(a) $r(X_\alpha) = r(Y_\alpha) = \alpha$;

(b) X_α *is scattered, and* Y_α *is not*;

(c) $|Y_\alpha| = |\alpha|+\omega$; *and*

(d) $|X_\alpha| = |\alpha|+\omega$ *unless* $\alpha = 0$, *in which case* $|X_\alpha| = 1$.

Proof. Let Z be the integers (with the usual order). Given X_α, let $X_{\alpha+1} = Z\times X_\alpha$, ordered lexicographically, and let $Y_\alpha = Q\times X_\alpha$, also ordered lexicographically. If α is a limit ordinal, let $X_\alpha = \{f\colon \alpha+1\to Z\colon f$ is continuous (with respect to order topologies on $\alpha+1$ and Z) and $f(\alpha) = 0\}$; if $f, g\in X_\alpha$ with $f\neq g$, let $\beta = \max\{\xi<\alpha\colon f(\xi)\neq g(\xi)\}$, and write $f<g$ if $f(\beta)<g(\beta)$. It is easily checked that X_α has the desired properties.

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On contractible fans

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Abstract. The purpose of this paper is to give a characterization of weakly confluent-contractible fans. After giving several definitions, it is shown that such a fan must be *pairwise smooth*, must contain no *ziz-zag*, and lastly must contain no *P-point*. It is then shown that a fan which satisfies these three properties must be monotone-contractible. This implies the fan is weakly confluent-contractible in as much as monotone functions are always weakly confluent. Hence these properties also yield a characterization of monotone contractible fans.

Introduction. Several mathematicians (see [1], [4], [5], [7]) in recent years have studied the contractibility of dendroids. We will use the term *dendroid* to designate a compact metric continuum which is arc-wise connected and is also hereditarily unicoherent. A *ramification point* of a dendroid is a point which is the intersection of three or more arcs. K. Borsuk [2] has described simple types of dendroids, containing only one ramification point, which are called *fans*. The ramification point is called the *top* of the fan.

A topological space X is *contractible* if there exists a continuous map $F\colon [0,1]\times X\to X$ such that $F(0,p)$ is p, for each point p of X; and there is a point q in X such that $F(1,p)$ is q for each point p of X. The map F is called a *contraction* of X.

Figure 1 in the Appendix is a contractible dendroid A with the surprising property that for each choice of a contraction F, there must be a time t in $[0, 1]$ for which $F(t\times A)$ is a *noncontractible* sub-dendroid of A. In order to restrict the spaces it was decided to place a stronger requirement on the maps involved. The property chosen was first defined by A. Lelek [9], that of *weak-confluence* of the maps. It was found that for dendroids, even with weakly-confluent maps, examples of the type found in Figure 1 are still admissible. The investigation was further restricted to the case of fans. It will be shown that a fan is weakly-confluent contractible if and only if it is confluent contractible, if and only if it is monotone contractible.

A continuous map is said to be *monotone* if the pre-image of each continuum lying in its image is itself a continuum. A contraction F on a space X is a *monotone contraction* provided that for each time t in $[0, 1]$, the map F restricted to $\{t\}\times X$ is monotone.

A continuous map is *confluent* if, for each continuum K lying in its image, it is true that every component of the pre-image of K is mapped *onto* K. A contraction F on a space X is a *confluent contraction* if F restricted to $\{t\}\times X$ is confluent for each t in $[0,1]$.

A continuous map is said to be *weakly-confluent* if, for each continuum K lying in its image, it is true that at least one component of the pre-image of K is mapped *onto* K. Given a space X and a contraction F of X, we will say that F is a *weakly-confluent contraction* provided that for each time t in $[0,1]$, the map F restricted to $\{t\}\times X$ is weakly-confluent.

The main result of this paper is a characterization of those fans which admit a weakly-confluent contraction. The following definitions will be used throughout the paper.

DEFINITION. Let X be a dendroid and let r be a point in X. Suppose there are two sequences $\{r(1,n)\}$, $\{r(2,n)\}$ $(n = 1, 2, 3 \ldots)$ of points of X, each converging to r. We say that the former sequence *dominates* the latter sequence provided that whenever there exists a point s in X and a sequence $\{s(1,n)\}$ converging to s, with the property that the arcs $[r(1,n), s(1,n)]$ converge to the arc $[r,s]$, then it follows that there also exists a sequence $\{s(2,n)\}$ converging to s such that the arcs $[r(2,n), s(2,n)]$ converge to $[r,s]$ set-wise.

DEFINITION. We say that a dendroid is *pairwise-smooth* provided that whenever a pair of sequences converge to a common point, then one of the pair dominates the other. Figures 2 and 3 in the Appendix illustrate fans which are *not* pairwise-smooth.

DEFINITION. We say that a dendroid X contains a *zig-zag* if there are distinct points a, b belonging to X and a sequence of arcs $[a_n, b_n, c_n, d_n]$, $n = 1, 2, \ldots$ (with endpoints a_n, d_n and interior points b_n, c_n in the order indicated) converging to the arc $[a,b]$ in such a way that $\{a_n\}_{n=1,2,\ldots}$ and $\{c_n\}_{n=1,2,\ldots}$ each converge to a, while $\{b_n\}_{n=1,2,\ldots}$ and $\{d_n\}_{n=1,2,\ldots}$ each converge to b. Figures 4 and 5 in the Appendix show some examples of a zig-zag.

The *P-point* defined next, is a slight modification of R. Bennett's *O-point*.

DEFINITION. Let X be a dendroid and let b be a point of X. We call b a *P-point* if there is a sequence of points in X $\{b_n\}_{n=1,2,\ldots}$ converging to b such that $\operatorname{Ls}[b, b_n]$ is not equal to b, and such that if $[b_n, x_n]$ denotes the arc irreducible between b_n and $\operatorname{Ls}[b, b_n]$, then it follows that $\{x_n\}_{n=1,2,\ldots}$ converges to b. A simple example of a P-point is given in Figure 6 of the Appendix.

We will show that a fan is weakly-confluent contractible if and only if it is pairwise smooth, contains no zig-zag, and contains no P-point.

The following notation will be used:

Cl = Closure,

$B(\,,\,)$ = Open ball of radius ---, centered at ---,

$[a,b]$ = Arc with endpoints a, b the order does not matter unless otherwise indicated,

$\langle a,b]$ = $[a,b]$ less $\{a\}$,

$[a,b\rangle$ = $[a,b]$ less $\{b\}$,

Bd = Boundary,

d, ϱ are used for distance functions,

$\mathscr{L}(b) = \{p\in X \mid p\leqslant b\}$,

$\mathscr{U}(b) = \{p\in X \mid p\leqslant b\}$,

$\operatorname{Ls}_n(\;)$ = Lim sup (as sets or else as points).

