ∇ such that $X = \bigcup \{C1(V_\alpha) \mid \alpha \in \nabla_\circ\}$, equivalently if every regular closed cover of X has a finite subcover,
- (3) mildly compact [32] if every clopen cover of X has a finite subcover.

Definition 4.3. A subset S of a space X is said to be

- (1) strongly compact relative to X [17] if every cover of S by preopen sets of X has a finite subcover,
- (2) strongly S -closed [3] if the subspace S is strongly S -closed.

Theorem 4.2. Let X be submaximal. If $f: X \rightarrow Y$ has a contra-preclosed graph, then the inverse image of a strongly S -closed set K of Y is closed in X .

Proof. Assume that K is a strongly S -closed set of Y and $x \notin f^{-1}(K)$. For each $k \in K, (x, k) \notin G(f)$. By Lemma 4.1, there exists $U_k \in PO(X, x)$ and $V_k \in C(Y, k)$ such that $f(U_k) \cap V_k = \emptyset$. Since $\{K \cap V_k \mid k \in K\}$ is a closed cover of the subspace K , there exists a finite subset $K_1 \subset K$ such that

$K \subset \bigcup \{V_k \mid k \in K_1\}$. Set $U = \bigcap \{U_k \mid k \in K_1\}$, then U is open since X is submaximal. Therefore $f(U) \cap K = \emptyset$ and $U \cap f^{-1}(K) = \emptyset$. This shows that $f^{-1}(K)$ is closed in X .

A space X is said to be *weakly Hausdorff* [31] if each point of X is an intersection of regular closed sets of X .

Corollary 4.1. *Let X be submaximal and Y be strongly S -closed weakly Hausdorff. The following properties are equivalent for a function $f : X \rightarrow Y$:*

- (1) f is contra-precontinuous;
- (2) $G(f)$ is contra-preclosed;
- (3) $f^{-1}(K)$ is closed in X for every strong S -closed set K of Y ;
- (4) f is contra-continuous.

Proof. (1) \Rightarrow (2): It is shown in [9, Theorem 3.7] that every S -closed weakly Hausdorff space is extremally disconnected. Since a strongly S -closed space is S -closed, Y is extremally disconnected and hence every regular closed set of Y is clopen. This shows that Y is Urysohn. By Theorem 4.1, $G(f)$ is contra-preclosed.

(2) \Rightarrow (3): This is a result of Theorem 4.2.

(3) \Rightarrow (4): First, we show that an open set of Y is strongly S -closed. Let V be an open set of Y and $\{H_\alpha \mid \alpha \in \nabla\}$ be a cover of V by closed sets H_α of the subspace V . For each $\alpha \in \nabla$, there exists a closed set K_α of X such that $H_\alpha = K_\alpha \cap V$. Then, the family $\{K_\alpha \mid \alpha \in \nabla\} \cup (Y - V)$ is a closed cover of Y . Since Y is strongly S -closed, there exists a finite subset $\nabla_\circ \subset \nabla$ such that $Y = \bigcup \{K_\alpha \mid \alpha \in \nabla_\circ\} \cup (Y - V)$. Therefore we obtain $V = (\bigcup \{K_\alpha \mid \alpha \in \nabla_\circ\}) \cap V = \bigcup \{H_\alpha \mid \alpha \in \nabla_\circ\}$. This shows that V is strongly S -closed. For any open set V , by (3) $f^{-1}(V)$ is closed in X and f is contra-continuous.

5. Strong forms of compactness

Theorem 5.1. *If $f : X \rightarrow Y$ is contra-precontinuous and K is strongly compact relative to X , then $f(K)$ is strongly S -closed in Y .*

Proof. Let $\{H_\alpha \mid \alpha \in \nabla\}$ be any cover of $f(K)$ by closed sets of the subspace $f(K)$. For each $\alpha \in \nabla$, there exists a closed set K_α of Y such that $H_\alpha = K_\alpha \cap f(K)$. For each $x \in K$, there exists $\alpha(x) \in \nabla$ such that

$f(x) \in K_{\alpha(x)}$ and by Theorem 3.1 there exists $U_x \in PO(X, x)$ such that $f(U_x) \subset K_{\alpha(x)}$. Since the family $\{U_x \mid x \in K\}$ is a preopen cover of K , there exists a finite subset K_0 of K such that $K \subset \bigcup\{U_x \mid x \in K_0\}$. Therefore, we obtain $f(K) \subset \bigcup\{f(U_x) \mid x \in K_0\}$ which is a subset of $\bigcup\{K_{\alpha(x)} \mid \alpha \in K_0\}$. Thus, $f(K) = \bigcup\{H_{\alpha(x)} \mid x \in K_0\}$ and hence $f(K)$ is strongly S -closed.

