Tôhoku Math. Journ.
Vol. 19, No. 2, 1967

ON CONTRACTION OF WALSH FOURIER SERIES

YASUO OKUYAMA

(Received December 10, 1966)

The purpose of this note is to prove the Walsh analogue of results due to M. Kinukawa [3] and an extension of the theorem on contraction of C. Watari [4].

We begin with some notations and definitions:

Following A. Beurling, $g(x)$ is called a contraction of $f(x)$ if

$$|g(x)-g(x')| \leqq |f(x)-f(x')| \quad \text{for} \quad x, x' \in (0,1).$$

A sequence $\{a_n\}$ is called a contraction of sequence $\{c_n\}$ if

$$|a_m-a_n| \leqq |c_m-c_n| \quad \text{for every } m \text{ and } n.$$

The Rademacher functions are defined by

$$\phi_0(x) = 1 \quad (0 \leqq x < 1/2), \qquad \phi_0(x) = -1 \quad (1/2 \leqq x < 1)$$
$$\phi_0(x) = \phi_0(x+1), \qquad \phi_n(x) = \phi_0(2^n \cdot x) \quad (n = 1, 2, \cdots).$$

The Walsh functions are then given by

$$\psi_0(x) \equiv 1, \quad \psi_n(x) = \psi_{n(1)}(x)\psi_{n(2)}(x) \cdots \psi_{n(r)}(x),$$

for $n=2^{n(1)} + 2^{n(2)} + \cdots + 2^{n(r)} \geqq 1$, where the integers $n(i)$ are uniquely determined by $n(i+1) < n(i)$. For basic properties of Walsh functions, the reader is referred to N. J. Fine [2]. Finally, A denotes a positive absolute constant not always the same. The author wishes to express his hearty thanks to Prof. C. Watari for his valuable suggestions and encouragements in the preparation of this paper. The author also thanks Prof. S. Igari for better presentation.

Our results are as follows:

THEOREM 1. *Let*

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x)$$

and

$$g(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x) \in L(0,1).$$

Suppose that

$$\int_0^1 |g(x+h) - g(x)|^2 dx \leqq \int_0^1 |f(x+h) - f(x)|^2 dx$$

for any h, and suppose that there exists a positive sequence $\{\gamma_n\}$ *such that*

(i) $$|c_n| \leqq \gamma_n \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n^p < \infty$$

(ii) $$\sum_{n=1}^{\infty} n^{-3p/2} \left(\sum_{\nu=1}^{n} \nu^2 \gamma_\nu^2 \right)^{p/2} + \sum_{n=1}^{\infty} n^{-p/2} \left(\sum_{\nu=n+1}^{\infty} \gamma_\nu^2 \right)^{p/2} < \infty,$$

then

$$\sum_{n=0}^{\infty} |a_n|^p < \infty$$

where $2/3 < p \leqq 2$.

The following theorem is the dual for Theorem 1.

THEOREM 2. *Let*

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x).$$

For a given $\{a_n\}$, *if* $a_n \to 0$ *and*

$$\sum_{n=0}^{\infty} |a_{m(n,j)} - a_n|^2 \leqq \sum_{n=0}^{\infty} |c_{m(n,j)} - c_n|^2 \quad \textit{for every integer } j$$

where $m(n,j) = n \dotplus 2^j$, *and if there exists a function* $\gamma(x)$ *such that*

(i) $$|f(x)| \leqq \gamma(x) \quad \text{and} \quad \gamma(x) \in L^p(0,1)$$

$$\text{(ii)} \qquad \int_0^1 x^{-p/2}\left(\int_x^1 \gamma^2(t)\,dt\right)^{p/2} dx + \int_0^1 x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)\,dt\right)^{p/2} dx < \infty$$

where $1 \leqq p \leqq 2$, *then there exists a function* $g(x)$ *belonging to* $L^p(0,1)$ *such that*

$$g(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x)\,.$$

The case $p=1$ of Theorem 1 was proved in [4], and the case $p=2$ is trivial. Hence in the proof of Theorem 1, we suppose that $2/3 < p < 2$. Moreover we suppose that $k = k(\nu)$ is the integer satisfying $2^k \leqq \nu < 2^{k+1}$ in Lemmas 1—3.

