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ON CONTRACTIONS SIMILAR TO ISOMETRIES AND TOEPLITZ OPERATORS

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1. Preliminaries and introduction

1. The (unitarily equivalent) canonical model of a completely non-unitary contraction T on a (separable, complex) Hilbert space is the operator $S(\Theta)$ on the space $\mathfrak{H}(\Theta)$, associated with a purely contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ [1] in the following manner [2]

$$\mathfrak{H}(\Theta) = \mathfrak{K}(\Theta) \ominus \{ \Theta w \oplus \Delta w : w \in H^2_{\mathfrak{E}} \}, \tag{1.1}$$

where

$$\mathfrak{K}(\Theta) = H^2_{\mathfrak{E}_*} \oplus \overline{\Delta L^2_{\mathfrak{E}}}, \qquad \Delta(e^{it}) = [I - \Theta(e^{it})^* \Theta(e^{it})]^{1/2}, \tag{1.2}$$

and

$$S(\Theta)(u \oplus v) = P_{\mathfrak{H}(\Theta)}(\chi u \oplus \chi v), \qquad u \oplus v \in \mathfrak{H}(\Theta), \tag{1.3}$$

χ denoting the function $\chi(\lambda) \equiv \lambda$; cf. [4], Chapter VI.

We have $\mathfrak{H}(\Theta) = \{0\}$ if and only if both $\mathfrak{E}$ and $\mathfrak{E}_*$ are zero (i.e. equal $\{0\}$); cf. [4], Proposition VI.3.2. On the other hand, $\mathfrak{H}(\Theta) = \mathfrak{K}(\Theta)$

[1] We denote by $\{\mathfrak{A}, \mathfrak{B}, \Phi(\lambda)\}$ an analytic function on the unit disc, whose values are operators from the Hilbert space $\mathfrak{A}$ into the Hilbert space $\mathfrak{B}$, both spaces being supposed complex and separable. This function is *bounded* if $\|\Phi(\cdot)\|_\infty = \sup_\lambda \|\Phi(\lambda)\|$ is finite; it is *contractive* if $\|\Phi(\lambda)\| \leqq 1$, and *purely* contractive if, moreover, $\|\Phi(0)a\| < \|a\|$ for all $a \in \mathfrak{A}$, $a \neq 0$. For a bounded analytic function the radial limits $\Phi(e^{it}) = \lim \Phi(re^{it})$ $(r \to 1-0)$ exist in the strong sense, almost everywhere on the unit circle.

[2] $L^2_{\mathfrak{E}}$ denotes the Hilbert space of $\mathfrak{E}$-vector valued, norm-square integrable functions on the unit circle, with respect to normalized Lebesgue measure. $H^2_{\mathfrak{E}}$ is its subspace of functions $u(e^{it}) \sim \sum_{k=0}^{\infty} a_k e^{ikt}$; these are radial limits a.e. of the corresponding analytic functions $u(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ in the unit disc.

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if and only if $\Theta w \oplus \Delta w = 0$, i.e. $w = 0$ for all $w \in H^2_{\mathfrak{E}}$, that is, if $\mathfrak{E}$ is zero. Thus the inequalities $0 \neq \mathfrak{H}(\Theta) \neq \mathfrak{K}(\Theta)$ simultaneously hold if and only if

$$\mathfrak{E} \neq \{0\}. \tag{1.4}$$

We shall assume (1.4) in the sequel.

The assumption that $\Theta(\lambda)$ be *purely* contractive has the effect that the operator $V(\Theta)$ defined on $\mathfrak{K}(\Theta)$ by

$$V(\Theta)(u \oplus v) = \chi u \oplus \chi v, \quad u \oplus v \in \mathfrak{K}(\Theta), \tag{1.5}$$

is the *minimal* isometric dilation of $S(\Theta)$.

Note that (1.2) is just the Wold decomposition of the space $\mathfrak{K}(\Theta)$ generated by the isometry $V(\Theta)$, that is,

$$\bigcap_{n \geqq 0} V(\Theta)^n \mathfrak{K}(\Theta) = \{0\} \oplus \overline{\Delta L^2_{\mathfrak{E}}}. \tag{1.6}$$

2. In any space L^2 of scalar or vector valued functions u on the unit circle we denote by

$$[u]_+ \quad \text{and} \quad [u]_-$$

the orthogonal projections of u to the subspaces

$$H^2 \quad \text{and} \quad L^2 \ominus H^2,$$

respectively.

