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On Convergence of q-Homotopy Analysis Method

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Abstract

The convergence of q- homotopy analysis method (q-HAM) is studied in the present paper. It is proven that under certain conditions the solution of the equation: $(1 - nq)[L(\emptyset(t,q)) - L(u_0)] - qhN[\emptyset(t,q)] = 0$ associated with the original problem exists as a power series in q.So,under a special constraint the q-homotopy analysis method does converge to the exact solution of nonlinear problems. An error estimate is also provided. The theorems outlined in the paper shows that the convergence of the q- homotopy analysis method is more accurate than the convergence of the homotopy analysis method (HAM). Illustrative examples are presented to illustrate the effectiveness of the theoretical results.

Keywords: q-Homotopy analysis method, Convergence

1. Introduction

In a series of papers[3-14], Liao developed and applied the homotopy analysis method(HAM) to deal with a lot of nonlinear problems. The HAM provides a simple way to ensure the convergence of a solution in a series form under certain conditions. The Homotopy Analysis Method (HAM) is based on homotopy, a fundamental concept in topology. Briefly, in HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of the problems to be considered. An auxiliary linear operator is chosen to construct such kind of continuous mapping and an auxiliary parameter is used to ensure convergence of the solution series. The method enjoys great freedom in choosing initial approximation and auxiliary linear operator.In 2004, Liao published the book [15] in which he summarized the basic ideas of the homotopy analysis method and gave the details of his approach both in the theory and on a large number of practical examples.

M. A. El-Tawil and S.N. Huseen [2] proposed a method namely q-homotopy analysis method (q-HAM) which is a more general method of HAM.The essential idea of this method is to introduce a homotopy parameter, say q, which varies from 0 to $1/n$, $n \geq 1$ and a nonzero auxiliary parameter h. At $q = 0$, the system of equations usually has been reduced to a simplified form which normally admits a rather simple solution. As q gradually increases continuously toward $1/n$, the system goes through a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation. Eventually at $q = 1/n$, the system takes the original form of the equation and the final stage of the deformation gives the desired solution.

2. The q-Homotopy Analysis Method

To illustrate the basic ideas of the q-homotopy analysis method (q-HAM), consider the nonlinear boundary value problem

$$N[u(t)] = 0 \,;\, t \in \Omega\,, \qquad B\left(u(t), \frac{du}{dt}\right) = 0 \,;\, t \in \mathbb{\Gamma} \tag{1}$$

where $u(t)$ defined over the region Ω is the function to be solved under the boundary constraints in B defined over the boundary $\mathbb{\Gamma}$ of Ω . The q- homotopy analysis technique defines a homotopy $\emptyset(t,q) : R \times [0,\frac{1}{n}] \rightarrow R$ so that

$$H(\emptyset,q) = (1 - nq)[L(\emptyset(t,q)) - L(u_0)] - qhN[\emptyset(t,q)] = 0 \tag{2}$$

where $q \in [0, \frac{1}{n}], n \geq 1$ denotes the so-called embedded parameter, $h \neq 0$ is an auxiliary parameter, L is a suitable auxiliary linear operator, u_0 is an initial approximation of equation (1) satisfying exactly the boundary conditions . It is obvious from equation (2) that

$$H(\emptyset, 0) = L(\emptyset(t, 0)) - L(u_0), \ \ H(\emptyset, \tfrac{1}{n}) = \tfrac{h}{n} N[\emptyset(t, \tfrac{1}{n})] \quad (3)$$

As q moves from 0 to $1/n$, $\emptyset(t, q)$ moves from $u_0(t)$ to $u(t)$. In topology, this called a deformation and $L(\emptyset(t, q)) - L(u_0)$ and $N[\emptyset(t, q)]$ are said to be homotopic. The solution of equation (2) exists as a power series in q as we proved in theorem 3.1.

$$\emptyset(t, q) = u_0(t) + q u_1(t) + q^2 u_2(t) + \cdots = \sum_{k=0}^{\infty} u_k(t)\, q^k \quad (4)$$

The appropriate solutions of the coefficients $u_k(t)$ in (4) can be found from the homotopy deformation equations, see [2]. Hence, the approximate solution of equation (1) can be readily obtained as

$$u(t) = \lim_{q \to \frac{1}{n}} \emptyset(t, q) = \sum_{k=0}^{\infty} U_k(t, n) = \sum_{k=0}^{\infty} u_k(t) \left(\frac{1}{n}\right)^k \quad (5)$$

It was found that the auxiliary parameters h and n can adjust and control the convergence region and rate of homotopy series solutions. It should be noted that in the case of $n = 1$, in equation (2) the standard homotopy analysis method (HAM) can be reached.

3. Main Result

Theorem 3.1: The solution of equation (2) together with equation (1) exists as a power series in$q$, i.e. $\emptyset(t, q) = \sum_{k=0}^{\infty} \emptyset_k(t)\, q^k$ If the nonlinear operator preserves on the power series in$q$.