Corollary 5.1. *If $f: X \rightarrow Y$ is contra-precontinuous surjection and X is strongly compact, then Y is strongly S -closed.*

Theorem 5.2. *A function $f: X \rightarrow Y$ is RC -continuous if and only if it is contra-precontinuous and semi-continuous.*

Proof. “Necessity”. Every RC -continuous function is contra-continuous and hence contra-precontinuous. Since every regular closed set is semi-open, RC -continuous functions are semi-continuous.

“Sufficiency”. For any open set V of Y , $f^{-1}(V)$ is preclosed and semi-open in X and hence we have $C1(\text{Int}(f^{-1}(V))) \subset f^{-1}(V) \subset C1(\text{Int}(f^{-1}(V)))$. Therefore, we obtain $C1(\text{Int}(f^{-1}(V))) = f^{-1}(V)$ and hence f is RC -continuous.

Remark 5.1. It follows from Examples 2.1 and 2.2 that contra-precontinuity and semi-continuity are independent of each other. Therefore, by Theorem 5.2 we had a decomposition of RC -continuity.

Theorem 5.3. *If $f: X \rightarrow Y$ is an RC -continuous surjection and X is S -closed, then Y is compact.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any open cover of Y . Then $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla_0\}$ is a regular closed cover of X and we have $X = \bigcup\{f^{-1}(V_\alpha) \mid \alpha \in \nabla_0\}$ for some finite subset ∇_0 of ∇ . Since f is surjective, $Y = \bigcup\{V_\alpha \mid \alpha \in \nabla_0\}$ and Y is compact.

A function $f: X \rightarrow Y$ is said to be α -continuous [18] if $f^{-1}(V) \in \alpha(X)$ for every open set V of Y . In [2, Theorem 2.9], Dontchev obtained decompositions of perfect continuity. The following is also a decomposition of perfect continuity.

Theorem 5.4. *A function $f: X \rightarrow Y$ is perfectly continuous if and only if it is contra-precontinuous and α -continuous.*

Proof. “Necessity”. This is obvious.

“Sufficiency”. Let f be contra-precontinuous and α -continuous. Let V be any open set of Y . Then $f^{-1}(V)$ is preclosed and α -open in X . Therefore, we have

$$\begin{aligned} \text{Int}(C1(\text{Int}(f^{-1}(V)))) &\subset C1(\text{Int}(f^{-1}(V))) \subset f^{-1}(V) \\ &\subset \text{Int}(C1(\text{Int}(f^{-1}(V)))) \subset C1(\text{Int}(f^{-1}(V))). \end{aligned}$$

This implies that $f^{-1}(V)$ is clopen in X . Thus, f is perfectly continuous.

Remark 5.2. Contra-precontinuity and α -continuity are independent of each other as shown by Examples 2.1 and 2.2.

Corollary 5.2. (Dontchev [3]). For a function $f : X \rightarrow Y$ the following are equivalent:

- (1) f is perfectly continuous;
- (2) f is continuous and contra-continuous;
- (3) f is α -continuous and contra-continuous.

Theorem 5.5. If $f : X \rightarrow Y$ is a perfectly continuous surjection and X is mildly compact, then Y is compact.

Proof. Let $f : X \rightarrow Y$ be a perfectly continuous surjection and X be mildly compact. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any open cover of Y . Then $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a clopen cover of X . Since X is mildly compact, there exists a finite subset ∇_\circ of ∇ such that $X = \bigcup \{f^{-1}(V_\alpha) \mid \alpha \in \nabla_\circ\}$. Since f is surjective, $Y = \bigcup \{V_\alpha \mid \alpha \in \nabla_\circ\}$ and Y is compact.