With a bounded analytic function $\{\mathfrak{A}, \mathfrak{B}, \Phi(\lambda)\}$ we associate the operator

$$T(\Phi): H^2_{\mathfrak{A}} \to H^2_{\mathfrak{B}}$$

defined by

$$(T(\Phi)u)(e^{it}) = [\Phi(e^{-it})u(e^{it})]_+;$$

such operators are also called co-analytic Toeplitz operators.

Observe that if W is the canonical unitary transformation $W: H^2 \to L^2 \ominus H^2$ defined by

$$W: u(e^{it}) \mapsto e^{-it}u(e^{-it}) \quad (u \in H^2),$$

then the transformed operator

$$T^\wedge(\Phi) = W_{\mathfrak{B}} T(\Phi) W_{\mathfrak{A}}^{-1}: L^2_{\mathfrak{A}} \ominus H^2_{\mathfrak{A}} \to L^2_{\mathfrak{B}} \ominus H^2_{\mathfrak{B}} \tag{1.7}$$

is given by

$$(T^\wedge(\Phi)\varphi)(e^{it}) = [\Phi(e^{it})\varphi(e^{it})]_- \quad (\varphi \in L^2_{\mathfrak{A}} \ominus H^2_{\mathfrak{A}}). \tag{1.8}$$

3. The principal aim of this paper is to derive a condition for an operator valued contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ in the unit disc to admit a "left-inverse", i.e. an operator valued bounded analytic function $\{\mathfrak{E}_*, \mathfrak{E}, D(\lambda)\}$ such that

$$D(\lambda)\,\Theta(\lambda) = I_{\mathfrak{E}},$$

and estimates for $\|D(\cdot)\|_\infty$. This condition will involve the operator $T(\Theta)$, or equivalently, its unitary transform $T^\wedge(\Theta)$.

The theorem obtained reduces in the particular case when $\Theta(\lambda)$ is a finite column vector over H^∞ to a recent result of Arveson [1], but it gives even in this case better estimates.

2. A general condition implying similarity

Proposition 1. *Suppose T is a contraction on a Hilbert space $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ such that*

(i) *the subspace $\mathfrak{H}'$ is invariant for T and $T|\mathfrak{H}'$ is isometric,*

(ii) $\inf\{\lim_{n\to\infty}\|T^n h\| : h \in \mathfrak{H}'', \|h\| = 1\} = \eta > 0$.

Then there exists an invertible operator X from $\mathfrak{H}$ onto some Hilbert space $\mathfrak{L}$ such that $X\,T\,X^{-1}$ is an isometry on $\mathfrak{L}$ and

$$\|X\|\,\|X^{-1}\| \leq 1/\eta. \tag{2.1}$$

Proof. Let $A = (\lim_{n\to\infty} T^{*n}T^n)^{1/2}$; the strong limit exists because T is a contraction. We have $T^*A^2T = A^2$, and therefore $\|A\,T\,h\| = \|A\,h\|$ for all $h \in \mathfrak{H}$. Thus there exists an isometry Z on $\mathfrak{L} = \overline{A\,\mathfrak{H}}$ such that

$$A\,T = Z\,A. \tag{2.2}$$

For $h' \in \mathfrak{H}'$ we have, by (i), $\|T^n h'\| = \|h'\|$ ($n = 0, 1, \ldots$). Hence,

$$\|A\,h'\|^2 = (A^2 h', h') = \lim_n \|T^n h'\|^2 = \|h'\|^2.$$

As $0 \leq A \leq I$, equality $\|A\,h'\| = \|h'\|$ implies $A\,h' = h'$. Therefore $\mathfrak{H}'$ is invariant for A, and hence so is $\mathfrak{H}''$: the decomposition $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ is reducing for A. Clearly,

$$\eta = \inf\{\|A\,h''\| : h'' \in \mathfrak{H}'', \|h''\| = 1\}.$$

Because $\|A\,(h' + h'')\|^2 = \|h' + A\,h''\|^2 = \|h'\|^2 + \|A\,h''\|^2$ and $0 \leq \eta \leq 1$, we infer that

$$\|A\,h\| \geq \eta\,\|h\| \quad \text{for all } h \in \mathfrak{H}. \tag{2.3}$$

Denote by X the operator $X : \mathfrak{H} \to \mathfrak{L}$ $(= \overline{A\,\mathfrak{H}})$ induced by A. Then, by (2.2), $X\,T\,X^{-1} = Z$. Moreover, $\|X\| \leq 1$, and by (2.3), $\|X^{-1}\| \leqq 1/\eta$. Thus (2.1) holds true.