Proof:
At $q = 0$, then $L(\emptyset(t, 0)) = L(u_0)$, hence $\emptyset(t, 0) = u_0 = \emptyset_0$.
At $q \notin \{0, \frac{1}{n}\}$ and using Picard approximations we have:

$$(1 - nq)[L(\emptyset_1(t, q)) - L(\emptyset_0)] - qhN(\emptyset_0) = 0$$

Therefore

$$\emptyset_1(t,q) = \emptyset_0 + \frac{q}{1-nq} h\, L^{-1}[N(\emptyset_0)] = \emptyset_0 + q[1 + nq + (nq)^2 + \cdots]h\ L^{-1}[N(\emptyset_0)]$$
$$= \emptyset_0 + \textstyle\sum_{i=1}^{\infty} \emptyset_1^i\,(t) q^i$$

Where $\emptyset_1^i(t) = n^{i-1}\, h\, L^{-1}[N(\emptyset_0)]$

$$\emptyset_2(t,q) = \emptyset_0 + \frac{q}{1-nq} h\, L^{-1}[N(\emptyset_1)]$$
$$= \emptyset_0 + q[1 + nq + (nq)^2 + \cdots]h\ L^{-1}\left[\textstyle\sum_{j=0}^{\infty} \varphi_1^j\, q^j\right]$$
$$= \emptyset_0 + h\ L^{-1}[\varphi_1^0]q + h\ L^{-1}[\varphi_1^1 + n\varphi_1^0]q^2 + h\ L^{-1}[\varphi_1^2 + n\varphi_1^1 + n^2\varphi_1^0]q^3 + h\ L^{-1}[\varphi_1^3 + n\varphi_1^2 + n^2\varphi_1^1 + n^3\varphi_1^0]q^4 + \cdots + h\ L^{-1}[\varphi_1^{k-1} + n\varphi_1^{k-2} + n^2\varphi_1^{k-3} + \cdots + n^{k-1}\varphi_1^0]q^k + \cdots$$
$$= \emptyset_0 + \textstyle\sum_{j=1}^{\infty} \emptyset_2^j(t)\, q^j$$

Where $\emptyset_2^j(t) = h\, L^{-1} \sum_{i=0}^{j-1} n^{j-1-i}\varphi_1^i$ and $N[\emptyset_1(t,q)] = \sum_{j=0}^{\infty} \varphi_1^j\, q^j$.
Proceeding in this way, the m-th approximation $\emptyset_m(t,q)$ is given by:

$$\emptyset_m(t,q) = \emptyset_0 + \textstyle\sum_{j=1}^{\infty} \emptyset_m^j(t)\, q^j$$

Where $\emptyset_m^j(t) = h\, L^{-1} \sum_{i=0}^{j-1} n^{j-1-i}\varphi_{m-1}^i$ and $N[\emptyset_{m-1}(t,q)] = \sum_{j=0}^{\infty} \varphi_{m-1}^j\, q^j$.
Now, if $\lim_{m\to\infty} \emptyset_m^i(t) = \emptyset^i\,(t)$ then:
$\lim_{m\to\infty} \emptyset_m(t,q) = \sum_{j=0}^{\infty}(\lim_{m\to\infty} \emptyset_m^j(t) q^j = \sum_{j=0}^{\infty} \emptyset^j(t) q^j = \emptyset(t,q)$
which is a power series in q.

Theorem 3.2. Suppose that $A \subset R$ be a Banach space denoted with a suitable norm $\|.\|$. Assume also that the initial approximation $u_0(t)$ remains inside the ball of the solution $u(t)$. Taking $r \in R$ be a constant, then for a prescribed value of h and $0 < r < n$, If
$\|U_{k+1}(t,n)\| \leq \frac{r}{n} \|U_k(t,n)\|$ for all k,
Then the series solution $\sum_{k=0}^{\infty} U_k(t,n) = \sum_{k=0}^{\infty} u_k(t)\, \left(\frac{1}{n}\right)^k$ is convergent over the domain of definition of t.

Proof:
In compliance with the ratio test for the power series in q, the proof is clear. However, in order to give an estimate to the error of q-homotopy analysis method, we give the whole proof here.

Let $S_j(t,n)$ denote the sequence of partial sum of the series (5), we need to show that $S_j(t,n)$ is a Cauchy sequence in A. For this purpose, consider,

$$\left\|S_{j+1}(t,n)-S_j(t,n)\right\| = \left\|U_{j+1}(t,n)\right\| \leq \frac{r}{n}\left\|U_j(t,n)\right\| \leq \left(\frac{r}{n}\right)^2 \left\|U_{j-1}(t,n)\right\| \leq \cdots \left(\frac{r}{n}\right)^{j+1} \|U_0(t,n)\| \qquad (6)$$

For every $i,j \in N$, $j \geq i$, making use of (6) and the triangle inequality successively, we have,

$$\begin{aligned}\left\|S_j(t,n)-S_i(t,n)\right\| &= \left\|(S_j(t,n)-S_{j-1}(t,n)) + (S_{j-1}(t,n)-S_{j-2}(t,n)) + \cdots + (S_{i+1}(t,n)-S_i(t,n))\right\| \\ &\leq \left\|S_j(t,n)-S_{j-1}(t,n)\right\| + \left\|S_{j-1}(t,n)-S_{j-2}(t,n)\right\| + \cdots + \|S_{i+1}(t,n)-S_i(t,n)\| \\ &\leq \left(\frac{r}{n}\right)^j \|U_0(t,n)\| + \left(\frac{r}{n}\right)^{j-1} \|U_0(t,n)\| + \cdots + \left(\frac{r}{n}\right)^{i+1} \|U_0(t,n)\| \\ &= \left(\frac{r}{n}\right)^{i+1} \|U_0(t,n)\| \sum_{m=0}^{j-i-1} \left(\frac{r}{n}\right)^m = \frac{1-\left(\frac{r}{n}\right)^{j-i}}{1-\frac{r}{n}} \left(\frac{r}{n}\right)^{i+1} \|U_0(t,n)\| \qquad (7)\end{aligned}$$

Since $0 < r < n$ and $n \geq 1$, we get from (7)

$$\lim_{j,i\to\infty}\left\|S_j(t,n)-S_i(t,n)\right\| = 0 \qquad (8)$$

Therefore, $S_j(t,n)$ is a Cauchy sequence in the Banach space A, and this implies that the series solution (5) is convergent.

