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ON L^1 CONVERGENCE OF CERTAIN COSINE SUMS

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ABSTRACT. Rees and Stanojević introduced a new class of modified cosine sums $\{g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a(k) + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a(j)\cos kx\}$ and found a necessary and sufficient condition for integrability of these modified cosine sums. Here we show that to every classical cosine series f with coefficients of bounded variation, a Rees-Stanojević cosine sum g_n can be associated such that g_n converges to f pointwise, and a necessary and sufficient condition for L^1 convergence of g_n to f is given. As a corollary to that result we have a generalization of the classical result of this kind. Examples are given using the well-known integrability conditions.

Theorem A gives a necessary and sufficient condition for a sine series with coefficients of bounded variation and converging to zero to be the Fourier series of its sum, or equivalently, for its sum to be integrable. Theorem B shows that if such a series is a Fourier series then its convergence is "good", that is, convergence in the L^1 metric.

THEOREM A [1]. *Let* $f(x) = \sum_{n=1}^{\infty} b(n)\sin nx$ *where* $\Delta b(n) \geqslant 0$ $[\Delta b(n) = b(n) - b(n+1)]$ *and* $\lim_{n\to\infty} b(n) = 0$. *Then* $f \in L^1[0,\pi]$ *or, equivalently,* $\sum_{n=1}^{\infty} b(n)\sin nx$ *is the Fourier series of* f *if and only if* $\sum_{n=1}^{\infty} |\Delta b(n)|\log n < \infty$.

THEOREM B [1]. *Let* $f(x)$ *be as in Theorem* A. *If* $f \in L^1[0,\pi]$ *then* $\sum_{k=1}^{n} b(k)\sin kx$ *converges to* f *in the* L^1 *metric.*

There is no known analogue of Theorem A for the cosine series. Theorems C and D only give sufficient conditions for the cosine series to be the Fourier series of its sum.

In what follows we will denote by C the cosine series

$$\frac{1}{2}a(0) + \sum_{n=1}^{\infty} a(n)\cos nx$$

where $\lim_{n\to\infty} a(n) = 0$ and $\sum_{n=1}^{\infty} |\Delta a(n)| < \infty$. Partial sums of C will be denoted by $S_n(x)$, and $f(x) = \lim_{n\to\infty} S_n(x)$.

THEOREM C [1]. *If* $\sum_{n=1}^{\infty} |\Delta a(n)|\log n < \infty$, *then* $f \in L^1[0,\pi]$ *or, equivalently, C is the Fourier series of* f.

THEOREM D [1]. *If* $\sum_{n=1}^{\infty} |\Delta^2 a(n)|(n+1) < \infty$, *then* $f \in L^1[0,\pi]$ *or, equivalently, C is the Fourier series of* f.

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[1] Portions of these results appear in a doctoral thesis of John W. Garrett at the University of Missouri-Rolla in 1974.

Theorem E is related to Theorem B. It shows that the classical cosine series is not as "well behaved" as the classical sine series.

THEOREM E [1]. *If* $\sum_{n=1}^{\infty} |\Delta^2 a(n)|(n+1) < \infty$, *then* S_n *converges to* f *in the* L^1 *metric if and only if* $\lim_{n\to\infty} a(n)\log n = 0$.

Rees and Stanojević introduced a new type of cosine sum and obtained a necessary and sufficient condition for integrability of its limit.

THEOREM F [2]. *Let*

$$g_n^*(x) = \sum_{k=1}^{n} \left[\frac{a(k)}{2} + \sum_{j=k}^{n} a(j)\cos kx \right]$$

where $\lim_{n\to\infty} a(n) = 0$ *and* $\Delta a(n) \geq 0$. *Then*

(i) $g^*(x) = \lim_{n\to\infty} g_n^*(x)$ *exists for* $x \in (0,\pi]$, *and*

(ii) $g^* \in L^1[0,\pi]$ *if and only if* $\sum_{n=1}^{\infty} a(n) < \infty$.