3. A connection between the operators $S(\Theta)$ and $T(\Theta)$

Consider the operator $S(\Theta)$ generated by a purely contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ as in 1.1, that is, with non-zero $\mathfrak{E}$.

First observe that the linear manifold

$$(3.1) \qquad \mathfrak{H}_0''(\Theta) = \{ [\Theta\,\varphi]_+ \oplus \Delta\,\varphi : \varphi \in L^2_{\mathfrak{E}} \ominus H^2_{\mathfrak{E}} \}$$

and its closure $\mathfrak{H}''(\Theta)$ are contained in $\mathfrak{H}(\Theta)$. Indeed, $\mathfrak{H}_0''(\Theta)$ is orthogonal to any vector of the form $\Theta\,w \oplus \Delta\,w$ $(w \in H^2_{\mathfrak{E}})$, because

$$\begin{aligned}([\Theta\,\varphi]_+, \Theta\,w) + (\Delta\,\varphi, \Delta\,w) &= (\Theta\,\varphi, \Theta\,w) + (\Delta\,\varphi, \Delta\,w)\\ &= ((\Theta^*\,\Theta + \Delta^2)\,\varphi, w) = (\varphi, w) = 0.\end{aligned}$$

Let $\mathfrak{H}'(\Theta) = \mathfrak{H}(\Theta) \ominus \mathfrak{H}''(\Theta)$. Clearly, we have:

$$u \oplus v \in \mathfrak{H}'(\Theta) \quad \Leftrightarrow \quad \begin{cases} u \in H^2_{\mathfrak{E}_*},\ v \in \overline{\Delta\,L^2_{\mathfrak{E}}},\ \Theta^*\,u + \Delta\,v \in L^2_{\mathfrak{E}} \ominus H^2_{\mathfrak{E}}, \\ 0 = (u \oplus v,\ [\Theta\,\varphi]_+ \oplus \Delta\,\varphi) \\ \quad = (\Theta^*\,u + \Delta\,v,\ \varphi) \quad \text{for all } \varphi \in L^2_{\mathfrak{E}} \ominus H^2_{\mathfrak{E}} \end{cases}$$

and therefore,

$$(3.2) \qquad \mathfrak{H}'(\Theta) = \{ u \oplus v \in \mathfrak{H}(\Theta) : \Theta^*\,u + \Delta\,v = 0 \}.$$

It follows that if $u \oplus v \in \mathfrak{H}'(\Theta)$ then $\chi\,u \oplus \chi\,v \in \mathfrak{H}'(\Theta)$, and hence, $\mathfrak{H}'(\Theta)$ is invariant for $S(\Theta)$; moreover, $S(\Theta) \mid \mathfrak{H}'(\Theta)$ is an isometry, namely multiplication by χ.

A straightforward computation shows that for any $u \oplus v \in \mathfrak{K}(\Theta)$ (cf. (1.2)) its projection to $\mathfrak{K}(\Theta) \ominus \mathfrak{H}(\Theta)$ equals

$$\Theta\,w \oplus \Delta\,w, \quad \text{where } w = [x]_+,\ x = \Theta^*\,u + \Delta\,v.$$

Therefore,

$$\|P_{\mathfrak{H}(\Theta)}\,(u \oplus v)\|^2 = \|u \oplus v\|^2 - \|\Theta\,w \oplus \Delta\,w\|^2 = \|u \oplus v\|^2 - \|w\|^2.$$

Apply this to $\chi^n\,u \oplus \chi^n\,v$ $(n = 0, 1, 2, \ldots)$ as well, and obtain

$$\lim_{n\to\infty} \|P_{\mathfrak{H}(\Theta)}\,(\chi^n\,u \oplus \chi^n\,v)\|^2 = \|u \oplus v\|^2 - \lim_{n\to\infty} \|[\chi^n\,x]_+\|^2;$$

the last limit obviously equals $\|x\|^2$. Thus we have for $u \oplus v \in \mathfrak{H}(\Theta)$:

$$\lim_{n\to\infty} \|S(\Theta)^n (u \oplus v)\|^2 = \|u \oplus v\|^2 - \|\Theta^* u + \Delta v\|^2 .$$