LEMMA 1. (ii) *is equivalent to* (ii′);

$$\text{(ii}'\text{)} \qquad \sum_{n=1}^{\infty} 2^{(1-3p/2)n}\left(\sum_{\nu=1}^{2^n-1} 2^{2k}\gamma_\nu^2\right)^{p/2} + \sum_{n=1}^{\infty} 2^{(1-p/2)n}\left(\sum_{\nu=2^n}^{\infty}\gamma_\nu^2\right)^{p/2} < \infty\,.$$

PROOF. The equivalence between

$$\sum_{n=1}^{\infty} n^{-p/2}\left(\sum_{\nu=n+1}^{\infty}\gamma_\nu^2\right)^{p/2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} 2^{n(1-p/2)}\left(\sum_{\nu=2^n}^{\infty}\gamma_\nu^2\right)^{p/2} < \infty$$

is nothing but Cauchy's condensation theorem. On the other hand

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-3p/2}\left(\sum_{\nu=1}^{n}\nu^2\gamma_\nu^2\right)^{p/2} \\
&\quad = \sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1} n^{-3p/2}\left(\sum_{\nu=1}^{n}\nu^2\gamma_\nu^2\right)^{p/2} \\
&\quad \leqq \sum_{j=0}^{\infty}(2^j)^{-3p/2}\sum_{n=2^j}^{2^{j+1}-1}\left(\sum_{\nu=1}^{2^{j+1}-1}\nu^2\gamma_\nu^2\right)^{p/2} \\
&\quad = \sum_{j=0}^{\infty}(2^j)^{-3p/2}\cdot 2^j\cdot\left(\sum_{\nu=1}^{2^{j+1}-1}\nu^2\gamma_\nu^2\right)^{p/2} \\
&\quad \leqq A\sum_{j=0}^{\infty} 2^{j(1-3p/2)}\left(\sum_{\nu=1}^{2^j-1}\nu^2\gamma_\nu^2\right)^{p/2} \\
&\quad \leqq A\sum_{j=1}^{\infty} 2^{j(1-3p/2)}\left(\sum_{\nu=1}^{2^j-1} 2^{2k}\gamma_\nu^2\right)^{p/2}
\end{aligned}$$

and

$$\sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1} n^{-3p/2}\left(\sum_{\nu=1}^{n}\nu^2\gamma_\nu^2\right)^{p/2}$$

$$\geqq A\sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1}(2^j)^{-3p/2}\left(\sum_{\nu=1}^{n}2^{2k}\gamma_\nu^2\right)^{p/2}$$

$$\geqq A\sum_{j=0}^{\infty}(2^j)^{-3p/2}\sum_{n=2^j}^{2^{j+1}-1}\left(\sum_{\nu=1}^{2^j-1}2^{2k}\gamma_\nu^2\right)^{p/2}$$

$$= A\sum_{j=0}^{\infty}2^{(1-3p/2)j}\left(\sum_{\nu=1}^{2^j-1}2^{2k}\gamma_\nu^2\right)^{p/2}, \qquad \text{Q. E. D.}$$

LEMMA 2. *For any sequence* $\{a_n\}$, *if the series on the right-hand side converges, then we have*

$$\sum_{\nu=1}^{\infty}|a_\nu|^p \leqq \sum_{n=1}^{\infty}2^{(1-3p/2)n}\left(\sum_{\nu=1}^{2^n-1}2^{2k}|a_\nu|^2\right)^{p/2}$$

where $2/3 < p < 2$.