This paper proves an analogue of Theorem B for this type of cosine sum. Indeed, these modified cosine sums approximate their limit "better" than the classical cosine series since they converge in the L^1 metric to their limit when the classical cosine series may not.

LEMMA 1. *Let*

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a(k) + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a(j)\cos kx.$$

Then $\lim_{n\to\infty} g_n(x) = f(x)$, *for* $x \in (0,\pi]$.

It will be shown in the proof of this lemma that

$$g_n(x) = S_n(x) - a(n+1)D_n(x).$$

We prefer the form given in the lemma, however, since it emphasizes better its use in [2].

PROOF. Denoting the Dirichlet kernel by $D_n(x)$ we get

$$\begin{aligned}\lim_{n\to\infty} g_n(x) &= \lim_{n\to\infty}\left[\frac{1}{2}\sum_{k=0}^{n}\Delta a(k) + \sum_{k=1}^{n}\sum_{j=k}^{n}\Delta a(j)\cos kx\right] \\ &= \lim_{n\to\infty}\left[\frac{a(0)}{2} + \sum_{k=1}^{n} a(k)\cos kx - a(n+1)D_n(x)\right] \\ &= \lim_{n\to\infty}\left[S_n(x) - a(n+1)D_n(x)\right] = f(x),\end{aligned}$$

$x \in (0,\pi]$ since $\lim_{n\to\infty} S_n(x) = f(x)$ and $\lim_{n\to\infty} a(n+1)D_n(x) = 0$, $x \in (0,\pi]$.

THEOREM 1. *Let* g_n *be as defined in Lemma* 1. *Then* g_n *converges to* f *in the* L^1 *metric if and only if given* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that* $\int_0^{\delta} |\sum_{k=n+1}^{\infty} \Delta a(k)D_k(x)| < \varepsilon$ *for all* $n \geq 0$.

PROOF. For the "if" part let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$\int_0^\delta |\sum_{k=n+1}^\infty \Delta a(k) D_k(x)| < \varepsilon/2$ for all $n \geq 0$. Then

$$\begin{aligned}
\int_0^\pi |f - g_n| &= \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta a(k) D_k(x) \right| \\
&= \int_0^\delta \left| \sum_{k=n+1}^\infty \Delta a(k) D_k(x) \right| + \int_\delta^\pi \left| \sum_{k=n+1}^\infty \Delta a(k) D_k(x) \right| \\
&< \frac{\varepsilon}{2} + \sum_{k=n+1}^\infty |\Delta a(k)| \int_\delta^\pi |D_k(x)| \\
&\leq \frac{\varepsilon}{2} + \sum_{k=n+1}^\infty |\Delta a(k)| \int_\delta^\pi \csc \tfrac{1}{2}x \\
&= \frac{\varepsilon}{2} + \sum_{k=n+1}^\infty |\Delta a(k)| [-2 \log|\csc \delta/2 - \cot \delta/2|] < \varepsilon
\end{aligned}$$

for sufficiently large n since $\sum_{k=0}^\infty |\Delta a(k)| < \infty$.

For the "only if" part, let $\varepsilon > 0$. Then there exists an integer M such that $\int_0^\pi |f(x) - g_n(x)| < \varepsilon/2$ if $n \geq M$. That is, $\int_0^\pi |\sum_{k=n}^\infty \Delta a(k) D_k(x)| < \varepsilon/2$ if $n \geq M$. Now if $\sum_{k=0}^M |\Delta a(k)| = 0$, then for $n > M$, $\int_0^\pi |\sum_{k=n}^\infty \Delta a(k) D_k(x)| < \varepsilon/2 < \varepsilon$ and, for $0 \leq n \leq M$,

$$\int_0^\pi \left| \sum_{k=n}^\infty \Delta a(k) D_k(x) \right| = \int_0^\pi \left| \sum_{k=M+1}^\infty \Delta a(k) D_k(x) \right| < \varepsilon/2 < \varepsilon.$$