Given a fan X with point c, the weak cut point order (with respect to c) is defined on X by: $p\leqslant q$ if p belongs to $[c,q]$ and $p<q$ if $p\leqslant q$ but p is distinct from q.

Given a fan X with a partial order $\leqslant$ defined on X a metric ϱ on X is radially convex provided that $p\leqslant q<z$ implies $\varrho(p,q)<\varrho(p,z)$.

A partial order $\leqslant$ on X is *closed* if the set $\{(a,b) \mid a\leqslant b\}$ is closed in $X\times X$.

Chapter 1. This section contains some basic results which will be needed to obtain the main theorem.

LEMMA 1.1. *Let X be a dendroid. Let $\{x_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ be sequences of points of X converging to x_0, r_0 respectively. Let b be a point of $\operatorname{Ls}_n[x_n, r_n]$ and let $\leqslant$ denote the weak-cut-point order, with respect to b defined on $\operatorname{Ls}_n[x_n, r_n]$. There exists a subsequence $\{[x_n(j), r_n(j)]\}_{j=1}^{\infty}$ and sequence $\{b_n(j)\}_{j=1}^{\infty}$ converging to b, with $b_n(j)$ contained in $[x_n(j), r_n(j)]$ such that $\operatorname{Ls}_j[x_n(j), b_n(j)]\leqslant b$ (if $x_0\leqslant b$) respectively, $\operatorname{Ls}_j[x_n(j), b_n(j)]\geqslant b$ (if $x_0\geqslant b$).*

Proof. We may as well assume that x_0 is distinct from b, that for each $n = 1, 2, \ldots, x_n$ is *not* contained in $\operatorname{Cl}(B(1,b))$, and that b belongs to $\lim_n[x_n, r_n]$. For each $j = 1, 2, \ldots$, there exists a subarc $[x_n, b_{(n,j)}]$ of $[x_n, r_n]$ which is irreducible between x_n and $\operatorname{Cl}(B(1/j, b))$, for each n greater than say N_j. Also, since b does not belong to $\operatorname{Ls}_n[x_n, b_{(n,j)}]$, it follows that $\operatorname{Ls}_n[x_n, b_{(n,j)}]\leqslant b$ (respectively, $\geqslant b$). Hence for each n larger than say M_j it must be true that $[x_n, b_{(n,j)}]$ is in the $1/j$ neighborhood of $\mathscr{L}(b)$ (respectively, of $\mathscr{U}(b)$). Choose $n(1)<n(2)<\ldots<n(k)<\ldots$, with $n(j)$ greater than M_j, such that $[x_n(j), b_{(n,j)}(j)]$ is in the $1/j$ neighborhood of $(\mathscr{L}(b))$ (respectively, of $(\mathscr{U}(b))$. Note that $b_{(n,j)}(j)$ is contained in $\operatorname{Cl}(B(1/j, b))$ for each j. Thus $\{b_{(n,j)}(j)\}_{j=1}^{\infty}$ converges to b and it is evident that $\operatorname{Ls}_j[x_n(j), b_{(n,j)}(j)]\leqslant b$ (respectively, $\geqslant b$).

LEMMA 1.2. *Let X be a dendroid and $\{x_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ be sequences of points of X, converging to x_0, r_0 respectively such that the arcs $[x_n, r_n]$ are pairwise disjoint or else $x_n = x_0$ for all n while $\langle x_n, r_n]$ are pairwise disjoint, and such that $\operatorname{Ls}_n[x_n, r_n] = [x_0, r_0]$. Define $y\leqslant z$ if y is contained in $[x_n, z]$ for some $n = 0, 1, 2, \ldots$ If X contains no zig-zags, then $\bigcup_{n=0}^{\infty}[x_n, r_n]$ admits a radially convex metric, with respect to $\leqslant$.*

Proof. It is known (see [3] and [10]) that the result follows if it can be shown that whenever a sequence of points $\{y_n\}_{n=1}^{\infty}$ contained in $\bigcup_{n=0}^{\infty}[x_n, r_n]$ converges to a point y_0, then it follows that $\operatorname{Ls}_n L(y_n)$ is included in $L(y_0)$. Suppose the lemma is false. There then exist points y_n in $[x_n, r_n]$, say, $\{y_n\}_{n=1}^{\infty}$ converging to y_0 in $[x_0, r_0]$ and there exist points p_n in $[x_n, y_n]$ with $\{p_n\}_{n=1}^{\infty}$ converging to a point p in $\langle y_0, r_0]$. We may as well suppose that p is $\max \operatorname{Ls}_n [x_n, y_n]$. Let q be $\min \operatorname{Ls}_n [p_n, r_n]$ and note that $q \leqslant y_0 < p$. By taking a subsequence and relabeling indices we can assume that there exist points q_n in $[p_n, r_n]$, $\{q_n\}$ converging to q. If $\hat{p}$ denotes $\max\limits_{[x_0, r_0]} \operatorname{Ls}_n [x_n, q_n]$ we find similarly, points $\hat{p}_n$ in $[p_n, q_n]$ with $\{\hat{p}_n\}_{n=1}^{\infty}$ converging to p. Note that $q \leqslant y_0 < p \leqslant \hat{p}$. Now $\hat{p}$ is contained in $[q, r_0]$ which is included in $\operatorname{Ls}_n [q_n, r_n]$ and, applying Lemma 1.1 we find (without loss of generality) points $\tilde{p}_n$ in $[q_n, r_n]$ with $\{\tilde{p}_n\}_{n=1}^{\infty}$ converging to $\hat{p}$ such that $\operatorname{Ls}_n [\tilde{p}_n, q_n] \leqslant \hat{p}$. But $\operatorname{Ls}_n [\tilde{p}_n, q_n] \geqslant q$ must follow from the minimality of q, hence $\operatorname{Ls}_n [\tilde{p}_n, q_n] = [\hat{p}, q]$. Also, since $q \in [x_0, \hat{p}] \leqslant \operatorname{Ls}_n [x_n, \hat{p}_n]$ it follows from Lemma 1.1 and the maximality of $\hat{p}$ that there exist points $\tilde{q}_n$ in $[x_n, \hat{p}_n]$, $\{\tilde{q}_n\}_{n=1}^{\infty}$ converging to q, such that $\operatorname{Ls}_n [\tilde{q}_n, \hat{p}_n] = [\hat{p}, q]$. The set $\operatorname{Ls}_n [q_n, \hat{p}_n]$ is equal to $[\hat{p}, q]$ because $\hat{p}$ is maximal and for almost all n, $\hat{p}_n$ is greater than or equal to p_n, while q is $\min \operatorname{Ls}_n [p_n, r_n]$. By relabeling indices on the appropriate subsequences, we obtain the fact that $\{[\tilde{p}_n, q_n, \hat{p}_n, \tilde{q}_n]\}_{n=1}^{\infty}$ converges to $[\hat{p}, q]$ in the manner required to form a zig-zag. This contradicts the hypothesis and the lemma must be true.