PROOF. By dissecting the range of summation and using Hölder's inequality, we have

$$\sum_{\nu=1}^{\infty}|a_\nu|^p = \sum_{j=0}^{\infty}\sum_{\nu=2^j}^{2^{j+1}-1}|a_\nu|^p$$

$$\leqq A\sum_{j=0}^{\infty}2^{jp}\sum_{n=j+1}^{\infty}2^{-np}\sum_{\nu=2^j}^{2^{j+1}-1}|a_\nu|^p$$

$$= A\sum_{n=1}^{\infty}2^{-np}\sum_{j=0}^{n-1}2^{jp}\sum_{\nu=2^j}^{2^{j+1}-1}|a_\nu|^p$$

$$= A\sum_{n=1}^{\infty}2^{-np}\sum_{j=0}^{n-1}\sum_{\nu=2^j}^{2^{j+1}-1}2^{kp}|a_\nu|^p$$

$$= A\sum_{n=1}^{\infty}2^{-np}\sum_{\nu=1}^{2^n-1}2^{kp}|a_\nu|^p$$

$$\leqq A\sum_{n=1}^{\infty}2^{-np}\left(\sum_{\nu=1}^{2^n-1}(2^{kp}|a_\nu|^p)^{2/p}\right)^{p/2}\left(\sum_{\nu=1}^{2^n-1}1^{(2/p)}\right)^{1/(2/p)'} \quad \left(\left(\frac{2}{p}\right)' = \frac{2}{2-p}\right)$$

$$= A \sum_{n=1}^{\infty} 2^{-np} \left(\sum_{\nu=1}^{2^n-1} 2^{2k} |a_\nu|^2 \right)^{p/2} \cdot (2^n)^{1-p/2}$$

$$= A \sum_{n=1}^{\infty} 2^{(1-3p/2)n} \left(\sum_{\nu=1}^{2^n-1} 2^{2k} |a_\nu|^2 \right)^{p/2}, \qquad \text{Q. E. D.}$$

LEMMA 3. *Under the assumption of Theorem* 1,

$$\sum_{\nu=1}^{2^n-1} 2^{2k} |a_\nu|^2 \leqq A \sum_{\nu=1}^{2^n-1} 2^{2k} |c_\nu|^2 + A\, 2^{2n} \sum_{\nu=2^n}^{\infty} |c_\nu|^2 .$$

This lemma is due to C. Watari [4], where there is an obvious misprint in the statement (see his proof in [4]), which we state in corrected form.

PROOF OF THEOREM 1. By lemmas 2 and 3, we have

$$\sum_{\nu=1}^{\infty} |a_\nu|^p \leqq A \sum_{n=1}^{\infty} 2^{(1-3p/2)n} \left(A \sum_{\nu=1}^{2^n-1} 2^{2k} |c_\nu|^2 + A\, 2^{2n} \sum_{\nu=2^n}^{\infty} |c_\nu|^2 \right)^{p/2}$$

$$\leqq A \sum_{n=1}^{\infty} 2^{(1-3p/2)n} \left(\sum_{\nu=1}^{2^n-1} 2^{2k} \gamma_\nu^2 \right)^{p/2} + A \sum_{n=1}^{\infty} 2^{(1-p/2)n} \left(\sum_{\nu=2^n}^{\infty} \gamma_\nu^2 \right)^{p/2}$$

which is convergent by lemma 1 and the assumption of Theorem 1.