It should be noted that, the series solution $\sum_{k=0}^{\infty} U_k(t,n) = \sum_{k=0}^{\infty} u_k(t) \left(\frac{1}{n}\right)^k$ is divergent if

$\|U_{k+1}(t,n)\| > \frac{n}{r}\|U_k(t,n)\|$ for all k

Remark 3.3: As special case, If $n=1$ in above theorem , then we get the same result of theorem (1) in [16].

Theorem 3.4: Assume that the series solution $\sum_{j=0}^{\infty} U_j(t,n) = \sum_{k=0}^{\infty} u_j(t) \left(\frac{1}{n}\right)^j$ is convergent to the solution $u(t)$ for a prescribed value of h. If the truncated series

$$\sum_{j=0}^{M} U_j(t,n) = \sum_{j=0}^{M} u_j(t) \left(\frac{1}{n}\right)^j$$

is used as an approximation to the solution $u(t)$ of problem (1), then an upper bound for the error, $E_M(t)$ is estimated as

$$E_M(t) \leq \frac{\left(\frac{r}{n}\right)^{M+1}}{1-\frac{r}{n}} \|U_0(t,n)\| \qquad (9)$$

Proof:

Since the series $\sum_{j=0}^{M} U_j(t,n) = \sum_{j=0}^{M} u_j(t) \left(\frac{1}{n}\right)^j$ is an approximation to the solution $u(t)$,then:

$$u(t) = \lim_{j\to\infty} S_j\,(t,n)$$

Where $S_j(t,n)$ denote the sequence of partial sum of the series

$\sum_{j=0}^{M} U_j(t,n) = \sum_{j=0}^{M} u_j(t) \left(\frac{1}{n}\right)^j$

Making use of the inequality (7) of Theorem (3.2), we immediately obtain

$$\|u(t) - S_M(t,n)\| \leq \frac{\left(\frac{r}{n}\right)^{M+1}}{1-\frac{r}{n}} \|U_0(t,n)\| \qquad (10)$$

Remark 3.5: As special case, If $n = 1$ in above theorem , then we get the same result of theorem (3) in [16].

It should be noted that as n increases the upper bound for the error (E_M) is decrease, hence the convergence of q-HAM is faster than the convergence of HAM (q-HAM ; $n = 1$).

4. Illustrative Examples

Example 4.1: Consider the nonlinear differential equation

$$\frac{du}{dt} = u^2\,, \quad u(0) = 1 \qquad (11)$$

We choose $u_0(t) = 1$ and the linear operator $L[\emptyset(t,q)] = \frac{d\emptyset(t,q)}{dt}$.

We define a nonlinear operator as:

$$N[\emptyset(t,q)] = \frac{d\emptyset(t,q)}{dt} - \emptyset^2(t,q),$$

Now the solution of equation (2) together with equation (11) for $m \geq 1$ becomes:

$$\emptyset_m(t,q) = u_0 + \frac{qh}{1-nq} L^{-1}[N(\emptyset_{m-1})]$$

Hence:

$$\emptyset_1(t,q) = 1 - htq - hntq^2 - hn^2tq^3 - hn^3tq^4 - hn^4tq^5 + O[q]^6$$

$$\emptyset_2(t,q) = 1 - htq + ht(-h-n+ht)q^2 + \left(-2h^2nt - hn^2t + 2h^2nt^2 - \frac{h^3t^3}{3}\right)q^3 + (-3h^2n^2t - hn^3t + 3h^2n^2t^2 - h^3nt^3)q^4 + (-4h^2n^3t - hn^4t + 4h^2n^3t^2 - 2h^3n^2t^3)q^5 + O[q]^6$$

$$\emptyset_3(t,q) = 1 - htq + ht(-h-n+ht)q^2 - ht(-h-n+ht)^2q^3 + \frac{1}{3}ht(-9h^2n - 9hn^2 - 3n^3 + 18h^2nt + 9hn^2t - 3h^3t^2 - 9h^2nt^2 + 2h^3t^3)q^4 - \frac{1}{6}(ht(36h^2n^2 + 24hn^3 + 6n^4 - 72h^2n^2t - 24hn^3t + 2h^4t^2 + 24h^3nt^2 + 36h^2n^2t^2 - 3h^4t^3 - 16h^3nt^3 + 2h^4t^4))q^5 + O[q]^6$$

$$\emptyset_4(t,q) = 1 - htq + ht(-h-n+ht)q^2 - ht(-h-n+ht)^2q^3 + ht(-h-n+ht)^3q^4 - \frac{1}{15}(h(60h^3nt + 90h^2n^2t + 60hn^3t + 15n^4t - 180h^3nt^2 - 180h^2n^2t^2 - 60hn^3t^2 + 30h^4t^3 + 180h^3nt^3 + 90h^2n^2t^3 - 40h^4t^4 - 60h^3nt^4 + 13h^4t^5))q^5 + O[q]^6$$

⋮

$$\emptyset_m(t,q) = 1 + ht\sum_{i=1}^{m}(-1)^i(-h-n+ht)^{i-1}q^i + O[q]^{m+1}$$

Now:

$$\lim_{m\to\infty}\emptyset_m(t,q) = 1 + ht\sum_{i=1}^{\infty}(-1)^i(-h-n+ht)^{i-1}q^i$$

Which is a power series in q.