LEMMA 1.3. *If X is a contractible dendroid and $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are sequences of points of X which converge to a_0, b_0 respectively, then $\operatorname{Ls}_n [a_n, b_n]$ is hereditarily locally connected.* (This is an unpublished result of Charatonik.)

Proof. Suppose F is a contraction of X with $F(\{1\} \times X) = z$ say. For each n let $[z, x_n]$ be irreducible between z and $[a_n, b_n]$. Now

$$\begin{aligned}
\operatorname{Ls}[a_n, b_n] &\subseteq \operatorname{Ls}([x_n, a_n] \cup [x_n, b_n]) \subseteq \operatorname{Ls}[x_n, a_n] \cup \operatorname{Ls}[x_n, b_n] \\
&\subseteq \operatorname{Ls}([z, x_n] \cup [x_n, a_n]) \cup \operatorname{Ls}([z, x_n] \cup [b, x_n]) \\
&= \operatorname{Ls}[z, a_n] \cup \operatorname{Ls}[z, b_n] \subseteq \operatorname{Ls} F([0, 1] \times a_n) \cup \operatorname{Ls} F([0, 1] \times b_n) \\
&\subseteq \operatorname{Ls} F(([0, 1] \times a_n) \cup ([0, 1] \times b_n)) \\
&= F(\operatorname{Ls}\{([0, 1] \times a_n) \cup ([0, 1] \times b_n)\}) \\
&= F(\operatorname{Ls}([0, 1] \times a_n)) \cup F(\operatorname{Ls}([0, 1] \times b_n)) \\
&= F([0, 1] \times \{a\}) \cup F([0, 1] \times \{b\}).
\end{aligned}$$

Now $F([0, 1] \times \{a\}) \cup F(0, 1] \times \{b\})$ is locally connected and hereditarily unicoherent and is thus *hereditarily locally connected*. The set $\operatorname{Ls}[a_n, b_n]$ therefore inherits the latter property.

Chapter 2. We are now prepared to show the necessity of each of the three conditions — pairwise smooth, no zig-zag, no P-point — in order that a fan be weakly-confluent contractible.

THEOREM 2.1. *A contractible dendroid does not contain a zig-zag.*

Proof. The zig-zag is a special case of a *continuum of type N* defined by Lex G. Oversteegen. See Theorem 2.1 of his paper *Non-contractibility of Continua* in Bull. Acad. Polon. Sci. (to appear 1978).

LEMMA 2.2. *If a contractible fan contains a P-point, then that point must be the top of the fan.*

Proof. Let X be a contractible fan with endpoints $\{e_\alpha\}_{\alpha \in A}$, top c, and let x be a P-point of X distinct from c. We wish to obtain a contradiction. There is a sequence of points $\{x_n\}_{n=1}^{\infty}$ converging to x for which the points x_n lie on distinct arcs $[c, e_\alpha(n)]$, such that if $[x_n, y_n]$ is irreducible between x_n and $\operatorname{Ls}_n [x, x_n]$, then the sequence of points $\{y_n\}_{n=1}^{\infty}$ converges to x. It follows that both x and c belong to $\operatorname{Ls}_n [x, x_n]$ and we know, as a result of Lemma 1.3, that $\operatorname{Ls}_n [x, x_n]$ is hereditarily locally connected. Therefore, it is also true that $\operatorname{Cl}\{\bigcup_{n=1}^{\infty}[c, y_n]\}$ is locally connected. Since x belongs to $\operatorname{Cl}\{\bigcup_{n=1}^{\infty}[c, y_n]\}$, there is a relatively open neighborhood $\mathcal{U}$ of x lying in $\operatorname{Cl}\{\bigcup_{n=1}^{\infty}[c, y_n]\}$ such that $\mathcal{U}$ is connected and does *not* contain c. However, $\mathcal{U}$ must contain y_n, for almost all n, and hence must contain the arcs $[y_n, x]$ for almost all n. This implies that y_n lies on $\langle c, x]$ for almost all n. The arc $[x_n, c]$ would then be a proper subcontinuum of $[x_n, y_n]$, joining x_n to $\operatorname{Ls}_n [x, x_n]$ for almost all n, contrary to the choice of the points y_n. This contradiction establishes the lemma.

THEOREM 2.3. *If a fan is weakly-confluent contractible, then it does not contain a P-point.*

Proof. Let X be a fan with top c, endpoints $\{e_\alpha\}_{\alpha \in A}$, and let F be a weakly-confluent contraction of X. If X contains a P-point, then c must be a P-point (Lemma 2.2). There is then a sequence of points $\{c_n\}_{n=1}^{\infty}$ converging to c, such that c_n is contained in $[c, e_\alpha(n)]$ $(n = 1, 2, \ldots)$ for distinct endpoints $\{e_\alpha(n)\}_{n=1}^{\infty}$, and possessing the property that $\operatorname{Ls}_n [c, c_n]$ contains at least one point, say y, different from c.

With the appropriate choice of subsequence $\{c_n(j)\}_{j=1}^{\infty}$ one can find a sequence of points $\{y_j\}_{j=1}^{\infty}$ converging to y such that y_j belongs to $[c, c_n(j)]$ $(j = 1, 2, \ldots)$. For each j let t_j be the greatest value of t in $[0, 1]$ for which $F(t, c_n(j))$ belongs to $[y_j, e_\alpha(n, j)]$. We may assume that the sequence $\{t_j\}_{j=1}^{\infty}$ converges, if not one uses a subsequence which does converge. Let t_0 be the limit of the sequence $\{t_j\}_{j=1}^{\infty}$. Now

$$F(t_0, c) = F\big(\operatorname{Ls}_j (t_j, c_n(j))\big) = \operatorname{Ls}_j F\big(t_j, c_n(j)\big) = \operatorname{Ls}_j \{y_j\} = y\,.$$

Let $\mathcal{U}$ be an open neighborhood of y which is small enough that it does not contain c and let

$$K = \left(\mathrm{Cl}\{\bigcup_{j=1}^{\infty} [c, F(t_0, c_n(j))]\}\right) - \mathcal{U}.$$

Let M be the component of K which contains c and note that M is a continuum which is *not* locally connected. (If M were locally connected, then it would be impossible for c to be a P-point.) Since c does not belong to $F_{t_0}^{-1}(M)$, each component of $F_{t_0}^{-1}(M)$ is an arc or a point. However, these are locally connected and thus cannot be mapped onto M. This is contradictory to the assumption that F is weakly-confluent. Therefore X must contain no P-point.