A space X is said to be *almost compact* [30] or *quasi H-closed* [27] if every open cover has a finite subfamily the closures of whose members cover X . We have the following implications:

$$\text{strongly } S\text{-closed} \Rightarrow S\text{-closed} \Rightarrow \text{almost compact} \Rightarrow \text{mildly compact}.$$

Corollary 5.3. (Dontchev [3]). The image of an almost compact space under contra-continuous nearly continuous (precontinuous) function is compact.

Proof. It is shown in [3, Theorem 2.9] that a function is contra-continuous and nearly continuous if and only if it is perfectly continuous. Then, the proof follows from Theorem 5.5.

6. Strong forms of connectedness

Definition 6.1. A space X is said to be

- (1) *hyperconnected* [33] if $C1(V) = X$ for every nonempty open set V of X ,
- (2) *θ -irreducible* [14] if every two nonempty regular closed sets intersect,
- (3) *preconnected* [24] if X cannot be expressed as the union of two nonempty preopen sets.

Theorem 6.1. Let X be preconnected and Y be T_1 . If $f: X \rightarrow Y$ is contra-precontinuous, then f is constant.

Proof. Since Y is T_1 -space, $\mathbf{U} = \{f^{-1}(y) \mid y \in Y\}$ is a disjoint preopen partition of X . If $|\mathbf{U}| \geq 2$, then X is the union of two nonempty preopen sets. Since X is preconnected, $|\mathbf{U}| = 1$. Therefore, f is constant.

A function $f: X \rightarrow Y$ is said to be *preclosed* [7] if the image $f(A)$ is preclosed in Y for every closed set A of X .

Theorem 6.2. Let $f: X \rightarrow Y$ be a contra-precontinuous and preclosed surjection. If X is submaximal, then Y is locally indiscrete.

Proof. Let V be any open set of Y . Since f is contra-continuous and X is submaximal, $f^{-1}(V)$ is closed in X and hence V is preclosed in Y . Therefore, $C1(V) = C1(\text{Int}(V)) \subset V$ and V is closed in Y . This shows that Y is locally indiscrete.

Theorem 6.3. If $f: X \rightarrow Y$ is a contra-precontinuous semi-continuous surjection and X is θ -irreducible, then Y is hyperconnected.

Proof. Suppose that Y is not hyperconnected. Then, there exists two disjoint nonempty open sets V, W and Y . By Theorem 5.2, f is an RC -continuous surjection and $f^{-1}(V), f^{-1}(W)$ are disjoint nonempty regular closed sets of X . Therefore, X is not θ -irreducible.

Theorem 6.4. If $f: X \rightarrow Y$ is a contra-precontinuous α -continuous surjection and X is connected, then Y has an indiscrete topology.

Proof. Suppose that there exists a proper open set V of Y . By Theorem 5.4, f is a perfectly continuous surjection and $f^{-1}(V)$ is a proper clopen set of X . This shows that X is not connected. Therefore, Y has an indiscrete topology.

Acknowledgement. The authors are very grateful to the referee for his careful work.