Example 4.2: Consider the nonlinear partial differential Burger's equation

$$u_t + uu_x = u_{xx}, \quad u(x,0) = 2x \tag{12}$$

That has been found to describe various kind of phenomena, such as a mathematical model of turbulence and the approximate theory of the flow through a shock wave traveling in a viscous fluid [1].

The exact solution of this problem is known to be

$$u(x,t) = \frac{2x}{1+2t} \tag{13}$$

For q- HAM solution we choose the linear operator :

$$L[\emptyset(x,t;q)] = \frac{\partial\emptyset(x,t;q)}{\partial t} \tag{14}$$

with the property :

$$L[c_1] = 0,$$

where c_1 is constant. Using initial approximation $u_0(x,t) = 2x$,
we define a nonlinear operator as

$$N[\emptyset(x,t;q)] = \frac{\partial\emptyset(x,t;q)}{\partial t} + \emptyset(x,t;q)\frac{\partial\emptyset(x,t;q)}{\partial x} - \frac{\partial^2\emptyset(x,t;q)}{\partial x^2}$$

We construct the zero order deformation equation:

$$(1-nq)L[\emptyset(x,t;q) - u_0(x,t)] = qhN[\emptyset(x,t;q)].$$

The m^{th} order deformation equation is:

$$L[u_m(x,t) - k_m u_{m-1}(x,t)] = hR_m(\vec{u}_{m-1}(x,t)) \tag{15}$$

with the initial conditions for $m \geq 1$

$$u_m(x,0) = 0 \tag{16}$$

where:

$$k_m = \begin{cases} 0 & m \leq 1 \\ n & otherwise \end{cases}$$

and

$$R_m(\vec{u}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\emptyset(x,t;q)]}{\partial q^{m-1}}\Big|_{q=0}$$
$$= \frac{\partial u_{m-1}(x,t)}{\partial t} + \sum_{i=0}^{m-1} u_i(x,t)\frac{\partial}{\partial x}u_{m-1-i}(x,t) - \frac{\partial^2}{\partial x^2}u_{m-1}(x,t)$$

Now the solution of equation (12) for $m \geq 1$ becomes

$$u_m(x,t) = k_m u_{m-1}(x,t) + h\int R_m(\vec{u}_{m-1}(x,s))ds + c_1$$

where the constant of integration c_1 is determined by the initial conditions (16).
Then, the components of the solution using q- HAM are:

$$u_m(x,t) = 4ht(n + h(1+2t))^{m-1}x\,, \quad \text{for } m = 1,2,3,\ldots$$

Hence the series solution expression by q- HAM can be written in the form:

$$u(x,t;n;h) \cong U_M(x,t;n;h) = \sum_{i=0}^{M} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \quad (17)$$

Equation (17) is an approximate solution to the problem (12) in terms of the convergence parameters h and n . To find the valid region of h , the h-curves given by the 10th order q-HAM approximation at different values of x , t and n are drawn in figures (1,2,3and 4). These figures show the interval of h at which the value of $U_{10}(x,t;n)$ is constant at certain values of x , t and n .We choose the horizontal line parallel to $x - axis$ (h) as a valid region which provides us with a simple way to adjust and control the convergence region of the series solution (17). From these figures , the valid intersection region of h for the values of x , t and n in the curves becomes larger as n increase. Figure (5) show the h-curves given by the 30th order q-HAM approximation at different values of x , t with $n = 100$.
Figure (6) show the comparison between U_{10} of HAM and U_{10} of q-HAM using different values of n with the exact solution (13), which indicates that the speed of convergence for q-HAM with $n > 1$ is faster in comparison with $n = 1$.
The Absolute errors of the 10th order solutions q-HAM approximate at $x = 1$ using different values of $n > 1$ compared with 10th order solutions HAM approximate at $x = 1$ are calculated by the formula

$$\text{Absolut Error} = \left|u_{exact} - u_{apprax}\right| \quad (18)$$

Figures (7,8,9 and 10) show that the series solution obtained by HAM is more accurate at $(0 < t \le 0.4)$ but at larger t the series solutions obtained by q-HAM at $n > 1$ converge faster than $n = 1$(HAM).

It should be noted that $\lim_{m\to\infty}\frac{|U_{m+1}|}{|U_m|} = |n + h(1+2t)| < n$. Thus, in accordance with Theorem 3.2, this holds for the values of $\frac{-1}{2} < t < \frac{-2n-h}{2h}$. It is clear that for the prescribed values of h, the range of t increase with the increasing of n.