THEOREM 2.4. *A weakly-confluent contractible fan is pairwise smooth.*

Proof. Let X be a weakly-confluent contractible fan with top c, endpoints $\{e_\alpha\}_{\alpha\in A}$, and let F be a contraction of X. Suppose that X is not pairwise smooth. There is then a point r belonging to X and sequences $\{r(1, n)\}$, $\{r(2, n)\}$ for $n = 1, 2, \ldots$, each converging to r, a point s and sequence $\{s(1, n)\}$ converging to s such that $\lim_n [r(1, n), s(1, n)] = [r, s]$ and a point q, sequence $\{q(2, n)\}$ converging to q such that $\lim_n [r(2, n), q(2, n)] = [r, p]$.

Because c is not a P-point (Theorem 2.3), we may choose the points s, r, q to lie (in that order) on an arc $[c, e_\beta]-\{c\}$ for some $\beta \in A$. Let $e_\alpha(1, n)$, $e_\alpha(2, n)$ be the endpoints of the arcs on which lie (respectively) the points $r(1, n)$, $r(2, n)$. We may assume that the points $s(1, n)$, $q(2, n)$ also belong to the arcs $[c, e_\alpha(1, n)]$, $[c, e_\alpha(2, n)]$ respectively.

It is important to note that the points $s(1, n)$, $q(2, n)$ may be chosen to lie on the arcs $[c, r(1, n)]$, $[c, r(2, n)]$ (respectively). For example, if the points $s(1, n)$ belong to $[r(1, n), e_\alpha(1, n)]$, and if $\mathrm{Ls}_n [c, r(1, n)] = [c, z]$ say, there are then points $z(1, n)$ in $[c, r(1, n)]$ converging to z such that $\lim_n [r(1, n), z(1, n)] = [r, z]$.

Actually there is a subsequence, but we relabel the indices. We have $\lim_n [r(1, n), z(1, n)] \leqslant z$ by choice of z and $\geqslant r$ by using Lemma 1.1. It is evident that $z < q$, or else $\{r(2, n)\}$ would *dominate* $\{r(1, n)\}$ since we could put a radially convex metric ϱ on the set

$$\{[r, z] \cup (\bigcup_{n=1}^{\infty} [r(1, n), z(1, n)])\}$$

(by Lemma 1.2) and $q \leqslant z$ would enable us to choose a sequence $[q(1, n)]$ converging to q such that $\lim_n [r(1, n), q(1, n)] = [r, q]$, the points $q(1, n)$ lying on $[r(1, n), z(1, n)]$ at the obvious correct distance $\varrho(r, q)$ from $r(1, n)$.

But with $z < q$, we can put a radially convex metric ϱ on the set

$$\{\bigcup_{n=1}^{\infty} [c, z(1, n)]\} \cup [c, z]$$

(by Lemma 1.2) and take points $s(3, n)$ (at distance $\varrho(s, z)$ from $z(1, n)$) lying on the arc $[c, z(1, n)]$ and take points $r(3, n)$ (at distance $\varrho(r, z)$ from $z(1, n)$) lying on $[c, z(1, n)]$ and find that $\lim_n [r(3, n), s(3, n)] = [r, s]$. Hence the note stated above is seen to be correct in this case. The other case, concerning the points $q(2, n)$ may be done using a symmetric argument (also reversing the direction of each inequality).

At least once during the contraction of X, the point r must be moved to the position s as well as to the position q, since

$$\begin{aligned} F([0, 1]\times\{r\}) &= F(\mathrm{Ls}_n [0, 1]\times\{r(1, n)\}) \\ &= \mathrm{Ls}_n F([0, 1]\times\{r(1, n)\}) \supseteq \mathrm{Ls}_n [F(0, r(1, n)), F(1, r(1, n))] \\ &\supseteq \mathrm{Ls}_n [r(1, n), c] \supseteq \mathrm{Ls}_n \{s(1, n)\} \\ &= \{s\}, \quad \text{and similarly for } q. \end{aligned}$$

Without loss of generality we suppose that r is mapped to s first before it is ever mapped to q. Let t_0 be the first time t in $[0, 1]$, such that $F(t, r) = s$. Let $F([0, t_0]\times\{r\}) =$ (say) $[s, w]$, where w must, of course, be less than q. Let t_1 be the last time t in $[0, t_0]$, such that $F(t, r)$ is w.

Now, without loss of generality, the arcs $[F(t_1, r(2, n)), F(t_0, r(2, n))]$ are contained in the arcs $[q(2, n), e_\alpha(2, n)]-\{q(2, n)\}$. (Since r has not yet moved to q, this *must* be true for almost all n.)

By the choice of t_0, t_1, and w, it is evident that

$$\lim_n [F(t_1, r(2, n)), F(t_0, r(2, n))] = [w, s].$$

By Lemma 1.2 we can put a radially convex metric ϱ on the union of these arcs. It is then true that the arcs $[r(2, n), F(t_0, r(2, n))]$ converge to $[r, s]$ and hence, $\{r(1, n)\}$ dominates $\{r(2, n)\}$ contrary to our initial supposition.

The theorem is thus proved.

Chapter 3. We shall now show that the three necessary conditions given in the previous section are in fact *sufficient* in order that a fan be not only weakly-confluent contractible, but also that it be monotone contractible.

Throughout this chapter we shall understand that X denotes a fan with top c and endpoints $\{e_\alpha\}_{\alpha\in A}$ which is pairwise smooth, contains no zig-zag, and contains no P-point.

Whenever we refer to the lim sup (Ls) of a sequence of arcs, it is to be considered that this set belongs to one of the arcs $[c, e_\alpha]$, or else c would be a P-point.

DEFINITION. Let n be an integer greater than zero. We say that an arc $[a, b]$ which, for some $\alpha \in A$, lies on $\langle c, e_\alpha]$ is a *partial n-hook* provided there exists a sequence $\{[c, e_\alpha(m)]\}_{m=1,2,\ldots}$ of arcs, each of which contains points

$$c = p(m, 0) < p(m, 1) < \ldots < p(m, n)$$

such that for each $j = 0, 1, 2, \ldots, n$, the sequence $\{p(m,j)\}_{m=1,2,\ldots}$ converges to the point p_j say; and such that for $j = 1, 2, \ldots, n$, the sequence

$$\{[p(m,j-1), p(m,j)]\}_{m=1,2,\ldots}$$

of arcs converges to the arc $[p_{j-1}, p_j]$, with the additional features:

(a) $[p_j, p_{j+1}]$ is properly contained in $[p_{j-1}, p_j]$ for $j = 1, 2, \ldots, n-1$;

(b) $p_{n-1} = b$, $p_n = a$; and finally,

(c) $\operatorname{Ls}_m [p(m,n), e_a(m)]$ is properly contained in $[p_{n-1}, p_n]$.

We call the point p_{n-1} the *top of the partial n-hook* and the point p_n the *bottom of the partial n-hook*.

Note. It follows from the definition of a partial n-hook that for a given n: either, for each partial n-hook $[a, b]$, $a<b$ (if n even) or, for each partial n-hook $[a, b]$, $b<a$ (if n odd). This is because in order to satisfy the portion of the definition concerning proper containment we must have $p_0<p_1$, $p_1>p_2$, $p_2<p_3$, etc.; that is, $p_{j-1}<p_j$ if j is odd, while $p_{j-1}>p_j$ if j is even.