Let, in particular, $h = u \oplus v \in \mathfrak{H}_0''(\Theta)$, say

$$h = [\Theta \varphi]_+ \oplus \Delta \varphi , \quad \varphi \in L^2_{\mathfrak{E}} \ominus H^2_{\mathfrak{E}} ,$$

then $\Theta^* [\Theta \varphi]_+ \oplus \Delta \Delta \varphi = (\Theta^* \Theta + \Delta^2) \varphi - \Theta^* [\Theta \varphi]_- = \varphi - \Theta^* [\Theta \varphi]_-$; hence,

$$\begin{aligned} \lim_{n\to\infty} \|S(\Theta)^n h\|^2 &= \|h\|^2 - \|\varphi - \Theta^* [\Theta \varphi]_-\|^2 \\ &= \|h\|^2 - \|\varphi - B \varphi\|^2 , \end{aligned} \tag{3.3}$$

where B denotes the operator on $L^2_{\mathfrak{E}} \ominus H^2_{\mathfrak{E}}$ defined by

$$B \varphi = \Theta^* [\Theta \varphi]_- .$$

B is selfadjoint and $0 \leq B \leq I$; indeed, we have

$$(B \varphi , \varphi) = ([\Theta \varphi]_- , \Theta \varphi) = ([\Theta \varphi]_- , [\Theta \varphi]_-) = \|T^\wedge(\Theta) \varphi\|^2 , \tag{3.4}$$

where $T^\wedge(\Theta)$ is the transformed Toeplitz operator defined by (1.8). We have

$$\begin{aligned} \|h\|^2 &= \|[\Theta \varphi]_+\|^2 + \|\Delta \varphi\|^2 = (\|\Theta \varphi\|^2 - \|[\Theta \varphi]_-\|^2) + \|\Delta \varphi\|^2 \\ &= \|\varphi\|^2 - \|[\Theta \varphi]_-\|^2 = \|\varphi\|^2 - (B \varphi , \varphi) = \|C \varphi\|^2 , \end{aligned} \tag{3.5}$$

where $C = (I - B)^{1/2}$, and

$$\begin{aligned} \|h\|^2 - \|\varphi - B \varphi\|^2 &= \|\varphi\|^2 - (B \varphi , \varphi) - \|\varphi\|^2 + 2 (B \varphi , \varphi) - \|B \varphi\|^2 \\ &= (B \varphi , \varphi) - (B \varphi , B \varphi) = (B C \varphi , C \varphi) . \end{aligned} \tag{3.6}$$

From (3.3), (3.5) and (3.6) we infer that the infima

$$\inf_h \{ \lim_{n\to\infty} \|S(\Theta)^n h\| : h \in \mathfrak{H}_0''(\Theta) , \ \|h\| = 1 \}$$

and

$$\inf_\psi \{ (B \psi , \psi) : \psi \in \text{range } C , \ \|\psi\| = 1 \}$$

are equal. They remain, by continuity, unchanged and therefore equal to each other if we allow h and ψ to run over all unit vectors in $\mathfrak{H}''(\Theta)$ and in the closure $\mathfrak{R}(C)$ of the range of C, respectively. Now $\mathfrak{R}(C)$ is obviously reducing B and for ψ in the orthogonal complement of $\mathfrak{R}(C)$ we have $C \psi = 0$, $B \psi = \psi$, $(B \psi , \psi) = \|\psi\|^2$. Hence we infer that the second infimum does not change even if we allow ψ to run over all unit vectors in $L^2_{\mathfrak{E}} \ominus H^2_{\mathfrak{E}}$.

Recalling (3.4) and observing that the Toeplitz operator $T(\Theta)$ clearly has the same lower bound as its unitary transform $T^{\wedge}(\Theta)$, on the respective unit spheres, we conclude:

Proposition 2. *For any purely contractive analytic* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ *the decomposition* $\mathfrak{H}(\Theta) = \mathfrak{H}'(\Theta) \oplus \mathfrak{H}''(\Theta)$ *of the space* $\mathfrak{H}(\Theta)$ *defined by (3.1) and (3.2) is such that*

(i) $S(\Theta) \mid \mathfrak{H}'(\Theta)$ *is an isometry on* $\mathfrak{H}'(\Theta)$,

(ii) *The infima*

$$\inf_h \{ \lim_{n\to\infty} \|S(\Theta)^n h\| : \ h \in \mathfrak{H}''(\Theta), \ \|h\| = 1 \}$$

and

$$\inf_u \{ \|T(\Theta) u\| : \ u \in H^2_{\mathfrak{E}}, \ \|u\| = 1 \}$$

are equal to the same value $\eta = \eta(\Theta)$.