If $\sum_{k=0}^M |\Delta a(k)| \neq 0$, let $\delta = \varepsilon/2M \sum_{k=0}^M |\Delta a(k)|$. For $n \geq M$,

$$\int_0^\delta \left| \sum_{k=n}^\infty \Delta a(k) D_k(x) \right| \leq \int_0^\pi \left| \sum_{k=n}^\infty \Delta a(k) D_k(x) \right| < \varepsilon/2 < \varepsilon.$$

For $0 \leq n < M$,

$$\begin{aligned}
\int_0^\delta \left| \sum_{k=n}^\infty \Delta a(k) D_k(x) \right| &\leq \int_0^\delta \left| \sum_{k=n}^{M-1} \Delta a(k) D_k(x) \right| + \int_0^\delta \left| \sum_{k=M}^\infty \Delta a(k) D_k(x) \right| \\
&\leq \int_0^\delta \sum_{k=n}^{M-1} k |\Delta a(k)| + \int_0^\pi \left| \sum_{k=M}^\infty \Delta a(k) D_k(x) \right| \\
&< \delta \sum_{k=0}^{M-1} k |\Delta a(k)| + \frac{\varepsilon}{2} \\
&\leq \delta M \sum_{k=0}^{M-1} |\Delta a(k)| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

So given $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_0^\delta |\sum_{k=n}^\infty \Delta a(k) D_k(x)| < \varepsilon$ for all $n \geq 0$.

If $\lim_{n\to\infty} \int_0^\pi |f(x) - g_n(x)| = 0$, it is clear that $f \in L^1[0, \pi]$.

$$\int_0^\pi |f(x)| \leq \int_0^\pi |f(x) - g_n(x)| + \int_0^\pi |g_n(x)| < \infty$$

since $g_n(x)$ is a finite cosine sum.

COROLLARY 1. *If for* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that*

$\int_0^\delta |\sum_{k=n}^\infty \Delta a(k) D_k(x)| < \varepsilon$ *for all* $n \geq 0$, *then* S_n *converges to* f *in the* L^1 *metric if and only if* $\lim_{n\to\infty} a(n) \log n = 0$.

Proof. Using g_n as defined in Lemma 1, we get

$$\begin{aligned} \int_0^\pi |f(x) - S_n(x)| &= \int_0^\pi |f(x) - g_n(x) + g_n(x) - S_n(x)| \\ &\leq \int_0^\pi |f(x) - g_n(x)| + \int_0^\pi |g_n(x) - S_n(x)| \\ &= \int_0^\pi |f(x) - g_n(x)| + \int_0^\pi |a(n+1) D_n(x)|. \end{aligned}$$

Also

$$\begin{aligned} \int_0^\pi |a(n+1) D_n(x)| &= \int_0^\pi |g_n(x) - S_n(x)| \\ &\leq \int_0^\pi |f(x) - S_n(x)| + \int_0^\pi |f(x) - g_n(x)|. \end{aligned}$$

Since $\int_0^\pi |a(n+1) D_n(x)|$ behaves like $a(n+1)\log n$ for large values of n, and $\lim_{n\to\infty} \int_0^\pi |f(x) - g_n(x)| = 0$, the corollary is proved.

The following examples show that known sufficient conditions for integrability of the limit of a cosine series are also sufficient for the L^1 convergence of g_n to that limit, since they imply the necessary and sufficient condition from Theorem 1.