(Recall that our partial order $\leqslant$ is defined as $p\leqslant q$ provided p weakly cuts q from c.)

LEMMA 3.1. *If a pair of partial n-hooks intersect, then their tops must coincide.*

Proof. The proof is handled by induction on n. For details see [6].

LEMMA 3.2. *Let ε be a positive real number and suppose that X contains no partial k-hook for $k = 2, 3, \ldots$ of diameter less than ε. If for a fixed k one chooses a sequence $\{[p_{k-1}(i), p_k(i)]\}_{i=1,2\ldots}$ of partial k-hooks such that $\{p_{k-1}(i)\} \to p_{k-1}(0)$, $\{p_k(i)\} \to p_k(0)$, then*

$$\lim_i [p_{k-1}(i), p_k(i)] = [p_{k-1}(0), p_k(0)]$$

and the latter set is also a partial k-hook.

Proof. One need only show the lemma is true when $k = 2$, since by definition, each sequence of partial n-hooks (for $n>2$) is embedded in a sequence of partial 2 hooks and with the lemma true for $k = 2$, we can put a radially convex metric ϱ on the closure of the sequence of partial 2-hooks, which is then inherited by the sequence of partial n-hooks. Using the radially convex metric ϱ, it is easy to show that the lemma then holds for $k = n$. We proceed by showing that if the lemma fails for $k = 2$ then one can find a zig-zag lying inside the fan, contrary to our general hypothesis. See [6] for further details.

LEMMA 3.3. *Let ε be a positive real number and suppose that X contains no partial k-hook for $k = 2, 3, \ldots$ of diameter less than ε. Then there exists a positive real number δ, called the nesting diameter of X such that for each partial k-hook, the diameter of $[p_{k-2}, p_{k-1}]$ is at least δ greater than the diameter of $[p_{k-1}, p_k]$, for $k = 2, 3, \ldots$* (using the same notation as in the definition).

Proof. If no such δ exists, then it is possible to consider two cases:

Case I. For some integer k greater than 1 there exists a sequence

$$\{[p_{k-1}(i), p_k(i)]\}_{i=1,2\ldots}$$

of partial k-hooks such that the difference in diameter between $[p_{k-2}(i), p_{k-1}(i)]$ and $[p_{k-1}(i), p_k(i)]$ is less than $(1/i)$. Now by taking subsequences and relabeling, we may assume (in view of Lemma 3.2) that

$$\{[p_{k-1}(i), p_k(i)]\} \to [p_{k-1}(0), p_k(0)],$$

where

$$p_{k-1}(0) = \lim_i p_{k-1}, \quad p_k(0) = \lim_i p_k(i)$$

are distinct points (using the ε hypothesis). Now if $k = 2$, it follows that the point c is a P-point. For $k>2$, we may "diagonalize" the three double-sequences

$$\{\{\{[p(i,m,j), p(i,m,j+1)]\}_{m=1}^{\infty}\}_{i=1}^{\infty}\}_{j=k-3,k-2,k-1}$$

(where $\{[p(i,m,j), p(i,m,j+1)]\} \underset{m}{\to} [p_j(i), p_{j+1}(i)]$ for $j = k-3$, $k-2$, $k-1$, $i = 1, 2, \ldots$). We may now suppose after some relabeling that

$$\{[p(i,m_i,k-3), p(i,m_i,k-2)]\} \to [p_{k-3}(0), p_{k-2}(0)],$$
$$\{[p(i,m_i,k-2), p(i,m_i,k-1)]\} \to [p_{k-2}(0), p_{k-1}(0)],$$

and

$$\{[p(i,m_i,k-1), p(i,m_i,k)]\} \to [p_{k-1}(0), p_k(0)],$$

where $p_{k-3}(0) = \lim_i p_{k-3}(i)$ and $p_{k-2}(0) = \lim_i p_{k-2}(i)$. However, because of our assumption on the diameters of $[p_{k-2}(i), p_{k-1}(i)]$, $[p_{k-1}(i), p_k(i)]$, having difference less than $(1/i)$, it is evident that $p_{k-2}(0)$ is identical to $p_k(0)$. Also, by using Lemma 1.2 to put a radially convex metric on $\operatorname{Cl}\{\bigcup_{i=1}^{\infty} [p(i,m_i,k-3), p(i,m_i,k-2)]\}$, we may find points z_i belonging to $[p(i,m_i,k-3), p(i,m_i,k-2])$ with $z_i \to p_{k-1}(0)$. The arcs

$$\{[z_i, p(i,m_i,k-2), p(i,m_i,k-1), p(i,m_i,k)]\}_{i=1,2\ldots}$$

then form a zig-zag, so this case cannot occur.

Case II. There exists a sequence $\{[p_{i-1}(i), p_i(i)]\}_{i=1,2\ldots}$ of partial i-hooks such that the difference in diameter between $[p_{i-2}(i), p_{i-1}(i)]$ and $[p_{i-1}(i), p_i(i)]$ is less than $[1/i)$. Using processes similar to those of Case I, we obtain sequences:

$$\{\{[p(i,m_i,j-1), p(i,m_i,j)]\}_{j=1}^{\infty}\}_{i=j}^{\infty}$$

such that for each j, the sequence

$$\{[p(i,m_i,j-1), p(i,m_i,j)]\}_{i=j}^{\infty}$$

converges to $[p_j(0), p_{j+1}(0)]$ where $p_j(0) = \lim_i p_{j-1}(i)$ (possible by Lemma 3.2).

(Note that for $i \geq j$, each partial i-hook is contained in a partial j-hook from which we obtain the points $p_{j-1}(i)$.).

We have:

$$p_0(0) < p_2(0) < p_4(0) < \ldots < p_{2n}(0) < \ldots < p_{2n-1}(0) < \ldots < p_3(0) < p_1(0) .$$

It follows from the ε hypothesis that $\{p_{2n}(0)\} \nearrow r$ while $\{p_{2n-1}(0)\} \searrow s$ such that s, r are distinct points (at least ε apart). But, without loss of generality, it follows that the "diagonal" sequences

$$\{p(i, m_i, i-3)\}, \quad \{p(i, m_i, i-2)\}, \quad \{p(i, m_i, i-1)\}, \quad \{p(i, m_i, i)\}, \quad i = 3, 4, \ldots,$$

each converge to s, r, s, r respectively, and the arcs formed by these four sequences yield a zig-zag.

Hence, the lemma is true.