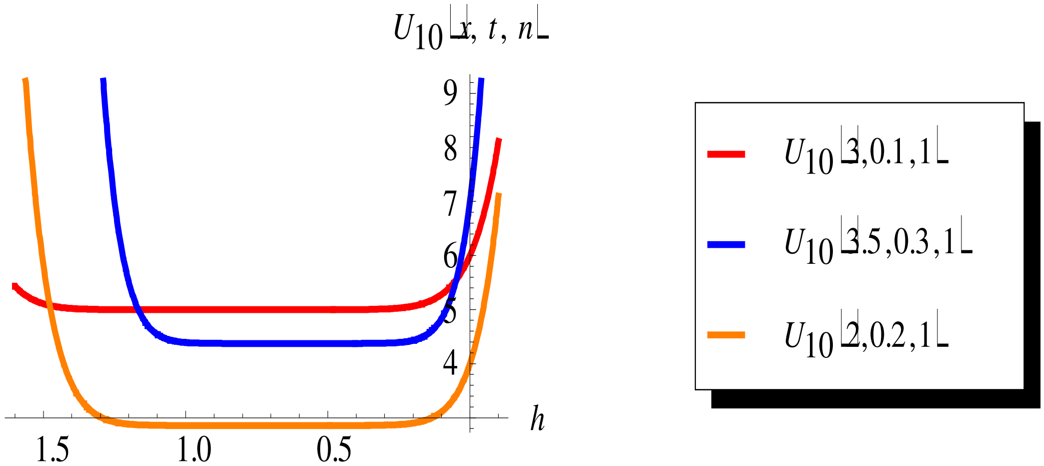

Figure (1) : h - curve for the HAM (q-HAM; $n = 1$) approximation solution $U_{10}(x, t; 1)$ of problem (12) at different values of x and t.

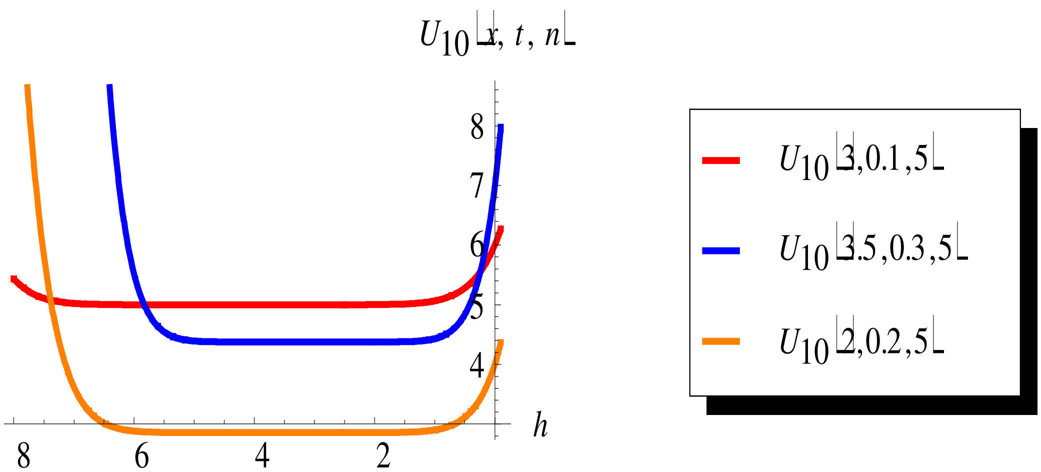

Figure (2) : h - curve for the (q-HAM; $n = 5$) approximation solution $U_{10}(x, t; 5)$ of problem (12) at different values of x and t.

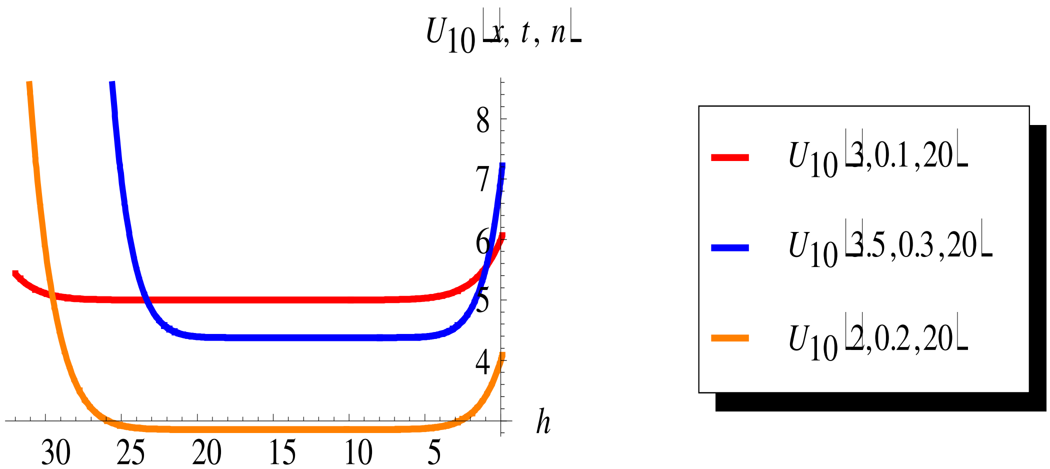

Figure (3) : h - curve for the (q-HAM; $n = 20$) approximation solution $U_{10}(x, t; 20)$ of problem (12) at different values of x and t.

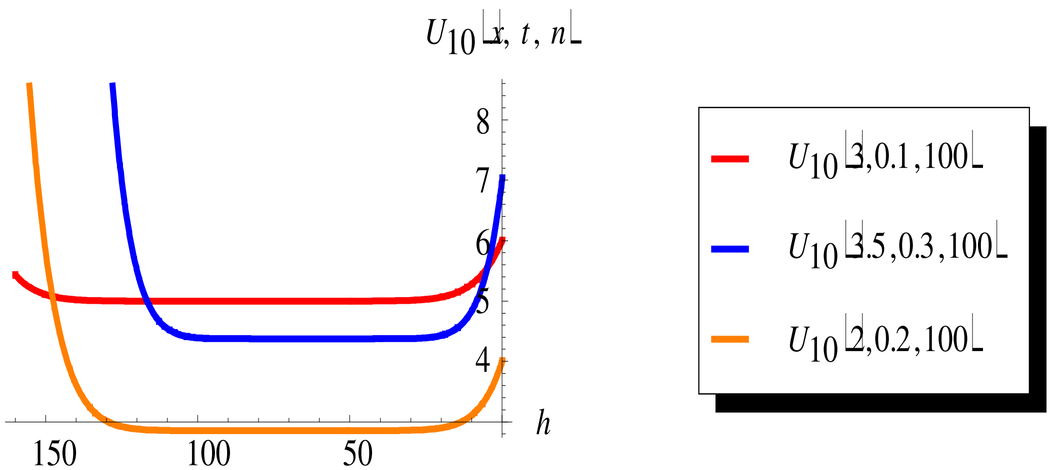

Figure (4) : h - curve for the (q-HAM; $n = 100$) approximation solution $U_{10}(x, t; 100)$ of problem (12) at different values of x and t.

