

ON CONTRACTIVE MAPPINGS

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ABSTRACT. The Cauchy condition for convergence of a contractive iteration in a complete metric space is replaced by an equivalent functional condition. Many generalizations of the Banach-Neumann contractive mapping principle follow.

Introduction. Let X be a complete metric space and let $f: X \rightarrow X$ be a contractive mapping or contraction, i.e.,

$$d(f(x), f(y)) < d(x, y) \quad \text{for all } x, y \text{ in } X.$$

Let x_0 be a chosen point in X and set $x_n = f(x_{n-1})$ for $n > 0$. Criteria for the sequence of iterates $\{x_n\}$ to be Cauchy are then of interest, for if it is Cauchy one can easily prove it converges to a unique fixed point x_∞ of the map f on X .

Many papers have presented such criteria, especially since the important paper of Rakotch [1]. In this note we first give in §1 a necessary and sufficient condition that a sequence of iterates be convergent. This condition is used in §2 to provide a criterion for the convergence of the iteration from an arbitrary initial point in X . The condition is then restated in a more familiar functional form, and some corollaries are then drawn in §3.

1. Main theorems. The following results, while stated for complete metric spaces, are readily extended to separated complete uniform spaces, as in [2, p. 197].

We shall use the following notation. For any pair of sequences x_n and y_n in X with $x_n \neq y_n$, we write

$$d_n = d(x_n, y_n) \quad \text{and} \quad \Delta_n = d(f(x_n), f(y_n))/d_n.$$

We then have the following theorem.

THEOREM 1.1. *Let X be a complete metric space. Let $f: X \rightarrow X$ with $d(f(x), f(y)) < d(x, y)$, for all x, y in X . Let $x_0 \in X$ and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \rightarrow x_\infty$ in X , with x_∞ a unique fixed point of f , iff for any*

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two subsequences x_{h_n} and x_{k_n} with $x_{h_n} \neq x_{k_n}$, we have that

$$\Delta_n \rightarrow 1 \quad \text{only if} \quad d_n \rightarrow 0.$$

PROOF. First assume that $x_n \rightarrow x_\infty$ in X and let x_{h_n} and x_{k_n} be any two subsequences. Then clearly $d_n = d(x_{h_n}, x_{k_n}) \rightarrow 0$ and so the condition is satisfied.

Next assume the condition is satisfied for a given initial point x_0 in X . Then $d_n = d(x_n, x_{n+1})$ is a decreasing sequence of nonnegative numbers and so approaches some $\varepsilon \geq 0$. Assume $\varepsilon > 0$. Then letting $h_n = n$ and $k_n = n+1$, we have $d_n \rightarrow \varepsilon > 0$ while $\Delta_n \rightarrow 1$. So the condition is violated. Thus $d(x_n, x_{n+1}) \rightarrow 0$.

Now assume the given sequence of iterates $\{x_n\}$ is not Cauchy. Then there exists some $\varepsilon > 0$ such that every tail $\{x_n\}_{n \geq N}$ of the sequence has diameter $D_N = \sup_{n, m \geq N} d(x_n, x_m) > \varepsilon$. Given this ε , we will construct a pair of subsequences violating the condition.

For any $n > 0$, let N_n be so large that $d(x_m, x_{m+1}) < 1/n$ for all $m \geq N_n$, as is possible since $d(x_m, x_{m+1}) \rightarrow 0$. Let $h_n \geq N_n$ be the lowest integer such that for some $k_n > h_n$, $d(x_{h_n}, x_{k_n}) > \varepsilon$. Such pairs exist by the above diameter condition. Next choose k_n to be the least such integer above h_n . Then either $k_n - 1 = h_n$ or else $d(x_{h_n}, x_{k_n-1}) \leq \varepsilon$. In either case we have $\varepsilon \leq d_n = d(x_{h_n}, x_{k_n}) < \varepsilon + 1/n$.

Moreover, using the triangular inequality on the contraction, we have

$$1 \geq \Delta_n = \frac{d(f(x_{h_n}), f(x_{k_n}))}{d_n} \geq \frac{d_n - 2/n}{d_n}.$$

So $\Delta_n \rightarrow 1$ while $d_n \rightarrow \varepsilon > 0$, again violating the condition.

So $\{x_n\}$ must be a Cauchy sequence in X and as X is complete, we have $x_n \rightarrow x_\infty$ for some x_∞ in X . Then by the usual arguments, x_∞ is a unique fixed point of f and the proof is complete.

The above proof is essentially that of Theorem 6.1 in [2, p. 197] with the added observation that on a particular iteration, the condition is necessary. A close inspection of the proof shows that the following slight improvement was really proven.

COROLLARY 1.2. *Let X be a complete metric space. Let $f: X \rightarrow X$ with $d(f(x), f(y)) < d(x, y)$ for all x, y in X . Let $x_0 \in X$ and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \rightarrow x_\infty$ in X , with x_∞ a unique fixed point of f , iff for any two subsequences x_{h_n} and x_{k_n} with $x_{h_n} \neq x_{k_n}$, we have that $\Delta_n \rightarrow 1$, with d_n decreasing, only if $d_n \rightarrow 0$.*

PROOF. Same as Theorem 1.1.

We now wish to convert this sequential condition to the more customary functional form. Following Rakotch [1], we define a class of test functions more general than his.

DEFINITION. S is the class of functions $\alpha: \mathbf{R}^+ \rightarrow [0, 1]$ with

- (i) $\mathbf{R}^+ = \{t \in \mathbf{R} \mid t > 0\}$,
- (ii) $\alpha(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$.