COROLLARY 3.4. *Let ε be a positive real number. If the fan X contains no partial k-hook for $k = 1, 2, \ldots$ of diameter less than ε, then there exists a positive integer n such that for each $k > n$, X contains no partial k-hook.*

Proof. Let the nesting diameter of X be δ. Since X is compact, we may suppose that diameter $X = 1$. Choose $[m-1]$ to be greater than say, $1/\delta$. If $[p_{m-1}, p_m]$ is a partial m-hook lying in X, then $[p_{m-1}, p_m]$ is properly contained in a partial $(m-1)$-hook which is properly contained in a partial $(m-2)$-hook ... which is properly contained in a partial 1-hook. By virtue of the property of the nesting diameter, it follows that the diameter of $[p_{m-1}, p_m]$ must be *less than zero*. This being impossible, we have an upper bound n as desired.

DEFINITION. Let X be a fan which satisfies the conditions of Corollary 3.4 (as well as satisfying the hypothesis of this chapter; namely, pairwise smooth, no zig-zag, and no P-point). We say that X is an *(ε, n)-fan.*

LEMMA 3.5. *Let X be an (ε, n)-fan. Let k be a positive integer less than or equal to n and let p_{k-1} be the top of a partial k-hook. We claim that the union of those partial k-hooks which contain the point p_{k-1} forms a closed set and is also a partial k-hook.*

Proof. Each summand of the union has the point p_{k-1} as its top, by virtue of Lemma 3.1. If k is even (respectively, k, odd), then the bottom of each summand belongs to $\langle c, p_{k-1}\rangle$ (respectively, $\langle p_{k-1}, e_\alpha\rangle$ for some $\alpha \in A$) and there is then a point q which is the infimum (respectively, supremum) of these bottom points. If we approach this limit point with a countable sequence $\{p(i, k)\}_{i=1}^{\infty}$ of the bottom points, then we have:

(a) $\{p(i, k)\}_{i=1}^{\infty} \to q$,

(b) $\{p(i, k-1)\}_{i=1}^{\infty} = \{p_{k-1}\} \to p_{k-1}$ implies

(c) $\{[p(i, k), p(i, k-1)]\}_{i=1}^{\infty} \to [p_{k-1}, q]$.

It follows from Lemma 3.2 that the set $[p_{k-1}, q]$ is also a partial k-hook. This set is thus contained in the union under consideration, but also contains the union

by choice of q. The union is therefore equal to the partial k-hook $[p_{k-1}, q]$ and is closed.

DEFINITION. A set of the form $[p_{k-1}, q]$ is called a *k-hook*. We drop the adjective "partial" since such a k-hook is complete in the sense that it does not properly lie in another partial k-hook. It should be noted, however, that each *k-hook* also satisfies the definition of a *partial k-hook* (Lemma 3.5), and every lemma or corollary we prove concerning a partial k-hook is also true for a k-hook. Now, for any pair of k-hooks, either the two are identical or else they do *not* intersect, in view of Lemma 3.1. We now refine this statement.

LEMMA 3.6. *Let X be an (ε, n)-fan. There exists a positive real number δ such that for each $k \leq n$, and for each $\alpha \in A$, for each pair of k-hooks lying on the arc $[c, e_\alpha]$, it is true that their δ-neighborhoods are mutually disjoint.*

Proof. If the lemma fails, then we may choose a k-hook $[p_{k-1}, p_k]$ on an arc $[c, e_\alpha]$ with a sequence $\{[p(i, k-1), p(i, k)]\}_{i=1}^{\infty}$ of k-hooks on $[c, e_\alpha]$, each one of which is mutually disjoint from $[p_{k-1}, p_k]$ with the property that

$$(*) \qquad d(p_{k-1}, p(i, k)) < (1/i)$$
$$\text{(respectively, } d(p_k, p(i, k-1)) < (1/i))$$

depending upon whether the k-hooks converge to $[p_{k-1}, p_k]$ from "above" or from "below." We may assume that $\{p(i, k)\}_{i=1}^{\infty}$ converges to $p(0, k)$ say, and that $\{p(i, k-1)\}_{i=1}^{\infty}$ converges to $p(0, k-1)$. From Lemma 3.2 we know that $\{[p(i, k-1), p(i, k)]\}_{i=1}^{\infty}$ converges to $[p(0, k-1), k(0, k)]$ and that the latter set is at least a partial k-hook (of diameter at least ε). Now the point $p(0, k-1)$ (respectively, $p(0, k)$) must lie outside the arc $[p_{k-1}, p_k]$, but it follows from $(*)$ that

$$p(0, k) = p_{k-1}$$
$$\text{(respectively, } p(0, k-1) = p_k) ,$$

which is contrary to Lemma 3.1. It therefore must be possible to find the desired number δ.

LEMMA 3.7. *Let X be an (ε, n)-fan. There exists a positive real number τ such that for each partial n-hook lying in X, it is true that for each $k < n$, and for each partial k-hook, neither the top nor the bottom of that k-hook may lie inside the τ-neighborhood of the n-hook.*

Proof. Let $[p_{n-1}, p_n]$ be a fixed partial n-hook, with top p_{n-1} and bottom p_n. If no τ works for this particular case, then there exists a sequence $\{[q(i, k-1), q(i, k)]\}_{i=1,2,\ldots}$ of partial k-hooks (for some $k < n$), which converges to say, $[q(0, k-1), q(0, k)]$, a partial k-hook itself (Lemma 3.2) and, with either $q(0, k-1$ or $q(0, k)$ belonging to $[p_{n-1}, p_n]$. But $[p_{n-1}, p_n]$ is contained in a partial k-hook whose top lies outside of $[p_{n-1}, p_n]$, by definition. Since $[q(0, k-1), q(0, k)]$ intersects this partial k-hook, Lemma 3.1 implies that the top $q(0, k-1)$ also must lie outside of $[p_{n-1}, p_n]$. We are led then to the case where the bottom $q(0, k)$ belongs to $[p_{n-1}, p_n]$. The partial n-hook $[p_{n-1}, p_n]$ is contained in a partial $(k+1)$-hook $[p_k, p_{k+1}]$ by definition, with, say, the sequence $\{[p(m, k), p(m, k+1)]\}_{m=1,2,\ldots}$ converging to $[p_k, p_{k+1}]$ in the usual way. As in the proof of Lemma 3.1, we may assume that there is a sequence $\{q_k(m)\}_{m=1,2,\ldots}$ of points belonging to

$$[p(m, k), p(m, k+1)]$$

for each m, which converges to $q(0, k)$. There is also a sequence $\{q(0, m, k)\}_{m=1,2,\ldots}$ of points, given by the definition of partial k-hooks, which converges to $q(0, k)$. It can be shown that neither of the pair $\{q_k(m)\}_{m=1}^{\infty}$, $\{q(0, m, k)\}_{m=1}^{\infty}$ *dominates* the other (using methods similar to those in the proof of Lemma 3.1), which is contrary to the pairwise smoothness of the fan X. We can therefore find a positive real number τ such that for each $k<n$, the top or bottom of no partial k-hook lies within τ of $[p_{n-1}, p_n]$. Moreover, the minimum value of τ we will need to choose as we let the partial n-hook vary, will be greater than zero. If it is equal to zero, we find a sequence $\{H_n\}_{n=1}^{\infty}$ of partial n-hooks requiring values of τ say, τ_n where $\{\tau_n\}_{n=1}^{\infty}$ converges to zero. But then $\lim_n \{H_n\}$, which is itself a partial n-hook (Lemma 3.2), will require a choice of $\tau = 0$ (using a diagonalization process as done previously), which is contrary to the proof, just completed, that each given partial n-hook admits a positive value of τ. The lemma is therefore proved as stated.