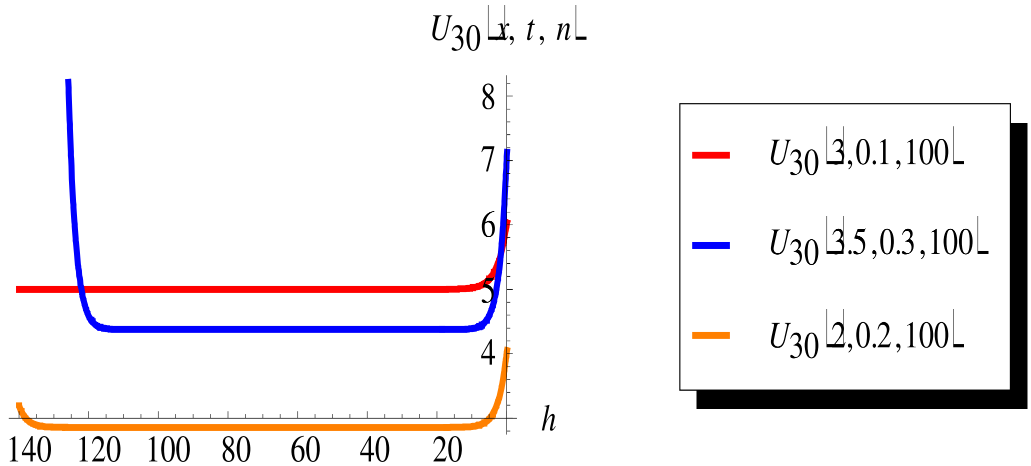

Figure (5) : h - curve for the (q-HAM; $n = 100$) approximation solution $U_{30}(x, t; 100)$ of problem (12) at different values of x and t.

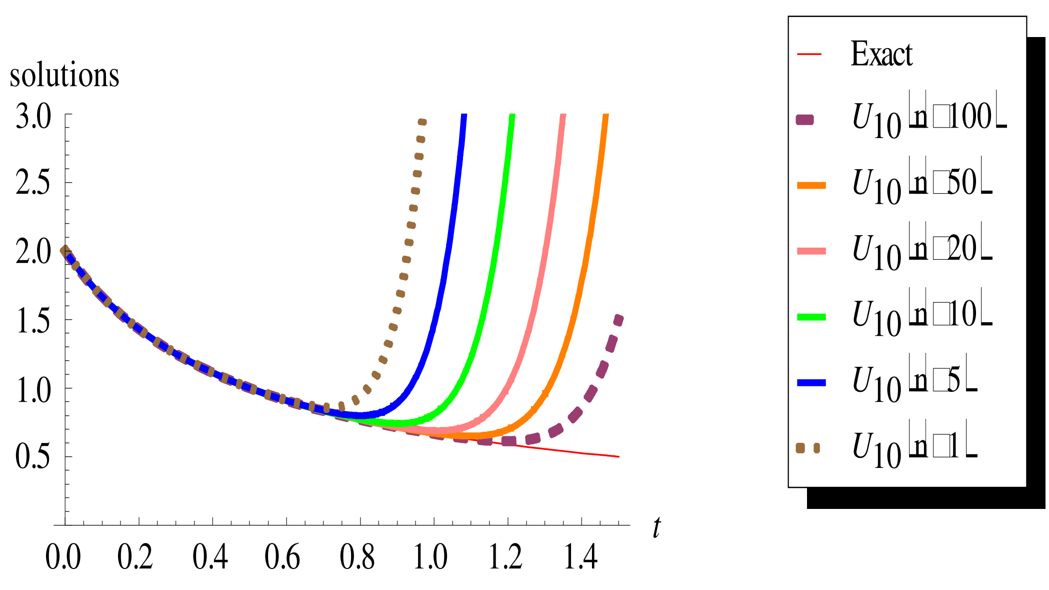

Figure (6): Comparison between U_{10} of HAM (q-HAM ($n = 1$)) and q-HAM($n = 5, 10, 20, 50, 100$) with the exact solution of (12) at $x = 1$ with ($h = -0.7, h = -3.25, h = -6, h = -11.1, h = -26.1, h = -49$).