Notes. (1) We do not assume that α is continuous in any sense.

(2) We only require that if α gets near one, it does so only near zero.

(3) Using Corollary 1.2, we could replace property (ii) with (ii') $\alpha(t_n) \rightarrow 1$ with t_n decreasing implies $t_n \rightarrow 0$.

THEOREM 1.3. Let $f: X \rightarrow X$ be a contraction on a complete metric space. Let $x_0 \in X$ and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \rightarrow x_\infty$, where x_∞ is a unique fixed point of f in X , iff there exists an α in S such that for all n, m ,

$$d(f(x_n), f(x_m)) \leq \alpha(d(x_n, x_m)) \cdot d(x_n, x_m).$$

PROOF. We need only show that the existence of such an α in S is equivalent to the sequential condition of Theorem 1.1. First assume such an α exists. Let x_{h_n} and x_{k_n} be subsequences with $x_{h_n} \neq x_{k_n}$. Assume that $\Delta_n \rightarrow 1$. Then it follows from the above inequality that $\alpha(d(x_{h_n}, x_{k_n})) \rightarrow 1$. But then since $\alpha \in S$, we have $d(x_{h_n}, x_{k_n}) \rightarrow 0$.

Next assume that the sequential condition holds. Define $\alpha: \mathbf{R}^+ \rightarrow \mathbf{R}$ as follows:

$$\alpha(t) = \sup \left\{ \frac{d(f(x_n), f(x_m))}{d(x_n, x_m)} \mid d(x_n, x_m) \geq t \right\}.$$

Since f is a contraction, the quotients are all below 1 and so α is defined for all $t > 0$ and $\alpha \leq 1$. Now assume that $\alpha(t_n) \rightarrow 1$ for $t_n \in \mathbf{R}^+$. We may further assume without loss of generality that $1 - 1/n < \alpha(t_n) \leq 1$. We must show $t_n \rightarrow 0$. But $\alpha(t_n)$ is the above least upper bound. So there is for each $n > 0$ a pair x_{h_n}, x_{k_n} in $\{x_n\}$ with

$$d(x_{h_n}, x_{k_n}) \geq t_n$$

and

$$1 - \frac{1}{n} < \frac{d(f(x_{h_n}), f(x_{k_n}))}{d(x_{h_n}, x_{k_n})} \leq \alpha(t_n).$$

So the sequence $\Delta_n \rightarrow 1$. But then by the sequential condition of Theorem 1.1, $d(x_{h_n}, x_{k_n}) \rightarrow 0$. So $t_n \rightarrow 0$, as was to be shown. This completes the theorem.

2. A general convergence criterion. We may now apply the results of §1 to obtain a criterion for convergence of the iteration from an arbitrary

starting point. Unfortunately, some generality is lost and the universal condition, while still sufficient, is no longer necessary.

THEOREM 2.1. *Let $f: X \rightarrow X$ be a contraction of a complete metric space satisfying*

$$d(f(x), f(y)) \leq \alpha(d(x, y)) \cdot d(x, y)$$

where $\alpha \in S$. Then for any choice of initial point x_0 , the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to the unique fixed point x_∞ of f in X .

PROOF. Simply apply Theorem 1.3 to any initial point x_0 .

It should be noted that Theorem 2.1 is essentially the same as Corollary 1 of [2, p. 197]. The improvement afforded by the above presentation is that the essential character of the condition on each orbit is brought out in §1.

3. Some corollaries. The usefulness of the contraction mapping principle has prompted many generalizations of the original result, which is Theorem 2.1 with $\alpha \in S$ being simply a constant function. A few of these are listed below, all of them following easily from Theorem 2.1 (which, as pointed out in §2, is really little more than Corollary 1 of [2, p. 197]).

COROLLARY 3.1 (RAKOTCH [1, p. 463]). *If $f: X \rightarrow X$ is a contraction of a complete metric space satisfying $d(f(x), f(y)) \leq \alpha(d(x, y)) \cdot d(x, y)$ where $\alpha: \mathbf{R}^+ \rightarrow [0, 1)$ and is monotone decreasing, then for any choice of x_0 in X , the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to a unique fixed point x_∞ of f in X .*

PROOF. Such an α is clearly in the class S .

COROLLARY 3.2. *If $f: X \rightarrow X$ is a contraction of a complete metric space satisfying $d(f(x), f(y)) \leq \alpha(d(x, y)) \cdot d(x, y)$ where $\alpha: \mathbf{R}^+ \rightarrow [0, 1)$ and is monotone increasing, then for any choice of x_0 in X , the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to a unique fixed point x_∞ of f in X .*

PROOF. As in Corollary 3.1.

COROLLARY 3.3 (BOYD-WONG [4, p. 331]). *If $f: X \rightarrow X$ is a contraction of a complete metric space satisfying $d(f(x), f(y)) \leq \alpha(d(x, y)) \cdot d(x, y)$ where $\alpha: \mathbf{R}^+ \rightarrow [0, 1)$ and is continuous, then for any choice of x_0 in X , the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to a unique fixed point x_∞ of f in X .*

PROOF. As in Corollary 3.1.

REMARK 1. The last corollary is derived from Theorem 1 of [3, p. 459]. This latter theorem also seems related to Theorem 2.1 and is a corollary,

if we were to use the slightly stronger form provided by the Note (3) on the class of functions S in §1.

REMARK 2. A paper investigating the relations among these various corollaries will shortly be submitted to the Journal for Mathematical Analysis and Applications.

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