LEMMA 4. *Let* $f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x)$ *with* $\sum_{n=0}^{\infty} |c_n|^2 < \infty$. *If the sequence* $\{a_n\}$ *is a contraction of a sequence* $\{c_n\}$, *then there exists a function* $g(x)$, *belonging to* $L^2(\varepsilon, 1)$ *for any positive number* ε, *such that*

$$g(x)(\phi_j(x) - 1) \sim \sum_{n=0}^{\infty} (a_{m(n,j)} - a_n)\, \psi_n(x) \quad \textit{for every integer } j.$$

Moreover, this function $g(x)$ *satisfies the following inequality*;

$$\int_0^x t^2 |g(t)|^2\, dt \leqq A \left\{ x^2 \int_x^1 |f(t)|^2\, dt + \int_0^x t^2 |f(t)|^2\, dt \right\}.$$

PROOF. Since $\{a_n\}$ is a contraction of $\{c_n\}$, we have, by Parseval theorem,

$$\sum_{n=0}^{\infty} |a_{m(n,j)} - a_n|^2 \leqq \sum_{n=0}^{\infty} |c_{m(n,j)} - c_n|^2 = \int_0^1 |f(x)(\phi_j(x)-1)|^2\, dx < \infty.$$

Thus we have

$$(*) \qquad \sum_{n=0}^{\infty} |a_{m(n,j)} - a_n|^2 < \infty \quad \text{for every integer } j.$$

Put $g_n(x) = \sum_{\nu=0}^{2^n-1} a_\nu \psi_\nu(x)$, then

$$\sum_{\nu=0}^{2^n-1} \psi_\nu(x)(a_{m(\nu,j)} - a_\nu) = (\phi_j(x) - 1)\, g_n(x) \quad (n > j).$$

Hence, by (*)

$$\underset{n\to\infty}{\overset{(2)}{\text{l.i.m.}}} \{g_n(x)(\phi_j(x)-1)\} = \underset{n\to\infty}{\overset{(2)}{\text{l.i.m.}}} \left\{\sum_{\nu=0}^{2^n-1} \psi_\nu(x)(a_{m(\nu,j)} - a_\nu)\right\}$$

which we can write

$$\equiv g^{(j)}(x)(\phi_j(x) - 1), \qquad \text{say,}$$

where $g^{(j)}(x)$ is defined for almost all x belonging to the set $E_j = \{x \mid (2\nu-1)/2^{j+1} \leqq x < 2\nu/2^{j+1},\ \nu = 1, 2, \cdots, 2^j\}$ and the Walsh Fourier series of $g^{(j)}(x)(\phi_j(x)-1)$ is $\sum_{n=0}^{\infty} (a_{m(n,j)} - a_n)\, \psi_n(x)$. Since

$$\phi_i(x) = \phi_j(x) = -1 \quad \text{for} \quad x \in E_i \cap E_j \quad (i, j = 1, 2, \cdots),$$

we have clearly

$$g^{(i)}(x) = g^{(j)}(x) \quad \text{for almost all} \quad x \in E_i \cap E_j \quad (i \neq j).$$

As the union of the sets E_j is $(0, 1)$, let us define the function $g(x)$ on the open interval $(0, 1)$ according to the following rule;

$$g(x) = g^{(j)}(x) \quad \text{for} \quad x \in E_j \quad (j = 1, 2, \cdots).$$

Thus $g(x)$ is well-defined almost everywhere in the open interval $(0, 1)$. It is clear from the definition of the function $g(x)$ that

$$g(x)(\phi_j(x)-1)\sim\sum_{n=0}^{\infty}(a_{m(n,j)}-a_n)\psi_n(x)\quad\text{for any integer } j$$

and

$$g(x)\in L^2(\varepsilon,1)\quad\text{for every positive number } \varepsilon.$$

Since

$$f(x)(\phi_k(x)-1)\sim\sum_{n=0}^{\infty}(c_{m(n,k)}-c_n)\psi_k(x)\quad\text{for each positive integer } k$$

and a sequence $\{a_n\}$ is a contraction of a sequence $\{c_n\}$, we have

$$\int_0^1 |g(x)(\phi_k(x)-1)|^2dx=\sum_{n=0}^{\infty}|a_{m(n,k)}-a_n|^2$$

$$\leqq\sum_{n=0}^{\infty}|c_{m(n,k)}-c_n|^2=\int_0^1|f(x)(\phi_k(x)-1)|^2dx.$$

By the definition of Rademacher function, we have

$$\phi_k(x)=1\quad(0\leqq x<2^{-k-1}),\quad \phi_k(x)=-1\quad(2^{-k-1}\leqq x<2^{-k}).$$