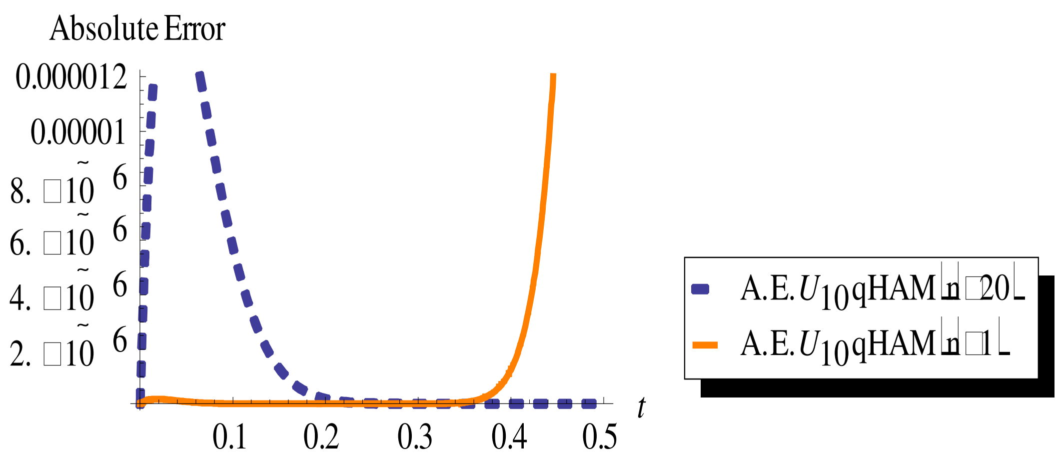

Figure (7): The Absolute error of U_{10} of q-HAM ($n = 1, n = 20$) for problem (12) at $0 < t \leq 0.5$ and $x = 1$ using $h = -0.7$ *and* $h = -11.1$.

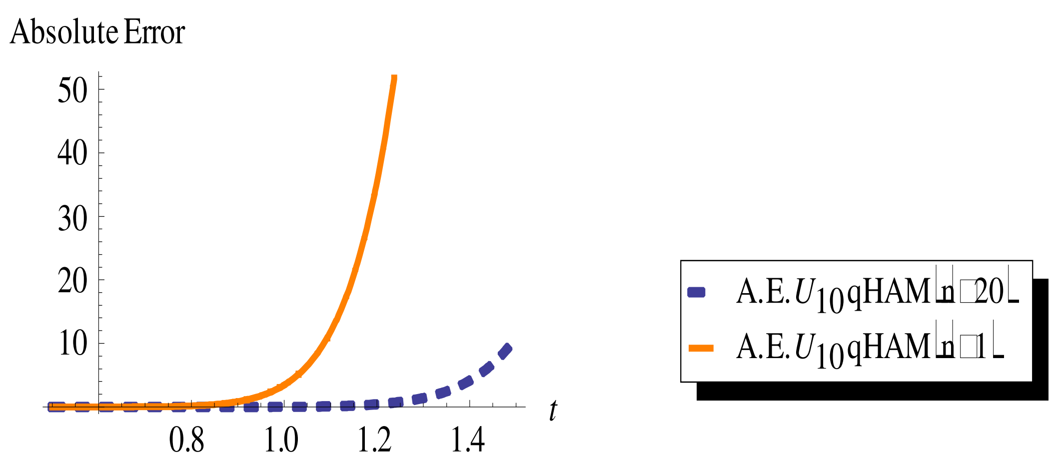

Figure (8): The Absolute error of U_{10} of q-HAM ($n = 1, n = 20$) for problem (12) at $0.5 < t \leq 1.5$ and $x = 1$ using $h = -0.7$ *and* $h = -11.1$.

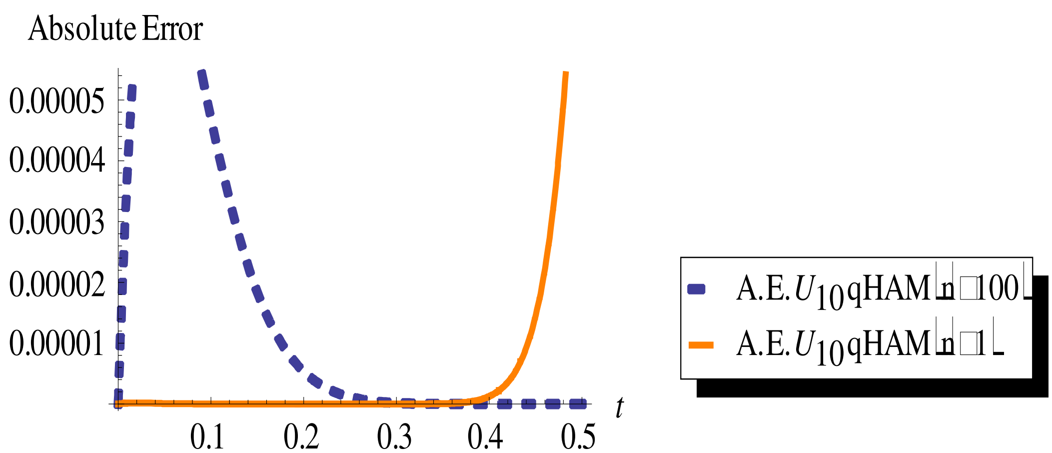

Figure (9): The Absolute error of U_{10} of q-HAM ($n = 1, n = 100$) for problem (12) at $0 < t \leq 0.5$ and $x = 1$ using $h = -0.7$ *and* $h = -49$.