Example 1. Let $\sum_{n=1}^\infty |\Delta^2 a(n)|(n+1) < \infty$. Then g_n converges to f in the L^1 metric space. Denoting the Fejér kernel by $F_n(x)$, we get

$$\begin{aligned} \int_0^\pi |f(x) - g_n(x)| &= \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta a(k) D_k(x) \right| \\ &= \int_0^\pi \left| \sum_{k=n}^\infty (k+1)\Delta^2 a(k) F_k(x) - (n+1)\Delta a(n) F_n(x) \right| \\ &\leq \sum_{k=n+1}^\infty (k+1)|\Delta^2 a(k)| \int_0^\pi F_k(x) + (n+1)|\Delta a(n)| \int_0^\pi F_n(x) \\ &\leq \pi \sum_{k=n}^\infty (k+1)|\Delta^2 a(k)| \end{aligned}$$

since $\int_0^\pi F_k(x) = \pi/2$ and

$$\begin{aligned} (n+1)|\Delta a(n)| &= \sum_{k=n}^\infty (n+1)[|\Delta a(k)| - |\Delta a(k+1)|] \\ &\leq \sum_{k=n}^\infty (n+1)|\Delta^2 a(k)| \leq \sum_{k=n}^\infty (k+1)|\Delta^2 a(k)|. \end{aligned}$$

Since $\sum_{n=1}^\infty (n+1)|\Delta^2 a(n)| < \infty$, then $\lim_{n\to\infty} \int_0^\pi |f(x) - g_n(x)| = 0$.

Example 2. Let $\sum_{k=1}^\infty |\Delta a(k)| \log k < \infty$. Then g_n converges to f in the L^1 metric space, for

$$\int_0^\pi |f(x) - g_n(x)| = \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta a(k) D_k(x) \right|$$
$$\leq \sum_{k=n+1}^\infty |\Delta a(k)| \int_0^\pi |D_k(x)|.$$

Since $\int_0^\pi |D_k(x)|$ behaves like $\log k$ for large k, and $\sum_{k=1}^\infty |\Delta a(k)| \log k < \infty$, we get $\lim_{n\to\infty} \int_0^\pi |f(x) - g_n(x)| = 0$. As a corollary of this example we have the well-known Theorem E.

Theorems C and D can be combined as in the following lemma.

LEMMA 2. *Let* $a(n) = b(n) + c(n)$ *where* $\sum_{n=1}^\infty |\Delta b(n)| \log n < \infty$, $\sum_{n=1}^\infty |\Delta^2 c(n)|(n+1) < \infty$, *and* $\lim_{n\to\infty} b(n) = \lim_{n\to\infty} c(n) = 0$. *Then* $f \in L^1[0,\pi]$.

It is interesting to note that in Lemma 2 we may have

$$\sum_{n=1}^\infty |\Delta a(n)| \log n = \sum_{n=1}^\infty |\Delta^2 a(n)|(n+1) = \infty.$$

EXAMPLE 3. Let $f(x)$ be as in Lemma 2. Then g_n converges to f in the L^1 metric. This follows from Examples 1 and 2, writing $a(n) = b(n) + c(n)$.

Stanojević combined Theorems C and D in a different way.

THEOREM G [3]. *Let* $a(n) = \alpha(n)\beta(n)$ *where* $\sum_{n=1}^\infty |\Delta\alpha(n)| < \infty$, $\sum_{n=1}^\infty |\Delta^2 \beta(n)|(n+1) < \infty$, $|\beta(n)| \leq M$, *and* $\sum_{n=1}^\infty |\beta(n)\Delta\alpha(n)| \log(n) < \infty$. *Then* $f \in L^1[0,\pi]$.

EXAMPLE 4. Let $f(x)$ be as in Theorem G. Then g_n converges to f in the L^1 metric. We get

$$\int_0^\pi |f(x) - g_n(x)| \leq M \sum_{k=n}^\infty |\beta(k)\Delta\alpha(k)| \log k$$
$$+ N \sum_{k=n}^\infty (k+1)\{|\alpha(k+1)\Delta^2\beta(k)| + |\Delta\alpha(k+1)\Delta\beta(k+1)|\}.$$

Since both series converge, we have $\lim_{n\to\infty} \int_0^\pi |f(x) - g_n(x)| = 0$.

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