Hence we have

$$4\int_{2^{-k-1}}^{2^{-k}}|g(x)|^2\,dx=\int_{2^{-k-1}}^{2^{-k}}|g(x)(\phi_k(x)-1)|^2\,dx$$

$$\leqq\int_0^1|g(x)(\phi_k(x)-1)|^2\,dx\leqq\int_0^1|f(x)(\phi_k(x)-1)|^2\,dx$$

$$=\int_{2^{-k-1}}^1|f(x)(\phi_k(x)-1)|^2\,dx\leqq 4\int_{2^{-k-1}}^1|f(x)|^2\,dx.$$

Thus we have

$$\int_{2^{-k-1}}^{2^{-k}}|g(x)|^2\,dx<\int_{2^{-k-1}}^1|f(x)|^2\,dx=\sum_{i=0}^{k}\int_{2^{-i-1}}^{2^{-i}}|f(x)|^2\,dx.$$

If $x=2^{-k}$, then we have

$$\int_0^{2^{-k}} t^2|g(t)|^2\,dt = \sum_{j=k}^{\infty}\int_{2^{-j-1}}^{2^{-j}} t^2|g(t)|^2\,dt \leqq \sum_{j=k}^{\infty} 2^{-2j}\int_{2^{-j-1}}^{2^{-j}} |g(t)|^2\,dt$$

$$\leqq A\sum_{j=k}^{\infty} 2^{-2j}\int_{2^{-j-1}}^{1} |f(t)|^2\,dt = A\sum_{j=k}^{\infty}\left(\sum_{i=0}^{k-1}+\sum_{i=k}^{j}\right)2^{-2j}\int_{2^{-i-1}}^{2^{-i}} |f(t)|^2\,dt$$

$$\leqq A\left\{2^{-2k}\int_{2^{-k}}^{1} |f(t)|^2\,dt + \sum_{i=k}^{\infty}\sum_{j=i}^{\infty} 2^{-2j}\int_{2^{-i-1}}^{2^{-i}} |f(t)|^2\,dt\right\}$$

$$\leqq A\left\{2^{-2k}\int_{2^{-k}}^{1} |f(t)|^2\,dt + \sum_{i=k}^{\infty}\int_{2^{-i-1}}^{2^{-i}} t^2|f(t)|^2\,dt\right\}$$

$$= A\left\{2^{-2k}\int_{2^{-k}}^{1} |f(t)|^2dt + \int_0^{2^{-k}} t^2|f(t)|^2\,dt\right\}.$$

Next, we suppose that

$$2^{-k-1} < x < 2^{-k}.$$

Then we have

$$\int_0^{x} t^2|g(t)|^2\,dt \leqq \int_0^{2^{-k}} t^2|g(t)|^2\,dt$$

$$\leqq A\left\{2^{-2k}\int_{2^{-k}}^{1} |f(t)|^2\,dt + \int_0^{2^{-k}} t^2|f(t)|^2\,dt\right..$$

On the other hand, we have

$$\int_0^{2^{-k}} t^2|f(t)|^2\,dt = \int_0^{x} t^2|f(t)|^2\,dt + \int_x^{2^{-k}} t^2|f(t)|^2\,dt\Big\}$$

$$\leqq \int_0^{x} t^2|f(t)|^2dt + 2^{-2k}\int_x^{2^{-k}} |f(t)|^2\,dt$$

$$\leqq A\left\{\int_0^{x} t^2|f(t)|^2\,dt + x^2\int_x^{2^{-k}} |f(t)|^2\,dt\right\}.$$

Thus we have generally

$$\int_0^{x} t^2|g(t)|^2\,dt \leqq A\left\{\int_0^{x} t^2|f(t)|^2\,dt + x^2\int_x^{1} |f(t)|^2\,dt\right\}. \qquad \text{Q. D. E.}$$

Our method of proof of Theorem 2 is based on that used by M. Kinukawa [3].