References

1. M.E. Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud, β -open sets and β -continuous mappings, *Bull. Fac. Sci. Assiut Univ.* **12** (1983), 77–90.
2. M. Caldas and S. Jafari, Some properties of contra- β -continuous functions, *Mem. Fac. Sci. Kochi Univ. (Math.)* **22** (2001), 19–28.
3. J. Dontchev, Contra-continuous functions and strongly S -closed spaces, *Internat. J. Math. Math. Sci.* **19** (1996), 303–310.
4. J. Dontchev, On door spaces, *Indian J. Pure Appl. Math.* **26** (1995), 873–881.
5. J. Dontchev and T. Noiri, Contra-semicontinuous functions, *Math. Pannonica* **10** (1999), 159–168.
6. J. Dontchev, S. Popvassilev and D. Stavrova, On the η -expansion topology for the co-semi-regularization and mildly Hausdorff spaces, *Acta Math. Hungar.* **80** (1998), 9–19.
7. N. El-Deeb, I.A. Hasanein, A.S. Mashhour and T. Noiri, On p -regular spaces, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **27** (1983), 311–315.
8. J. Foran and P. Liebnitz, A characterization of almost resolvable spaces, *Rend. Circ. Mat. Palermo* **40** (1991), 136–141.
9. R.A. Herrmann, RC -convergence, *Proc. Amer. Math. Soc.* **75** (1979), 311–317.
10. S. Jafari and T. Noiri, On almost precontinuous functions, *Internat. J. Math. Math. Sci.* **24** (2000), 193–201.
11. S. Jafari and T. Noiri, Contra- α -continuous functions between topological spaces, *Iranian Int. J. Sci.* **2** (2001), 153–167.
12. S. Jafari and T. Noiri, Contra-super-continuous functions, *Annales Univ. Sci. Budapest* **42** (1999), 27–34.
13. D.S. Jankovic, θ -regular spaces, *Internat. J. Math. Math. Sci.* **8** (1985), 615–619.
14. D. Jankovic and P.E. Long, θ -irreducible spaces, *Kyungpook Math. J.* **26** (1986), 63–66.
15. N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly* **70** (1963), 36–41.
16. A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt* **53** (1982), 47–53.
17. A.S. Mashhour, M.E. Abd El-Monsef, I.A. Hasanein and T. Noiri, Strongly compact spaces, *Delta J. Sci.* **8** (1984), 30–46.
18. A.S. Mashhour, I.A. Hasanein and S.N. El-Deeb, α -continuous and α -open mappings, *Acta Math. Hungar.* **41** (1983), 213–218.
19. M. Mrsevic, On pairwise R_0 and pairwise R_1 bitopological spaces, *Bull. Math. Soc. Sci. Math. R.S. Roumanie* **30** (1986), 141–148.
20. A.A. Nasef and T. Noiri, Some weak forms of almost continuity, *Acta Math. Hungar.* **74** (1997), 211–219.
21. T. Nieminen, On ultrapseudocompact and related spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **3** (1977), 185–205.
22. O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.* **15** (1965), 961–970.
23. T. Noiri, Super-continuity and some strong forms of continuity, *Indian J. Pure Appl. Math.* **15** (1984), 241–250.
24. V. Popa, Properties of H -almost continuous functions, *Bull. Math. Soc. Sci. Math. R.S. Roumanie* **31** (78) (1987), 163–168.
25. V. Popa and T. Noiri, Almost weakly continuous functions, *Demonstratio Math.* **25** (1992), 241–251.
26. V. Popa, T. Noiri and M. Ganster, On upper and lower almost precontinuous multifunctions, *Far East J. Math. Sci. Special* **Vol. I** (1997), 49–68.

27. J. Porter and J. Thomas, On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.* **138** (1969), 159–170.
28. I.L. Reilly and M.K. Vamanamurthy, Connectedness and strong semi-continuity, *Casopis Pest. Mat.* **109** (1984), 261–265.
29. I.L. Reilly and M.K. Vamanamurthy, On some questions concerning preopen sets, *Kyungpook Math. J.* **30** (1990), 87–93.
30. M.K. Singal and A. Mathur, On nearly-compact spaces, *Boll. Un. Mat. Ital.* (4) **2** (1969), 702–710.
31. T. Soundararajan, Weakly Hausdorff spaces and the cardinality of topological spaces, General Topology and its Relations to Modern Analysis and Algebra III, *Proc. Conf. Kampur* (1968), *Academia, Prague* (1971), 301–306.
32. R. Staum, The algebra of bounded continuous functions into a nonarchimedean field, *Pacific J. Math.* **50** (1974), 169–185.
33. L.A. Steen and J.A. Seebach Jr., *Counterexamples in Topology*, Holt, Rinehart and Wiston, New York, 1970.
34. T. Thompson, S-closed spaces, *Proc. Amer. Math. Soc.* **60** (1976), 335–338.
35. J. Tong, On decomposition of continuity in topological spaces, *Acta Math. Hungar.* **54** (1998), 51–55.

Keywords and phrases: contra-precontinuous, preopen set, mildly compact.

1991 AMS Subject Classification: 54C08