4. Similarity of $S(\Theta)$ to an isometry

In case the quantity $\eta = \eta(\Theta)$ defined in Proposition 2 is non-zero we can apply Proposition 1 and conclude that $S(\Theta)$ is similar to some isometry Z on a space $\mathfrak{L}$, i.e. there exist operators

$$X: \ \mathfrak{H}(\Theta) \to \mathfrak{L}, \quad X': \ \mathfrak{L} \to \mathfrak{H}(\Theta)$$

such that

$$(4.1) \qquad ZX = XS(\Theta), \quad S(\Theta)X' = X'Z, \quad X' = X^{-1}$$

and moreover,

$$(4.2) \qquad \|X'\|\,\|X\| \leq 1/\eta .$$

Now the following is true:

Proposition 3. *From (4.1), Z an isometry, it follows that there exists a bounded analytic function* $\{\mathfrak{E}_*, \mathfrak{E}, D(\lambda)\}$ *such that*

$$D(\lambda)\,\Theta(\lambda) = I_{\mathfrak{E}}, \quad \|D(\cdot)\|_\infty \leq \|X'\|\,\|X\| .$$

Proof. The existence of a bounded analytic $D(\lambda)$ with the property $D(\lambda)\,\Theta(\lambda) = I$ is proved in Theorem 2.4 of [5], and an estimate for $\|D(\cdot)\|_\infty$ can also be deduced from the proof of that theorem. For convenience, we give a direct and complete proof.

This proof is based upon the "commutant lifting theorem" of [3]; see also [4], Sec.II.2.3. Since Z is its own minimal isometric dilation this theorem asserts in this case that there exist operators

$$Y:\ \mathfrak{K}(\Theta) \to \mathfrak{L}, \qquad Y':\ \mathfrak{L} \to \mathfrak{K}(\Theta)$$

such that (using the notations of Sec. 1.1):

$$ZY = YV(\Theta), \qquad V(\Theta)Y' = Y'Z, \tag{4.3}$$

$$YP_{\mathfrak{H}(\Theta)^{\perp}} = 0,\text{[3]} \tag{4.4}$$

$$X = Y \mid \mathfrak{H}(\Theta), \qquad X' = P_{\mathfrak{H}(\Theta)}Y', \tag{4.5}$$

and

$$\|Y\| = \|X\|, \qquad \|Y'\| = \|X'\|. \tag{4.6}$$

Moreover, $X'X = I_{\mathfrak{H}(\Theta)}$ implies by (4.5):

$$P_{\mathfrak{H}(\Theta)}Y'Y \mid \mathfrak{H}(\Theta) = I_{\mathfrak{H}(\Theta)};$$

on account of (4.4) this is equivalent to the condition

$$P_{\mathfrak{H}(\Theta)}(I - Y'Y)k = 0 \quad \text{for all } k \in \mathfrak{K}(\Theta). \tag{4.7}$$

From (4.4) and (4.7) it easily follows that the operator

$$F = I - Y'Y \tag{4.8}$$

satisfies the conditions

$$F^2 = F \quad \text{and} \quad F\mathfrak{K}(\Theta) = \mathfrak{H}(\Theta)^{\perp}. \tag{4.9}$$

Thus F is a (bounded) parallel projection of $\mathfrak{K}(\Theta)$ onto $\mathfrak{H}(\Theta)^{\perp}$.

Observe that

$$\omega:\ w \mapsto \Theta w \oplus \Delta w \qquad (w \in H^2_{\mathfrak{E}})$$

is a unitary operator $\omega: H^2_{\mathfrak{E}} \to \mathfrak{H}(\Theta)^{\perp}$, which commutes with multiplication by the scalar function χ. As (4.3) implies $FV(\Theta) = V(\Theta)F$, the operator F also commutes with multiplication by χ. As a consequence we have

$$\begin{aligned} F\bigcap_{n\geqq 0} V(\Theta)^n\mathfrak{K}(\Theta) &\subset \bigcap_{n\geqq 0} V(\Theta)^n F\mathfrak{K}(\Theta) = \bigcap_{n\geqq 0} V(\Theta)^n\mathfrak{H}(\Theta)^{\perp} = \bigcap_{n\geqq 0}\chi^n\cdot\omega H^2_{\mathfrak{E}} \\ &= \omega\bigcap_{n\geqq 0}\chi^n H^2_{\mathfrak{E}} = \omega\{0\} = \{0\}, \end{aligned}$$

thus by (1.6)

$$F(0 \oplus v) = 0 \quad \text{for any } v \in \overline{\Delta L^2_{\mathfrak{E}}}.$$