PROOF OF THEOREM 2. We may suppose that $1 \leqq p < 2$. By the assumption of Theorem 2, we have

$$f(x) \in L^2(\varepsilon, 1) \quad \text{for every positive number } \varepsilon.$$

Hence

$$f(x)(\phi_j(x)-1) \in L^2(0,1) \qquad \text{for every integer } j.$$

Since $\{a_n\}$ is a contraction of $\{c_n\}$, we have, by Parseval theorem,

$$\sum_{n=0}^{\infty} |a_{m(n,j)} - a_n|^2 \leqq \sum_{n=0}^{\infty} |c_{m(n,j)} - c_n|^2 = \int_0^1 |f(x)(\phi_j(x)-1)|^2 dx < \infty.$$

Thus we have

$$\sum_{n=0}^{\infty} |a_{m(n,j)} - a_n|^2 < \infty \quad \text{for every integer } j.$$

Therefore, by lemma 4, we have a function $g(x)$ which is well-defined almost everywhere on the open interval $(0,1)$. Now we shall show that this function $g(x)$ belongs to $L^p(0,1)$. It is clear from lemma 4 that

$$\int_0^1 x^2 |g(x)|^2 dx < \infty,$$

and hence, by Hölder's inequality

$$\text{(1)} \qquad \int_0^1 x^p |g(x)|^p dx \leqq \left(\int_0^1 x^2 |g(x)|^2 dx\right)^{p/2} \left(\int_0^1 dx\right)^{1-p/2} < \infty.$$

Thus we have only to show that $g(x) \in L^p(0, 1/2)$. By (1) we can define

$$h_p(x) = \int_0^x t^p |g(t)|^p dt.$$

Then by Hölder's inequality and lemma 4,

$$
\begin{aligned}
|h_p(x)| &\leqq x^{1-p/2}\left(\int_0^x t^2|g(t)|^2\,dt\right)^{p/2} \\
&\leqq A\,x^{1-p/2}\left(\int_0^x t^2|f(t)|^2\,dt + x^2\int_x^1 |f(t)|^2\,dt\right)^{p/2} \qquad (2)\\
&\leqq A\,x^{1-p/2}\left\{\left(\int_0^x t^2\gamma^2(t)\,dt\right)^{p/2} + x^p\left(\int_x^1 \gamma^2(t)\,dt\right)^{p/2}\right\}
\end{aligned}
$$

by Jensen's inequality. By the assumption (i), for each $\alpha \in (0, 1/2]$, we have

$$
\begin{aligned}
A &\geqq \int_\alpha^{2\alpha} x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)\,dt\right)^{p/2} dx \\
&\geqq \int_\alpha^{2\alpha} x^{-3p/2}\left(\int_0^\alpha t^2\gamma^2(t)\,dt\right)^{p/2} dx \\
&= A_p\,\alpha^{1-3p/2}\left(\int_0^\alpha t^2\gamma^2(t)\,dt\right)^{p/2}.
\end{aligned}
$$

Hence

$$
(3) \qquad \left(\int_0^x t^2\gamma^2(t)\,dt\right)^{p/2} \leqq A\,x^{3p/2-1}, \quad \text{for} \quad 0 < x \leqq 1/2 .
$$

Also

$$
\begin{aligned}
A &\geqq \int_{\alpha/2}^{\alpha} x^{-p/2}\left(\int_x^1 \gamma^2(t)\,dt\right)^{p/2} dx \\
&\geqq \int_{\alpha/2}^{\alpha} x^{-p/2}\left(\int_\alpha^1 \gamma^2(t)\,dt\right)^{p/2} dx \\
&= \left(\int_\alpha^1 \gamma^2(t)\,dt\right)^{p/2} \cdot \int_{\alpha/2}^{\alpha} x^{-p/2}\,dx ,
\end{aligned}
$$

and hence we have

$$
(4) \qquad \left(\int_x^1 \gamma^2(t)\,dt\right)^{p/2} \leqq A\,x^{p/2-1}, \text{ for } 0 < x < 1 .
$$