LEMMA 3.8. *Let X be an (ε, n)-fan with $n>1$. There exists a map F from* $[0, 1]\times X$ *into X such that for each point p of X we have $F(0, p) = p$ and such that $F(\{1\}\times X)$ is an $(\varepsilon, n-1)$-fan and, moreover, for each time t in* $[0, 1]$, *the map F restricted to $\{t\}\times X$ is monotone.*

Proof. Can be found in [6].

LEMMA 3.9. *Let X be a fan which is pairwise smooth, contains no zig-zag, and no P-point. There exists a map F from $[0, 1]\times X$ into X such that for each x in X, $F(0, x) = x$ and $F(\{1\}\times X)$ contains no partial k-hooks except $k = 1$. Moreover, F may be chosen so that F restricted to $\{t\}\times X$ is monotone for each t belonging to* $[0, 1]$.

Proof. In view of Corollary 3.4, we may assume that for $k = 1, 2, \ldots$, the fan X contains no partial k-hook of diameter less than $1/k$. For some $n>1$ we assume that we have a map F so that the image $F(1/n, X)$ contains no partial k-hook for each $k\geqslant n$. That is to say $F(1/n, X)$ is a $(1/n, n)$-fan. We may then apply Lemma 3.8 to obtain a $(1/(n-1), n-1)$-hook during the time between $t = 1/n$ and $t = 1/(n-1)$. By composing these maps in the appropriate fashion and setting $F(0, X)$ to be the identity mapping, we obtain the desired result.

THEOREM 3.10. *Let X be a fan which is pairwise smooth, contains no zig-zag and no P-point. Then X is monotone contractible.*

Proof. In view of Lemma 3.9, we may assume that for $n>1$ it is true that X contains no n-hook. Let $\leqslant$ denote the weak cut point order with respect to the top c of X. We claim that X admits a metric ϱ which is radially convex with respect to $\leqslant$. It follows from [3] and [10] that this will be the case provided we show:

If a sequence $\{p_n\}_{n=1}^{\infty}$ of points of X converges to a point p_0 in X, then it follows that $\mathrm{LS}_n\{L(p_n)\}$ is contained in $L(p_0)$. Let us say that p_n belongs to $[c, e_\alpha(n)]$ for $n = 0, 1, 2, \ldots$ Let $z_0 = \max_n \mathrm{Ls}\{c, e_\alpha(n)]$. We may as well assume that there are points z_n belonging to $[c, e_\alpha(n)]$ such that $\{z_n\}_{n=1}^{\infty}$ converges to z_0. If $\mathrm{Ls}_n [z_n, e_\alpha(n)]$ contains a point different from z_0, then we let y_0 be the least such point. It follows that $[y_0, z_0]$ is a 2-hook, contrary to our above assumption. If, on the other hand, $\mathrm{Ls}_n [z_n, e_\alpha(n)] = z_0$, then for almost all n, p_n belongs to the arc $[c, z_n]$. Since $\{[c, z_n]\}_{n=1}^{\infty}$ converges to $[c, z_0]$, Lemma 1.2 shows us that $\mathrm{Ls}_n [c, p_n]$ is contained in $[c, p_0]$, which was to be shown. It is now easy to see that X is monotone contractible.

Appendix. This section is devoted to a discussion of various examples. The first one we wish to consider is a dendroid which has the property that, although it is contractible — no matter which choice of a contraction is made, there must be a time at which the image is a noncontractible sub-dendroid.

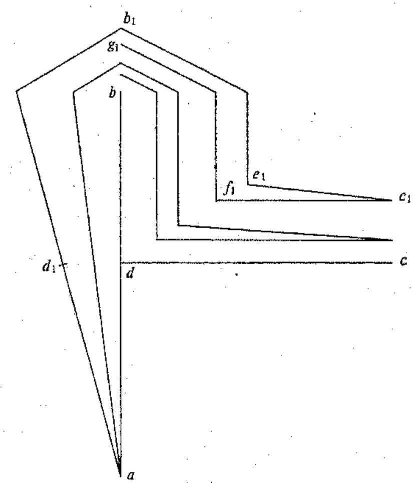

Fig. 1

The dendroid A consists of the triod abc with center d, together with a sequence of arcs $\{[a, d_n, b_n, e_n, c_n, f_n, g_n]\}$ where $n = 1, 2, \ldots$, converging to the triod in the manner indicated.

It is possible to contract A to the point $\{a\}$ by the following informal recipe.

Step 1. While keeping the points a, b, c fixed in place you must push the point d along the arc $[db]$ all the way up to b, contracting the points in front of d ($\langle d, b]$) into b, and stretching out those behind d ($[a, d\rangle$, $[c, d\rangle$). At the same time and in the same manner you must move the points $\{d_n\}$ for $n = 1, 2, \ldots$, $\{e_n\}_{n=1,2,\ldots}$, and $\{f_n\}_{n=1,2,\ldots}$ up to the points $\{b_n\}$, $\{b_n\}$, and $\{g_n\}$ respectively.

Step 2. Keep the arc $[d_n, e_n]$ together as a single point (for each n) and move this point down from b_n to e_n. At the same time you keep the arc $[g_n, f_n]$ together as a single point (for each n) and move it from g_n down to f_n. This forces the arc $[d, b]$ to remain together as a single point and to move from b down to d. The arc $[a, d_n]$ is now stretched out to cover $[a, d_n, b_n, e_n]$ while the arcs $[c, d]$, $[c_n, e_n]$, $[c_n, f_n]$ slip back to their original positions. The arc $[a, d]$ covers $[a, b, d]$ by going up to b and folding back over itself to d.

At the end of this step the arcs $\langle f_n, g_n]$ have vanished.

Step 3. Push d along the arc $[d, c]$, all the way out to c. At the same time you must move the points e_n (now the image of $[d_n, e_n]$) out to c_n, and move the points f_n (now the image of $[g_n, f_n]$) out to c_n. This action collapses the arcs $[d_n, b_n, e_n, c_n, f_n, g_n]$ down to the single point c_n (for each n) and leaves the arcs $[a, d_n]$ stretched all the way along $[a, d_n, b_n, e_n, c_n]$. The arc $[ad]$ now is stretched up to b, folds back to d, and then out to c. The arcs $\langle c_n, f_n, g_n]$ have now vanished.