Combining this with (4.9) we get in particular

[3] $\mathfrak{H}(\Theta)^{\perp} = \mathfrak{K}(\Theta) \ominus \mathfrak{H}(\Theta) = \{\Theta w \oplus \Delta w:\ w \in H^2_{\mathfrak{E}}\}$; cf. (1.1).

$$(4.10) \qquad \Theta u \oplus \Delta u = F(\Theta u \oplus \Delta u) = F(\Theta u \oplus 0) + F(0 \oplus \Delta u) = F(\Theta u \oplus 0)$$

for any $u \in H^2_{\mathfrak{E}}$.

Applying (4.9) again, we see that for every $k \in \mathfrak{R}(\Theta)$ there exists a unique $w \in H^2_{\mathfrak{E}}$ such that $F k = \omega w$. Choosing in particular $k = u_* \oplus 0$, $u_* \in H^2_{\mathfrak{E}_*}$, equation

$$(4.11) \qquad F(u_* \oplus 0) = \omega \cdot D u_*$$

defines an operator $D: H^2_{\mathfrak{E}_*} \to H^2_{\mathfrak{E}}$; clearly

$$(4.12) \qquad \|D\| \leqq \|F\|.$$

For F, the inequality $\|F\| \leq 1 + \|Y'\| \|Y\|$ is immediate from the definition (4.8). But we have even

$$(4.13) \qquad \|F\| = \|I - F\|,$$

and hence the inequality [4]

$$(4.14) \qquad \|F\| \leqq \|Y'\| \|Y\|.$$

Indeed, (4.13) holds for any (bounded) parallel projection of a Hilbert space onto a non-trivial subspace. This follows, namely, from the relation [5]

$$\|F\|^{-2} = 1 - \sup |(h, g)|^2$$

where h, g run over the sets of unit vectors satisfying $(I - F) h = 0$ and $F g = 0$, respectively, and from the symmetry of this relation in F and $I - F$. Note that in the case under consideration F projects indeed to a *non-trivial* subspace of $\mathfrak{R}(\Theta)$, because our assumption $\mathfrak{E} \neq \{0\}$ assures that $\{0\} \neq \mathfrak{H}(\Theta) \neq \mathfrak{R}(\Theta)$.

Thus, taking account of (4.6) and (4.12), we have

$$(4.15) \qquad \|D\| \leq \|X'\| \|X\|.$$

Next observe that as F and ω commute with multiplication by χ so does D too. Hence it follows (cf. Lemma V.3.1 in [4]) that D itself is a multiplication operator, viz.

$$(D u_*)(\lambda) = D(\lambda) u_*(\lambda),$$

[4] The authors are indebted for this ingenious and useful remark to Professor T. Ando from Sapporo (Japan), presently visiting Szeged (Hungary).

[5] This formula for F is due to V. È. Ljance and is reproduced in the book of I. C. Gohberg and M. G. Kreĭn [2], VI.5.4.

where $\{\mathfrak{E}_*, \mathfrak{E}, D(\lambda)\}$ is a bounded analytic function,

$$\|D(\cdot)\|_\infty = \|D\| . \tag{4.16}$$

On account of definition (4.11) of D we have in particular $F(\Theta u \oplus 0) = \omega \cdot D\,\Theta u$ for any $u \in H^2_{\mathfrak{E}}$, while (4.10) means $F(\Theta u \oplus 0) = \omega \cdot u$. Therefore, $u = D\,\Theta u$, and hence

$$D(\lambda)\,\Theta(\lambda) = I_{\mathfrak{E}} .$$

Recalling (4.15) and (4.16) the proof of Proposition 3 is done.