Combining (2), (3) and (4), we have

$$(5) \qquad |h_p(x)| \leqq Ax^{1-p/2}(x^{3p/2-1} + x^p \cdot x^{p/2-1}) \leqq Ax^p .$$

Now we can show that $|g(x)|^p \in L(0, 1/2]$; that is,

$$\int_0^{1/2} |g(x)|^p \, dx = \int_0^{1/2} x^{-p} \, dh_p(x)$$

$$= [x^{-p} h_p(x)]_0^{1/2} + p\int_0^{1/2} x^{-p-1} h_p(x) \, dx ,$$

where the first part is $O(1)$ by (5) and the second part is, by (2), less than

$$A\int_0^{1/2} x^{-p-1} \cdot x^{1-p/2} \left\{ \left(\int_0^x t^2 \gamma^2(t) \, dt \right)^{p/2} + x^p \left(\int_x^1 \gamma^2(t) \, dt \right)^{p/2} \right\} dx$$

$$\leqq A \left\{ \int_0^1 x^{-3p/2} \left(\int_0^x t^2 \gamma^2(t) \, dt \right)^{p/2} dx + \int_0^1 x^{-p/2} \left(\int_x^1 \gamma^2(t) \, dt \right)^{p/2} dx \right\}$$

which is finite because of the assumption (ii). Therefore we see that $g(x)$ belongs to $L^p(0, 1)$ and whose Walsh Fourier series we shall denote by

$$g(x) \sim \sum_{n=0}^{\infty} a_n^* \psi_n(x) .$$

Now $\sum_{n=0}^{\infty} (a_{m(n,j)} - a_n) \psi_n(x)$ and $\sum_{n=0}^{\infty} (a^*_{m(n,j)} - a^*_n) \psi_n(x)$ are the Walsh Fourier series of the same function $g(x)(\phi_j(x) - 1)$ which belongs to $L^2(0, 1)$. By the completeness of the system of Walsh functions, we have

$$a_{m(n,j)} - a_n = a^*_{m(n,j)} - a^*_n$$

for every pair (n, j) of integers. Letting $j \to \infty$ and observing $a_{m(n,j)} \to 0$, $a^*_{m(n,j)} \to 0$, we see

$$a_n = a_n^* ,$$

which completes the proof of Theorem 2.

Added in proof [May 29, 1967]. Recently Prof. G. Sunouchi pointed out that the convergence of two series $\sum_{n=1}^{\infty} n^{-\beta/2} \left(\sum_{\nu=n}^{\infty} a^2 \nu \right)^{\beta/2}$ and $\sum_{n=1}^{\infty} n^{-3\beta/2} \left(\sum_{\nu=1}^{n} \nu^2 a_\nu^2 \right)^{\beta/2}$ is equivalent, so the hypotheses of our theorems may be accordingly modified.

REFERENCES

[1] R. P. BOAS, Beurling's test for absolute convergence of Fourier series, Bull. Amer. Math. Soc., 66 (1960), 24-27.
[2] N. J. FINE, On Walsh functions, Trans. Amer. Math. Soc., 65 (1949), 372-414.
[3] M. KINUKAWA, Contraction of Fourier coefficients and Fourier integrals, Journ. d'Analyse Math., 8 (1960/1961), 377-406.
[4] C. WATARI, Contraction of Walsh Fourier series, Proc. Amer. Math. Soc., 15 (1964), 189-192.

DEPARTMENT OF TECHNOLOGY,
SHINSHU UNIVERSITY,
NAGANO, JAPAN