Step 4. Let the arcs $[a, d_n]$ snap back from $[a, d_n, b_n, e_n, c_n]$ to cover just $[a, d_n, b_n, e_n]$. This makes the arcs $\langle dc]$, $\langle e_n, c_n, f_n, g_n]$ vanish.

Step 5. Let the arcs $[a, d_n]$ continue to reverse their stretching process so that they are back to covering just $[a, d_n]$. The arcs $\langle d, b]$, $\langle d_n, b_n, e_n, c_n, f_n, g_n]$ have now vanished.

Step 6. Finally, contract $[a, d_n]$ down to a (for each b), thus collapsing $[da]$ to the single point $\{a\}$. The dendroid A is therefore a contractible space. There follows a *sketch* of the proof that for each possible contraction F of A, there exists a time t in $[0, 1]$ such that $F(t \times A)$ is *not* contractible.

Proof of the above remark. Let the set S consist of those times t in $[0, 1]$ such that the arc $[b, d]$ intersects $\mathrm{Ls}\{F(t \times A) \cap [c_n, g_n]\}$. Let the set T include the times t in $[0, 1]$ for which b belongs to $\mathrm{Ls}\{F(t \times A) \cap [c_n, g_n]\}$. Now T is evidently a proper subset of S. For t belonging to $S - T$, the set $F(t \times A)$ would look like:

The dendroid (see Fig. 2) is *not* contractible since any contraction of it would involve moving d along $[dc]$ to c. Since the sequence $\{d_n\}$ converges to d, almost all of the points d_n must slide up through j_n, b_n before d is moved to c or else down through $\{a\}$ before d is moved to c. But this it not possible because the point d would then move to b or a without the points $\{f_n\}$ being able to follow (since the arcs $[h_n, g_n]$ are no longer in the space). Therefore, the remark above is correct.

The fan below was inspired by the example of F. Burton Jones in [7].

The fan (see Fig. 3) consists of a countable sequence of arcs $\{[c, b_n]\}$ converging to $[c, a]$, together with a countable sequence of 2-hooked arcs $[c, d_n, a_n]$.

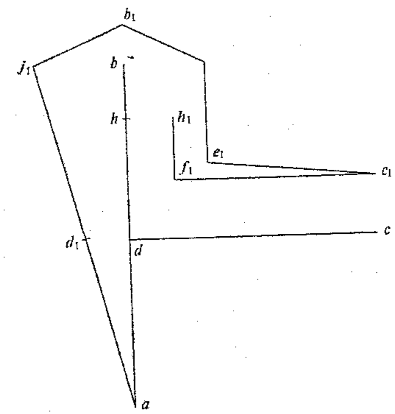

Fig. 2

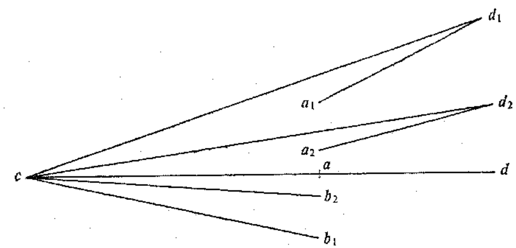

Fig. 3

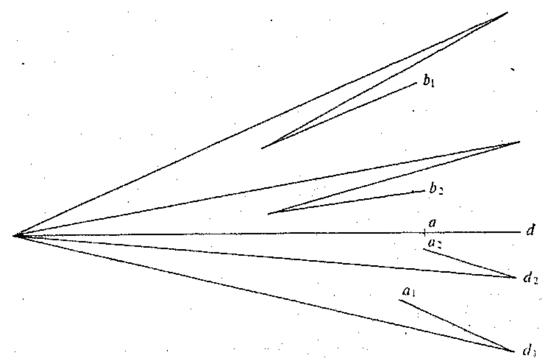

Fig. 4

Out of the pair $\{a_n\}$, $\{b_n\}$ each of which converge to $\{a\}$, it is *not* possible to find one which dominates the other.

In fact, the arcs $[a_n, d_n]$ converge to $[a, d]$ with no similar capability with respect to the sequence $\{b_n\}$ and the arcs $[c, b_n]$ converge to $[c, b]$ with no similar capability on the part of the sequence $\{a_n\}$. Hence this fan is *not* pairwise smooth.

This fan (see Fig. 4) consists of a countable sequence of 2-hooked arcs $[c, a_n]$ and a countable sequence of 3-hooked arcs $[c, b_n]$. Out of the pair $\{a_n\}$, $\{b_n\}$, it is *not* possible to find one which dominates the other. The arcs $[a_n, d_n]$ converge to $[a, d]$ with nothing similar for $\{b_n\}$ and the arcs $[b_n, e_n]$ converge to $[a, e]$ with nothing similar for $\{a_n\}$.

Our next example is a fan which contains a zig-zag. The sequence

$$\{[a_n, b_n, c_n, d_n]\}_{n=1}^{\infty}$$

of arcs converges to the arc $[a, b]$ in the manner required by the definition of a zig-zag.

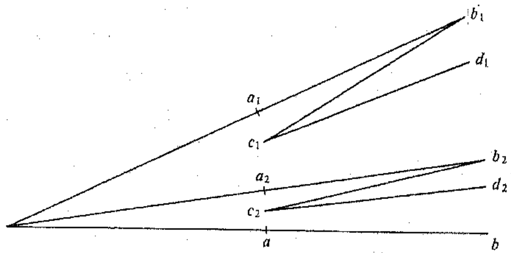

Fig. 5

Another possible way in which a fan may contain a zig-zag is illustrated below.

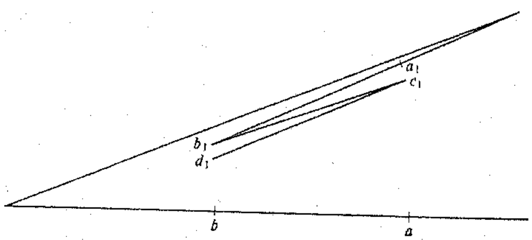

Fig. 6

Once again the sequence $\{[a_n, b_n, c_n, d_n]\}_{n=1}^{\infty}$ of arcs converges to the arc $[a, b]$ in the appropriate manner.

In the fan below, the point b is an example of a P-point. The set $\operatorname{Ls}_n[b, b_n]$ is just the arc $[b, d]$. For $n = 1, 2, \ldots$, the arc which is irreducible between the point b_n and the set $\operatorname{Ls}_n[b, b_n]$ is simply the arc $[b_n, b]$.

This concludes our collection of examples.

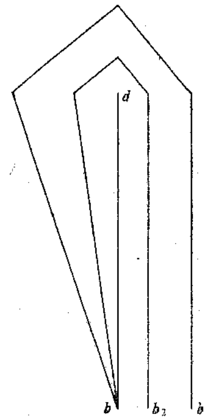

Fig. 7

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