5. Conclusions

Combining Propositions $1-3$ we conclude that *if* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ *is a purely contractive analytic function for which* [6]

$$\inf\{\, \|T(\Theta)\,u\| : u \in H^2_{\mathfrak{E}},\ \|u\| = 1 \,\} = \eta > 0 , \tag{5.1}$$

then there exists an analytic function $\{\mathfrak{E}_*, \mathfrak{E}, D(\lambda)\}$ *such that*

$$D(\lambda)\,\Theta(\lambda) = I_{\mathfrak{E}} \quad \textit{and} \quad \|D(\cdot)\|_\infty \leqq 1/\eta . \tag{5.2}$$

Now it is easy to get rid of the restriction "purely contractive". Indeed, every contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ is, according to Proposition V.2.1 of [4], direct sum of a purely contractive analytic function $\{\mathfrak{E}^0, \mathfrak{E}^0_*, \Theta^0(\lambda)\}$ and of a unitary valued constant function $\{\mathfrak{E}', \mathfrak{E}'_*, \Theta'\}$ ($\mathfrak{E} = \mathfrak{E}^0 \oplus \mathfrak{E}'$, $\mathfrak{E}_* = \mathfrak{E}^0_* \oplus \mathfrak{E}'_*$). Hence it follows for any $u = u^0 \oplus u' \in H^2_{\mathfrak{E}}$ with components $u^0 \in H^2_{\mathfrak{E}^0}$, $u' \in H^2_{\mathfrak{E}'}$ that

$$T(\Theta)\,u = T(\Theta^0)\,u^0 \oplus \Theta'\,u' , \quad \|T(\Theta)\,u\|^2 = \|T(\Theta^0)\,u^0\|^2 + \|u'\|^2 .$$

In the case the first component is missing, but (5.1) is fulfilled, then, necessarily, $\mathfrak{E}' \neq \{0\}$ and $\eta = 1$, and a trivial solution for $D(\lambda)$ in (5.2) is the constant function $\{\mathfrak{E}'_*, \mathfrak{E}', \Theta'^*\}$. If both components are present then η equals the analogous quantity η^0 formed for Θ^0 (because $\eta^0 \leqq 1$) and hence (5.1) implies the existence of an analytic function $\{\mathfrak{E}^0_*, \mathfrak{E}^0, D^0(\lambda)\}$ satisfying $D^0(\lambda)\,\Theta^0(\lambda) = I_{\mathfrak{E}^0}$, $\|D^0(\cdot)\|_\infty \leqq 1/\eta^0 = 1/\eta$. Setting $D(\lambda) = D^0(\lambda) \oplus \Theta'^*$ we get a solution for (5.2).

So we can formulate our main result:

T h e o r e m. *If the contractive analytic function* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ *is such that*

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[6] This condition obviously implies $\mathfrak{E} \neq \{0\}$.

$$(*) \qquad \|T(\Theta)\,u\| \geqq \eta\,\|u\| \quad \textit{for an } \eta > 0 \textit{ and all } u \in H^2_{\mathfrak{E}}\,,$$

then there exists an analytic function $\{\mathfrak{E}_*\,, \mathfrak{E}\,, D(\lambda)\}$ *such that*

$$D(\lambda)\,\Theta(\lambda) = I_{\mathfrak{E}} \quad \textit{for } |\lambda| < 1\,, \quad \textit{and } \|D(\cdot)\|_\infty \leqq 1/\eta\,.$$

Remark 1. In the special case of a function $\{E^1, E^N, \Theta(\lambda)\}$, where

$$\Theta(\lambda) = \begin{bmatrix} \vartheta_1(\lambda) \\ \vdots \\ \vartheta_N(\lambda) \end{bmatrix},$$

the theorem can be given the following from: *If*

$$\sum_{k=1}^N |\vartheta_k(\lambda)|^2 \leqq 1 \quad \textit{for } |\lambda| < 1\,,$$

and

$$\sum_{k=1}^N \|T(\vartheta_k)\,u\|^2 \geqq \eta^2\,\|u\|^2 \quad \textit{for an } \eta > 0 \textit{ and all } u \in H^2\,,$$

then there exist $d_k \in H^\infty$ $(k = 1, \dots, N)$ *such that, for* $|\lambda| < 1$,

$$(5.3) \qquad \sum_{k=1}^N d_k(\lambda)\,\vartheta_k(\lambda) = 1 \quad \textit{and } |d_k(\lambda)| \leqq 1/\eta \quad (k = 1, \dots, N)\,.$$