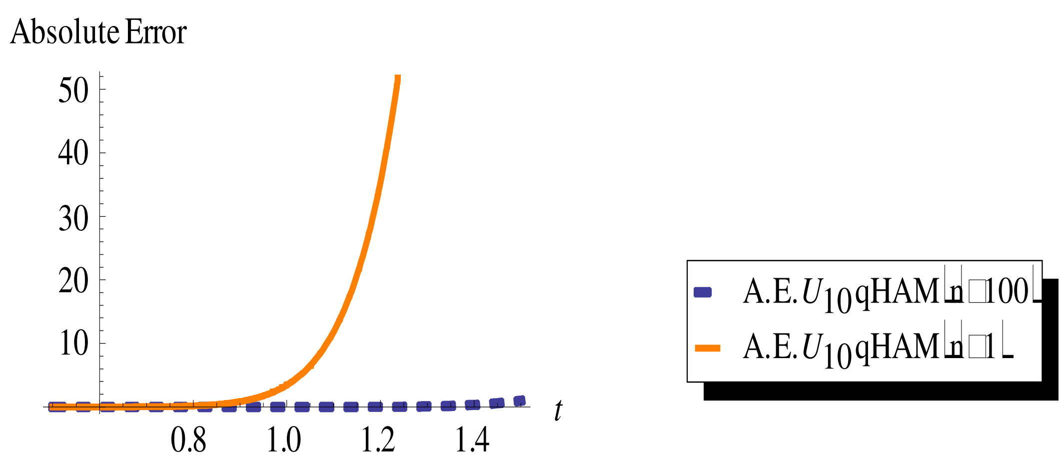

Figure (10): The Absolute error of U_{10} of q-HAM ($n = 1, n = 100$) for problem (12) at $0.5 < t \leq 1.5$ and $x = 1$ using $h = -0.7$ *and* $h = -49$.

It should be noted that the absolute error is highly decreased when modifying the solution by taking more terms into consideration. Figures (11,12) illustrate this fact.

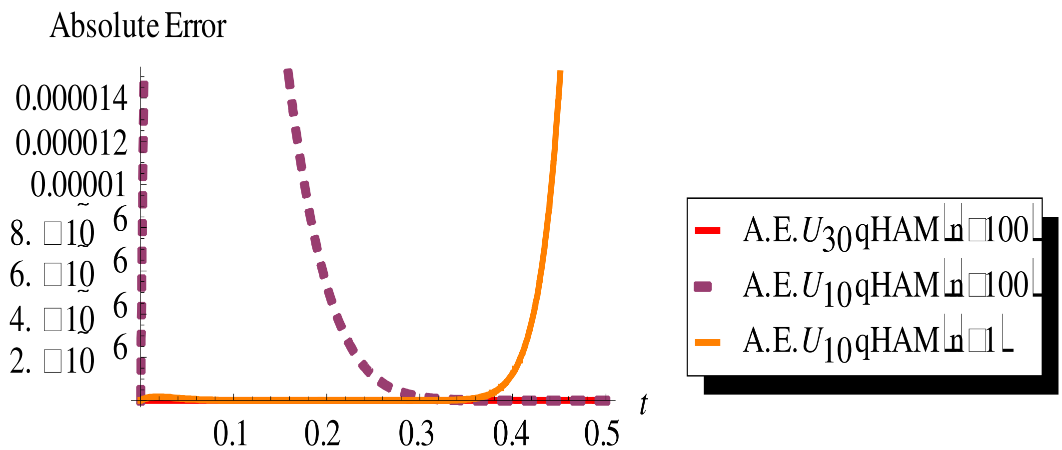

Figure (11): The Absolute error of U_{10} ($n = 1, n = 100$) and U_{30}($n = 100$) of q-HAM for problem (12) at $0 \le t \le 0.5$ using $h = -0.7$, $h = -49$ *and* $h = -40$ respectively.

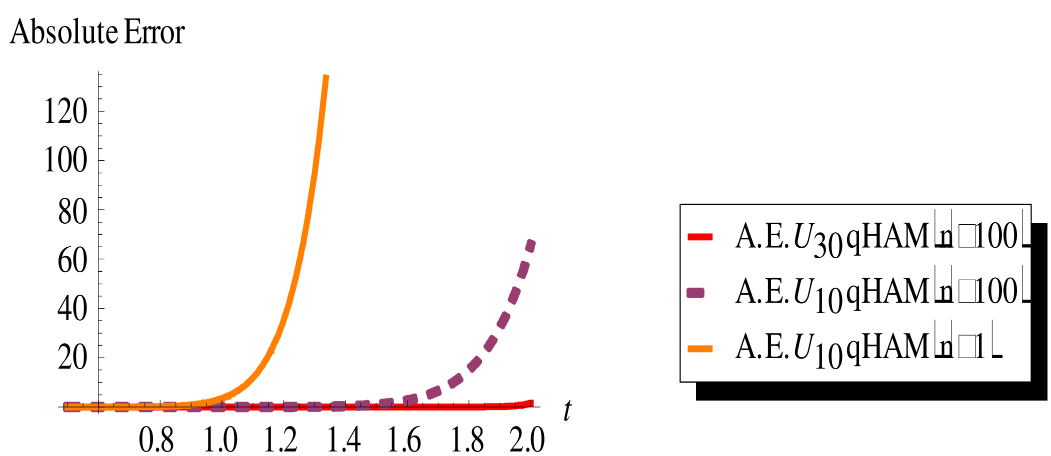

Figure (12): The Absolute error of U_{10} ($n = 1, n = 100$) and U_{30}($n = 100$) of q-HAM for problem (12) at $0.5 \le t \le 2$ using $h = -0.7$, $h = -49$ *and* $h = -40$ respectively.

5. Conclusion

In this paper, the q-homotopy analysis method has been analyzed with an aim to investigate the conditions which result in the convergence of the generated homotopy solutions of the nonlinear problems. The theorems provided here, have proved that the solution of the zeroth-order deformation equation together with the original problem exists as a power series in q .So, if specific values are assigned to the auxiliary parameters in the q-homotopy analysis method, then the approximate homotopy results successfully converge to the exact solution, and the upper bound for the error is decreasing as q is decreased.

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