Observe that if

$$(5.4) \qquad f_k \in H^\infty\,, \ \|f_k\|_\infty \leqq 1 \quad (k = 1, \dots, N)$$

and

$$(5.5) \qquad \sum_{k=1}^N \|T(f_k)\,u\|^2 \geqq \varepsilon^2\,\|u\|^2 \quad \text{for some } \varepsilon > 0 \text{ and all } u \in H^2\,,$$

then the functions $\vartheta_k(\lambda) = f_k(\lambda)\,/\sqrt{N}$ $(k = 1, \dots, N)$ satisfy the above requirements, with $\eta = \varepsilon/\sqrt{N}$. Hence there exist functions $d_k(\lambda)$ as in (5.3), and therefore the functions $g_k(\lambda) = d_k(\lambda)\,/\sqrt{N}$ satisfy

$$(5.6) \qquad \sum_{k=1}^N g_k(\lambda)\,f_k(\lambda) = 1 \quad \text{and} \quad \|g_k\|_\infty \leqq 1/\varepsilon \ (k = 1, \dots, N)\,.$$

The fact that assumptions (5.4) and (5.5) imply the existence of $g_k \in H^\infty$ satisfying (5.6) was also proved by Arveson [1], Theorem 6.3, however with the estimate $\|g_k\|_\infty \leqq 4\,N\,\varepsilon^{-3}$ only.

Remark 2. The functions $k_\mu(\lambda) = (1 - \mu\,\lambda)^{-1}$ $(|\mu| < 1)$ span the space H^2 and therefore the functions $k_\mu(\lambda)\,a$ $(|\mu| < 1\,, a \in \mathfrak{E})$ span $H^2_{\mathfrak{E}}$. Thus $(*)$ holds for every $u \in H^2_{\mathfrak{E}}$ if and only if it holds for the finite linear combinations of these functions. Now observe that

$$(T(\Theta)\, k_\mu\, a)(e^{it}) \;=\; [\Theta(e^{-it})\, k_\mu(e^{it})\, a]_+ \;=\; k_\mu(e^{it})\, \Theta(\mu)\, a$$

and that $(k_\mu , k_\nu) \;=\; (1 - \bar\nu\, \mu)^{-1}$ $(\,|\mu| < 1\,,\; |\nu| < 1)$. Thus we infer that condition $(*)$ is equivalent to the condition that the kernel

$$K(\mu\,, \nu) \;=\; (1 - \bar\nu\, \mu)^{-1}\, (\Theta(\nu)^*\, \Theta(\mu) \,-\, \eta^2\, I) \qquad (\,|\mu| < 1\,,\; |\nu| < 1\,)$$

be positive definite, i.e.

$$\sum_{i=1}^{N} \sum_{j=1}^{N} (K(\mu_i\,, \mu_j)\, a_j \;,\; a_i) \;\geqq\; 0$$

for any finite set of points μ_i in the unit disc and vectors a_i in E.

Remark 3. Our Theorem has a rather immediate converse. Indeed if there exists an analytic function $D(\lambda)$ such that

$$D(\lambda)\, \Theta(\lambda) \;=\; I_{\mathfrak{E}} \quad \text{and} \quad \|D(\cdot)\|_\infty \;=\; 1/\eta \;<\; \infty$$

then we have for $u \in H^2_{\mathfrak{E}}$

$$u \;=\; [u]_+ \;=\; [D(e^{-it})\, \Theta(e^{-it})\, u(e^{it})]_+ \;=\; [D(e^{-it})\, [\Theta(e^{-it})\, u(e^{it})]_+]_+\,,$$

and hence

$$\eta^2\, \|u\|^2 \;\leqq\; \eta^2 \int_0^{2\pi} \|D(e^{-it})\, [\Theta(e^{-it})\, u(e^{it})]_+\|^2\, \frac{dt}{2\pi} \;\leqq\; \|T(\Theta)\, u\|^2\,.$$

Remark 4. From Propositions $1-3$ and Remark 3 it readily results the equality of the infima

$$\inf\{\, \|X^{-1}\|\, \|X\| :\; X^{-1}\, S(\Theta)\, X \;\text{ is an isometry}\,\}$$

and

$$\inf\{\, \|D(\cdot)\|_\infty :\; D(\lambda)\, \Theta(\lambda) \;=\; I_{\mathfrak{E}}\,\}\,.$$

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[7] We use this opportunity to correct some deficiencies of the paper [5].

Page 230: at the end of the 9th row change "$\hat{\Theta}_1$" for "$\hat{\Delta}_1$".

Page 231: insert between the 12th and 13th rows:

$$\text{"} (\delta)_0 \quad B' A_* + C^{-1} B = -\hat{\Delta}_1 D \text{"}.$$

Page 231: 8th row from below, insert "$B' = 0$".

Page 254: papers [6]—[8] in the References have the authors "B. Sz.-Nagy—C. Foiaș".